Peak and RMS Voltage, Power

The DC power dissipated in a resistor is

$$P = IV = \frac{V^2}{R} = I^2 R$$

For sinusoidal alternating current, with a period t, the voltage has the form

$$V = V_0 \cos \omega t, \qquad t = \frac{1}{f} = \frac{2\pi}{\omega}$$

And the thermal power *P* deposited in the resistor is

$$P = \frac{1}{Rt}\int V^2(t)\,dt = \frac{1}{2}\frac{V_0^2}{R}$$

(The integral of $\cos^2$ is ½ .)

Here, V0 is the peak voltage of the sine wave. The ½ in the equation is a result of the definition of the amplitude of the sinusoidal waveform.

The usual definition of the amplitude of a sinusoidal AC waveform is the RMS (root mean square) value, which is

$$V_{rms} = \frac{1}{\sqrt{2}} V_{peak}$$

Then

$$P_{rms} = \frac{V_{rms}^2}{R}$$

Peak fields are of interest in accelerators, as they define the energy gain across a gap, for example, but the conventional definition is the RMS value of the voltage or current.

What is the peak and peak-and-peak value of 120 volt AC line voltage?

Power Factor

Let's see how much power it would take to excite an accelerator gap.

A typical 200 MHz accelerator has an average field strength of 2 MV/m. Our linac is injected with by a 460 keV proton source, with a velocity β = 0.031. Since λ = 1.5 meters, the cell length is 4.7 cm. The accelerating gap is typically ¼ of the cell length, or 1.17 cm. The diameter of a typical drift tube is 21 cm.

We can calculate the capacitance of the gap.

$$C = \epsilon_0 \frac{A}{d} = 26\, pF$$

The voltage on the gap is 2 MV/m times the cell length, or 94 kV.

The capacitive reactance of the gap is

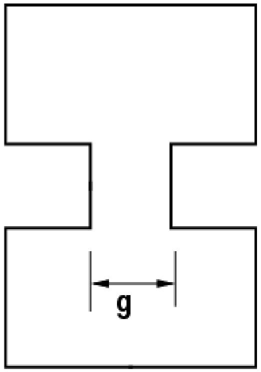
$$X_{gap} = \frac{1}{j\omega C} = -j30.6\, ohms$$

And the peak current at 94 kV to charge this capacitance is V/Xgap = 3070 amperes.

The RMS power is ½ V I = 144 Megawatts! (This is a lot of power!)

Resonant Energy Storage

To realize the benefit of a resonant structure, we will calculate the same configuration, but now included in a single-cell linac cavity.   The SUPERFISH code will calculate the actual parameters of the cavity.



For the same peak voltage across the gap, 94 kV, including the power loss on all the walls, only 29 kilowatts is required.

If the end walls are removed, as in a longer structure of several cells, this cell will require only 7.74 kW.

This is a huge reduction in power, compared to exciting a capacitive gap in a non-resonant system, a savings of about 18000.

The drive power to the linac cell is stored and built up over a period of time, the filling time, to produce a high gap field.

Power Factor

In our example of driving a 26 picoFarad gap, notice that the driving voltage and current are 90 degrees out of phase.

The power delivered by a voltage source supplying a current is actually the vector product. For voltage and current expressed as RMS quantities,

$$P = I\,V\cos\phi$$

where ϕ is the phase difference between the voltage and current waveforms.

If ϕ = 90 degrees, no actual power is delivered to the load. However, the power company is still supplying volts, and the wires are still carrying current, which spin the wattmeter. The term *cos* ϕ is the **power factor** of the load.

Power Factor = 100% *cos* ϕ

And the units are volt-amp, (KVA, MVA). The most efficient load has a 100% power factor: the voltage and current are in phase.

Resonant Cavities

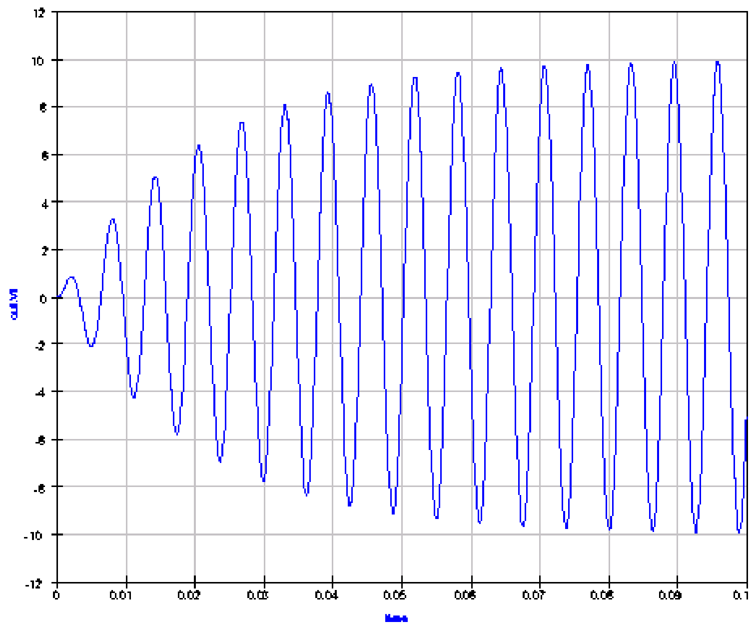
We saw that at resonance, a system can be driven to large amplitude with less power than a non-resonant system, as energy is stored in the oscillator during the build-up.

If the oscillator has no dissipation (loss), the stored energy will increase indefinitely. If there is energy loss in the structure, it will be proportional to the stored energy in the structure, which is proportional to the square of the amplitude of the fields, and the fields will approach an asymptotic value.

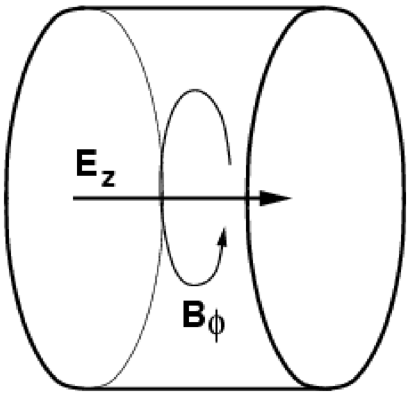
At the asymptotic field level, the energy loss per cycle is equal to the energy from the source per cycle.

These structures are **narrow band** and support fields at one particular frequency.

Some accelerator systems are **wide band**, similar to the charged capacitor example where special waveforms are required. They are much less efficient.



Pillbox Cavity



A simple resonant cavity is the pillbox cavity.

The cavity supports an E-field along the axis, and

$$\nabla \times \vec{H} = \epsilon_0 \dot{\vec{E}}$$

indicates that a B-field circulates around the axis in the azimuthal (ϕ) direction.

The pillbox cavity forms the basis of the Alvarez accelerator cavity and a typical buncher cavity.

We will analyze the fields and their modes in the pillbox cavity.

Boundary Conditions

The E and B (H) fields are subject to **boundary conditions** on metallic surfaces.

No component of the E vector may be parallel to a metallic surface.
The E field vector is perpendicular to the surface.

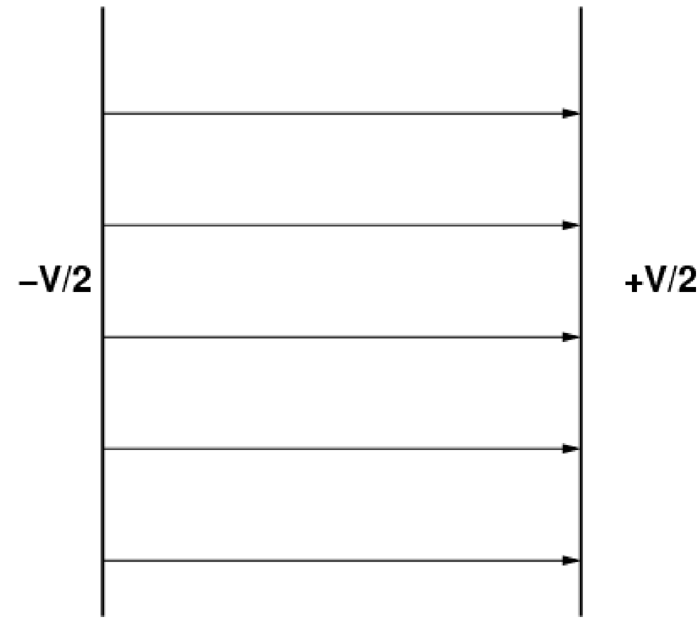
No component of the B vector may be perpendicular to a metallic surface.
The B field vector is parallel to the surface.

The H field at the surface is mirrored by an equivalent current density J in the surface (amps/meter), oriented 90 degrees in the metal to the direction of H at the surface.

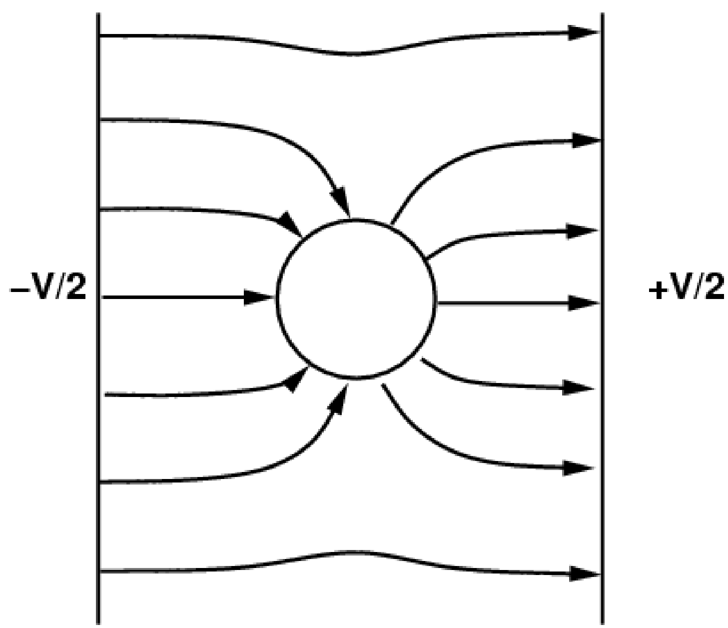
The surface current J will flow in the metal, and if the surface is lossy, will result in power being dissipated in the material.

E-Field on Metallic Boundary

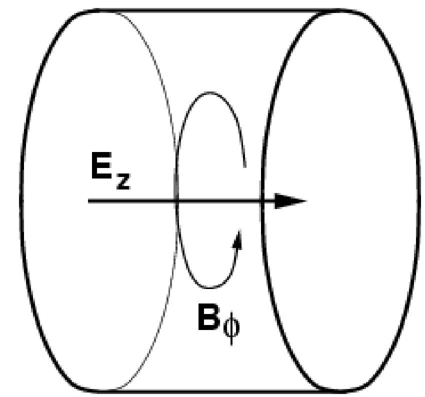
Between two parallel plates, the E-field is perpendicular to the plates. (There may be fringe fields at the edges of the plates, but the E-vector is still perpendicular.)



If a conducting rod or sphere is inserted between the plates, the E-field vector must terminate on the sphere at right angles to the surface.



Analysis of the Pillbox Cavity



We will use cylindrical coordinates $r,\ \phi,\ z$

The E-field vector is everywhere perpendicular to the walls.

The only field component is E_z

$E_z = 0$ on the sidewalls

$E_\phi = 0$ on the sidewalls

The H-field vector has no component perpendicular to any wall.

$H_z = 0$ on the endwalls

$H_r = 0$ on the sidewalls

Only H_ϕ is present.

We have not said anything yet about the variation of $E_z(z)$ along the cavity.

The Wave Equation for the Pillbox Cavity

The wave equation in cylindrical coordinates is (Wangler page 28)

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{\partial^2 E_z}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

There are two sets of solutions to this equation

Transverse magnetic (TM) solutions: Pillbox cavities, Alvarez linacs

Transverse electric (TE) solutions: Deflecting cavities, RFQ linacs

The wave equation is of the form that has Bessel functions as its solution.

$$z^2\frac{d^2 x}{dz^2} + z\frac{dx}{dz} + (z^2 - m^2)x = 0$$

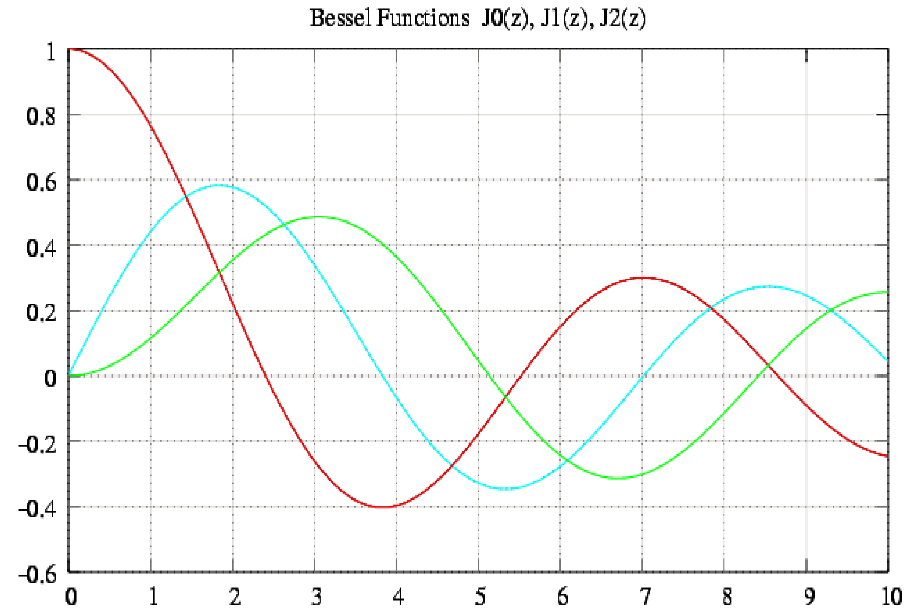
Most accelerators are constructed with some sort of cylindrical symmetry, so we can use the same set of coordinates for both analyses.

The TM Modes

The TM solution to the wave equation in cylindrical coordinates has the form (Wangler, page 30-31), with the sinusoidal time dependence removed:

$$E_z \propto J_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z$$
$$E_r \propto p J'_m(k_{mn} r) \quad \cos m\theta \quad \sin k_z z$$
$$E_\theta \propto \frac{p}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \sin k_z z$$

$$B_z = 0$$
$$B_r \propto -\frac{i}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \cos k_z z$$
$$B_\theta \propto -i J'_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z$$



m, n, p are integers that describe the mode of the solution.
The J_m are Bessel functions of the first kind. J'_m is the derivative of the Bessel function.

$$k_z = \frac{\pi p}{L_{cav}}, \qquad \frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2 \qquad J'_0(z) = -J_1(z), \quad J'_1(z) = J_0(z) - \frac{1}{z} J_1(z), \ldots$$

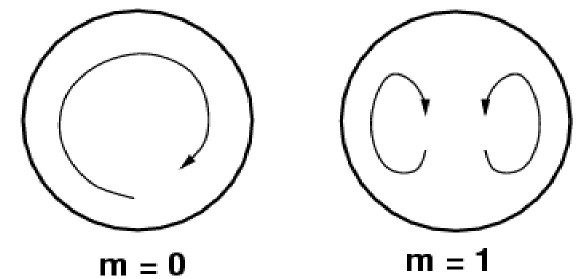
The *i* in the equations for *B* is $\sqrt{-1}$ and indicates that the E and B fields are 90 degrees offset from each other in RF phase.

TM Mode Indices *m, n, p* TM_{mnp}

The modes are described by three indices.

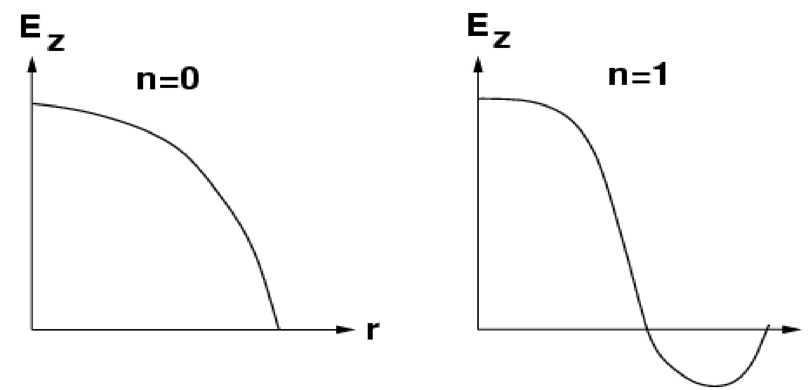
m is the number of variation of field
of the azimuthal variable ϕ:

m = 0, 1, 2, ...



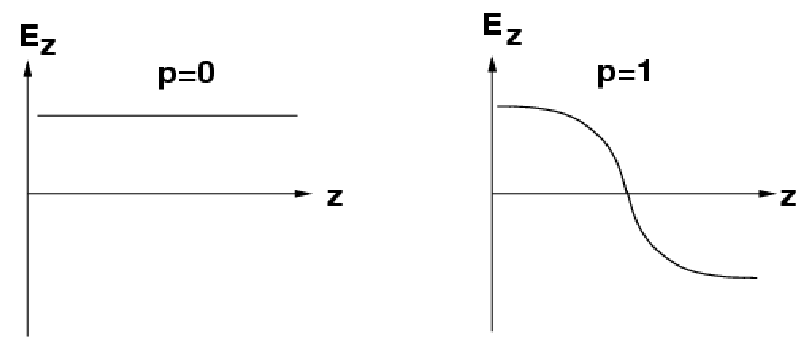
n is the number of nulls in E_z along the
radial direction

n = 1, 2, 3, ...



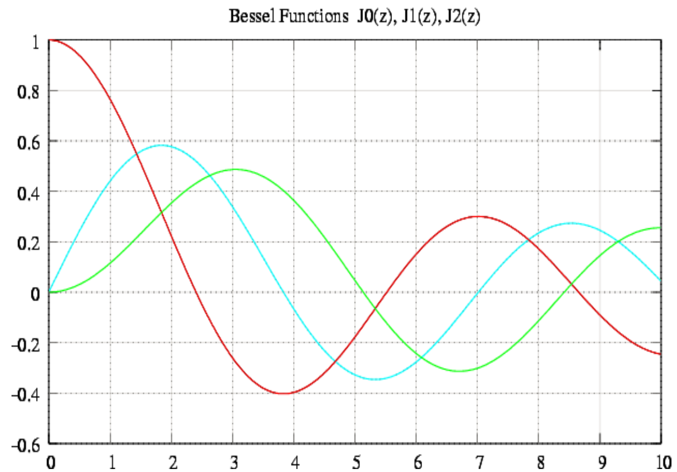
p is the number of nodes of E_z along
the z-axis.

p = 0, 1, 2, ...



The TM_{010} Pillbox Mode

The radius of the cavity is *a*.
m = 0, *n* = 1, *p* = 0.



$$E_z(r) = A J_0(k_{01} r)$$

The boundary condition that $E_z(a) = 0$ is satisfied if $k_{01}\, a$ = the first zero of J_0. $(k_{01}\, a) = 2.405$.

$$k_z = \frac{\pi p}{L_{cav}} = 0, \qquad \frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2 = k_{01}^2$$

We can solve this for the resonant frequency of the TM_{010} mode

$$k_{01} = \frac{2.405}{a}, \quad \omega = k_{01} c, \quad f = \frac{\omega}{2\pi} = \frac{2.405}{2\pi}\frac{c}{a}$$

For a pillbox cavity with a radius *a* = 1 meter, the TM_{010} mode frequency is

f = 114.9 MHz

and is independent of the length of the cavity.

The TM_{010} Fields

$m = 0$, $n = 1$, $p = 0$

$p = 0$ so E_r and $E_\theta = 0$

$$E_z(r) = E_0 J_0\left(2.405\frac{r}{a}\right)$$

$m = 0$ so $\sin m\theta = 0$ so $B_r = 0$

B_z is always zero

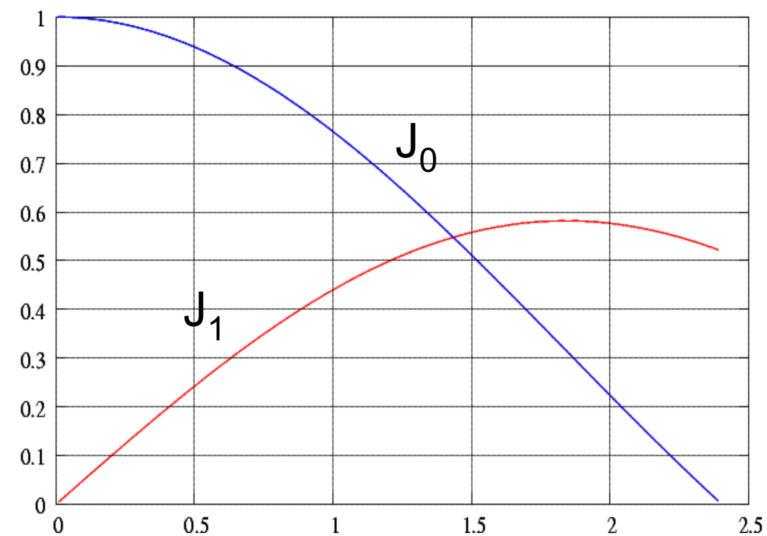
$$B_\theta(r) = i B_0 J_1\left(2.405\frac{r}{a}\right)$$

Note that E_z is at a maximum on the axis and zero at $r = a$, and that B_θ is maximum about ¾ of the way out.

E_0 and B_0 are constants.

$$E_z \propto J_m(k_{mn}r) \quad \cos m\theta \quad \cos k_z z$$
$$E_r \propto pJ'_m(k_{mn}r) \quad \cos m\theta \quad \sin k_z z$$
$$E_\theta \propto \frac{p}{r}J_m(k_{mn}r) \quad \sin m\theta \quad \sin k_z z$$

$$B_z = 0$$
$$B_r \propto -\frac{i}{r}J_m(k_{mn}r) \quad \sin m\theta \quad \cos k_z z$$
$$B_\theta \propto -iJ'_m(k_{mn}r) \quad \cos m\theta \quad \cos k_z z$$



TM_{0np} Mode Spectrum

For m = 0, the modes are azimuthally symmetric (no θ dependence).

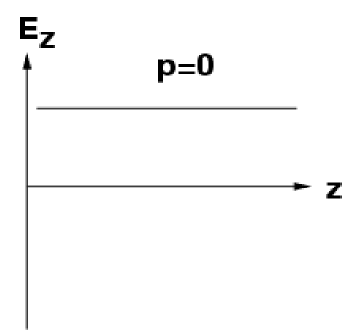
$$E_z \propto J_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z$$
$$E_r \propto p J'_m(k_{mn} r) \quad \cos m\theta \quad \sin k_z z$$
$$E_\theta \propto \frac{p}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \sin k_z z$$

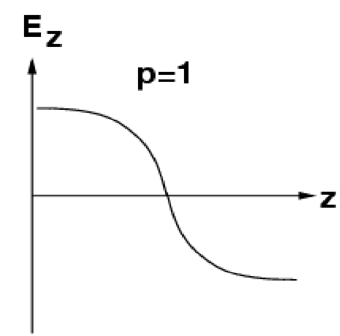
$$B_z = 0$$
$$B_r \propto -\frac{i}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \cos k_z z$$
$$B_\theta \propto -i J'_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z$$

The TM_{0n0} modes show a radial dependence of $E_z(r)$ that has n zeros (including the zero at the outer radius). These modes are not harmonically related, but lie along the zeros of $J_0(k_{0n})$.
Those values of k_{0n} are 2.405, 5.520, 8.654...

For $p > 0$, $E_z(z)$ has p nodes (zeros) along the z-axis. The frequency of the TM_{0np} modes for $p > 0$ depend on the length of the cavity, and E_r and E_θ have components which are non-zero, except at the outer radius.

$$\frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2, \qquad f = \frac{c}{2\pi}\sqrt{k_{mn}^2 + \left(\frac{\pi p}{L_{cav}}\right)^2}$$

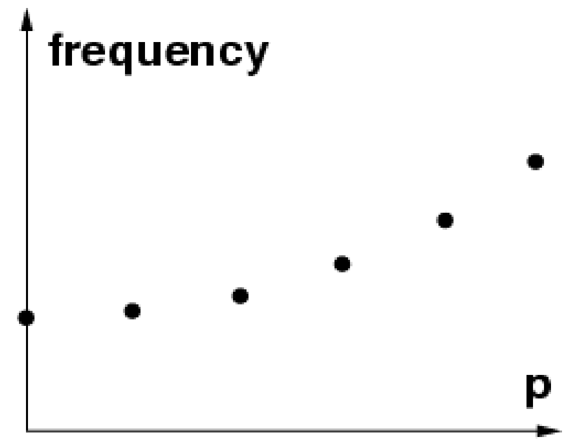




Mode Mixing

These structures can be characterized as having a spectrum of frequencies.

$$\frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2, \qquad f = \frac{c}{2\pi}\sqrt{k_{mn}^2 + \left(\frac{\pi\, p}{L_{cav}}\right)^2}$$



This plot is a type of **dispersion plot**, which relates the resonant frequency to the phase shift along the axis of the field.

The lowest mode, *p* = 0, has a uniform $E_z(z)$ distribution along the axis of the accelerator. Notice that the next higher mode is only slightly removed from the fundamental mode, and has a $E_z(z)$ distrbution that has one node halfway down the linac.

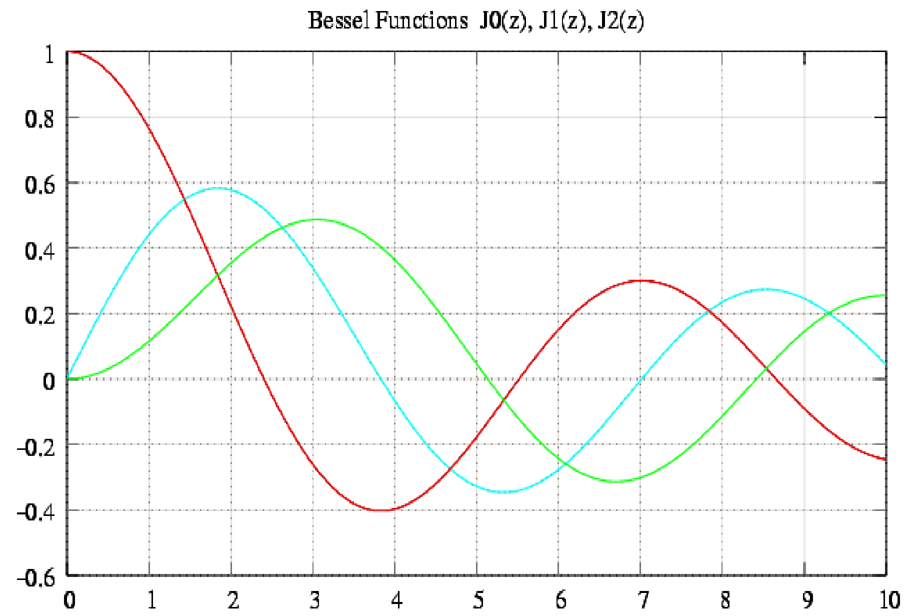
Energy can be coupled into this and higher modes by several methods, such as construction errors or beam loading, and alter the desired field configuration of the linac. We will discuss this further.

The TE_{mnp} Modes

These are the transverse electric modes.

$$E_z = 0$$
$$E_r \propto \frac{i}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \sin k_z z$$
$$E_\theta \propto i J'_m(k_{mn} r) \quad \cos m\theta \quad \sin k_z z$$

$$B_z \propto J_m(k_{mn} r) \quad \cos m\theta \quad \sin k_z z$$
$$B_r \propto p J_m(k_{mn} r) \quad \cos m\theta \quad \cos k_z z$$
$$B_\theta \propto -\frac{p}{r} J_m(k_{mn} r) \quad \sin m\theta \quad \cos k_z z$$

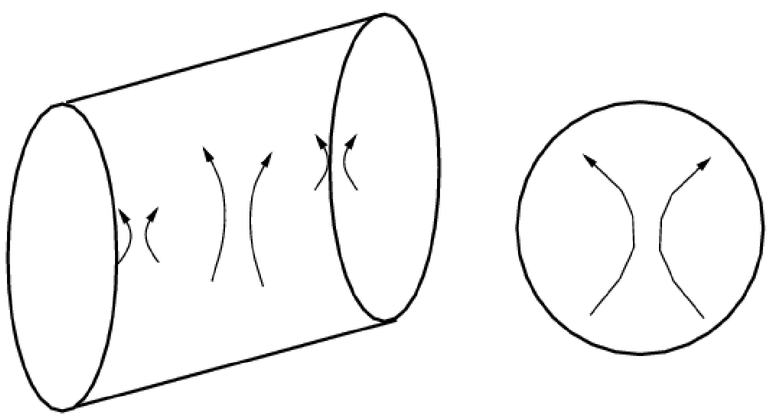


$$k_z = \frac{\pi p}{L_{cav}}, \qquad \frac{\omega^2}{c^2} = k_{mn}^2 + k_z^2$$

m = 0, 1, 2,... azimuthal
n = 1, 2, 3,... radial
p = 1, 2, 3,... longitudinal (why not 0?)

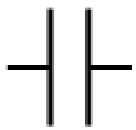
Here, $k_{mn} = x'_{mn}/R_{cavity}$ and the x'_{mn} are the zeros of J'_m.

$x'_{01} = 3.832$, $x'_{02} = 7.016$, $x'_{03} = 10.174$,...



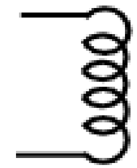
The RFQ uses a TE_{210} mode of operation.

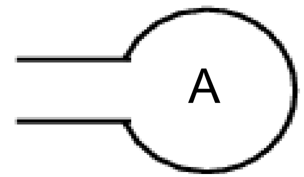
Lumped-Circuit Equivalent of a Pillbox Cavity



A capacitor stores energy

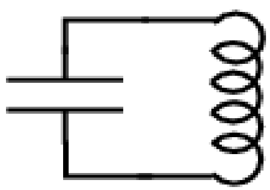
$$U = \tfrac{1}{2}CV^{2}, \qquad C = \epsilon_0 \frac{A}{d}$$



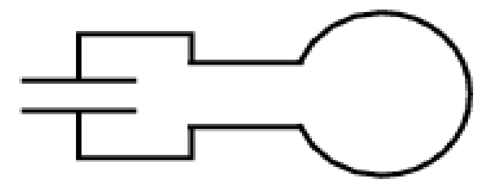


The inductance of an inductor is a comples function of its area A.

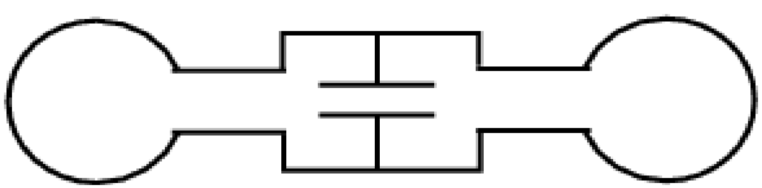
The energy stored is $U = \tfrac{1}{2} L I^{2}$



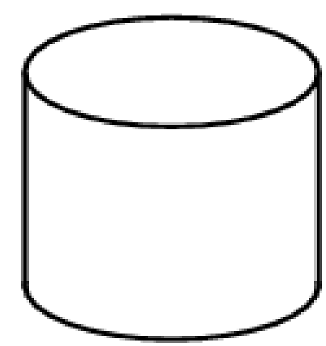
Connect to form a resonant circuit, and make the inductor smaller to raise the frequency.



Put more conductors in parallel across the capacitance, further raising the frequency.



Eventually, the inductors form a wall around the capacitance, forming a pillbox cavity.



Energy Relations in a Cavity

Energy stored in a capacitor and inductor: $U_{cap} = \frac{1}{2} C V^2, \quad U_{ind} = \frac{1}{2} L I^2$

The electric and magnetic fields in a cavity are 90 degrees apart in RF phase. At one instant in time, all the energy is stored in the electric field, and 90 RF degrees later in the magnetic electric field.

The stored electric and magnetic energies are integrals over the cavity volume, each when the stored energy in the magnetic and electric fields is zero:

$$U_E = \frac{\epsilon_0}{2} \int_{cavity} E^2 dV \, , \qquad U_H = \frac{\mu_0}{2} \int_{cavity} H^2 dV = \frac{1}{2\mu_0} \int_{cavity} B^2 dV$$

By conservation of energy (no losses) $U_E = U_H$

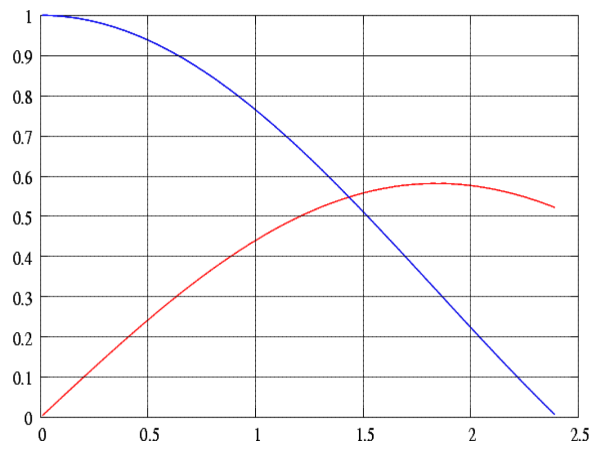
So the total stored energy can be written, taking the values of the E and H fields at their respective peak values

$$U_{cav} = \frac{1}{4} \int_{cavity} \left(\epsilon_0 E^2 + \mu_0 H^2 \right) dV$$

Field Balance and Frequency Perturbation

Resonance may be defined as $U_E = U_H$, the frequency where the stored electric field energy integral is equal to the magnetic field energy integral.

However, the *distrbution of fields* may be very different: in a pillbox cavity, the electric field is concentrated near the axis, and the magnetic field further out near the sidewall.

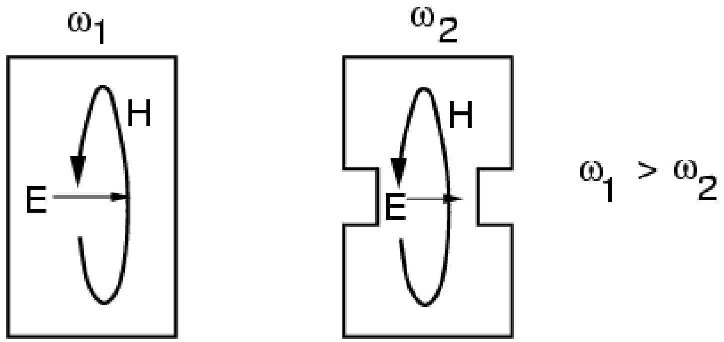


We have the opportunity of **tuning** a cavity by varying its geometry either near the axis or the outer wall.

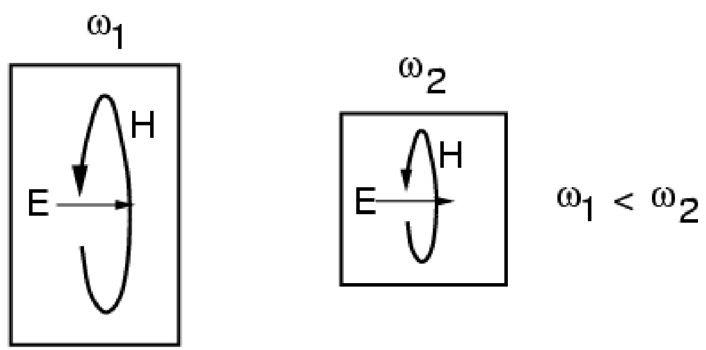
In analogy to lumped circuit models, we can associate the high E-field regions with electrical capacitance, and high H-field regions with electrical inductance.

Moving Walls

If the endwalls of the cavity near the axis are moved together, the frequency will decrease.



The capacitance between the endplates will increase, reducing the resonant frequency. Recall that for the TM_{010} mode, the cavity frequency is independent of cavity length. However, if we move the walls near the axis where E_z predominates, the frequency will decrease. *Here, we have removed volume that is occupied by E-field.*



If the sidewalls of the cavity are pushed in, the equivalent loop area of an inductor is decreased, and the frequency will increase. *Here, we have removed volume that is occupied by H-field.*

Remove E-field volume to decrease frequency, remove H-field volume to increase frequency.

Slater Perturbation Theorem and Bead Pulling

How can we measure the actual field distribution in a cavity?

We cannot just put a voltmeter test probe in the cavity. (A probe measures potential, anyway, not field, and it will disturb the field configuration in the cavity.)

By removing small volumes of E-field or H-field, or both, we can upset the energy balance $U_E = U_H$ in such a way that a new resonant frequency will be re-established, restoring the energy balance. This shift will be proportional to the volume of field energy removed.

This is known as the **Slater Perturbation Theorem**, and the technique of its use is known as **Bead Pulling,** (Wangler, page 162).

A metallic or dielectric bead is suspended on a thin thread and moved around inside the cavity. The frequency perturbation is then measured. The angular frequency is shifted, depending on the volume of *E* or *H* field that is removed by the bead.

$$\omega^2 = \omega_0^2 \left(1 + k \frac{\int_{bead} (\mu_0 H^2 - \epsilon_0 E^2)\, dV}{\int_{cavity} (\mu_0 H^2 + \epsilon_0 E^2)\, dV} \right)$$

The frequency shift is proportional to the difference in H and E-field energy removed by a bead.

Bead Pulling

The electric and magnetic field can be separately measured in the same location by using both a metallic bead, which removes E and H-field volume, and then retracing the path with a dielectric bead, which alters the E-field only.

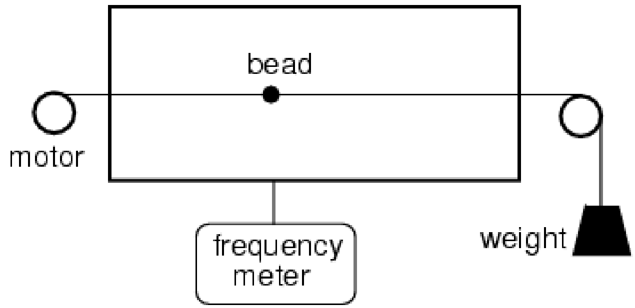
Subtracting one measurement from the other will separate the E and H fields in the path of the bead.

The constant *k* depends on the geometry of the perturber. For a sphere, *k* = 3.

$$\omega^2 = \omega_0^2 \left(1 + k \frac{\int_{bead} (\mu_0 H^2 - \epsilon_0 E^2)\, dV}{\int_{cavity} (\mu_0 H^2 + \epsilon_0 E^2)\, dV} \right)$$

For small perturbation in frequency, where τ is the volume of the bead. The frequency shift is proportional to the square of the field intensity.

$$\frac{\Delta\omega}{\omega_0} = \frac{3\tau}{4U_{cavity}} \left(\epsilon_0 E^2 - \frac{1}{2} \mu_0 H^2 \right)$$



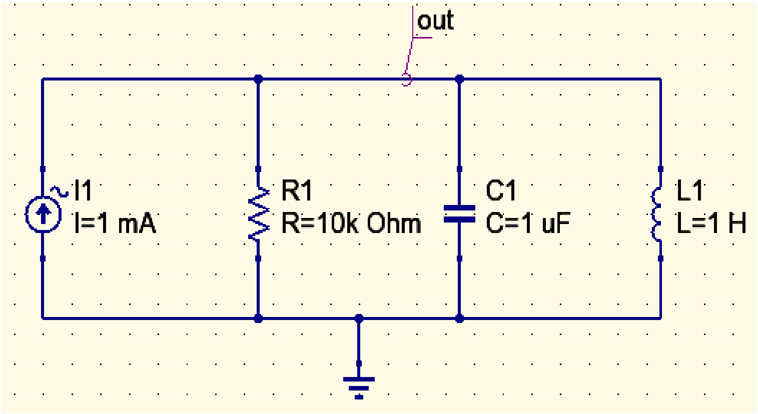
The beam is usually drawn through by a motor drive, and the measured frequency shift recorded on a computer.

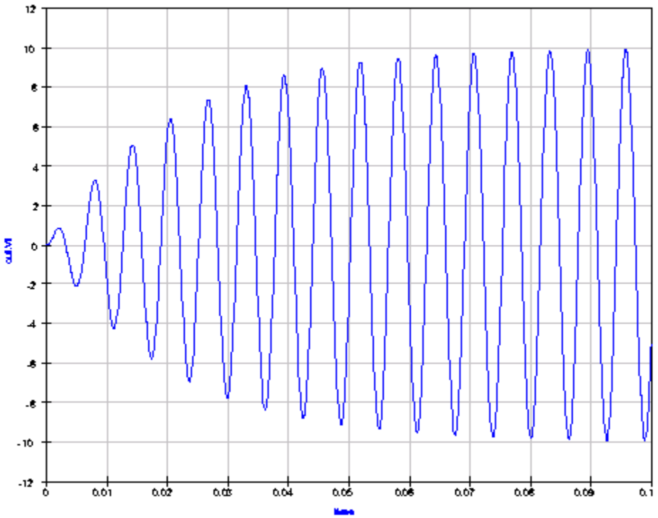
Shunt Impedance of a Lumped Circuit

Consider a lumped-circuit model of a lossy resonantor.

I1 is a current source with infinite internal impedance. It feeds energy into the LC resonant circuit.

A resistor R1 shunts the circuit, with a loss V^2/R.





When the generator is turned on, the voltage in the circuit builds up to an asymptotic limit. At this limit, the energy supplied to the circuit equals the dissipation in the resistor.

The stored energy in the resonant circuit is

$$U = \tfrac{1}{2} C V^2 = \tfrac{1}{2} L I^2$$

Which is larger than the energy delivered by the generator in one RF cycle.

Circuit Q

The quality factor, or Q of a resonant circuit is proportional to the total stored energy of the circuit divided by the power lost per cycle.

$$Q = \frac{\omega U}{P}$$

The Q of a resonant cavity is a measure of the power loss in the walls of the cavity due to the current flowing through walls of finite resistivity.

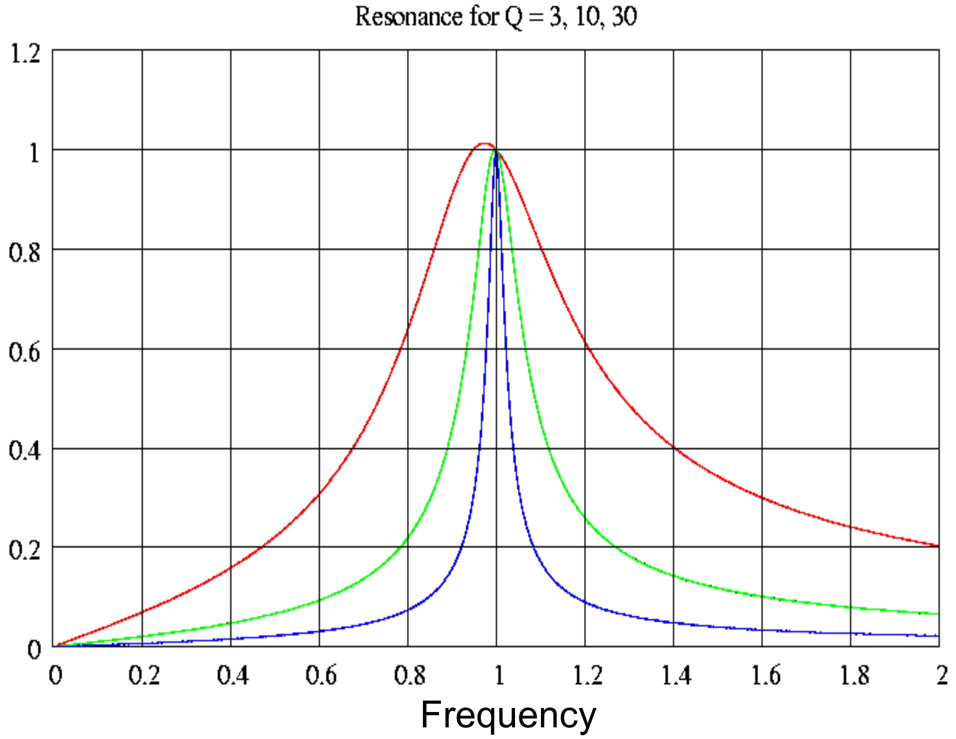
The reactance of the L and C in the lumped-element circuit are

$$X_L = \omega L, \qquad X_C = \frac{1}{\omega C}$$

At resonance, $X_L = X_C$. For a shunt resistor R, the Q of the circuit is

$$Q = \frac{R}{X_C} = \frac{R}{X_L}$$

The width of the resonance widens with lower Q. The approximate bandwidth is f/Q.



Shunt Impedance of an Accelerator Cavity

The shunt impedance of an accelerator measures the effectiveness of transforming input RF power to accelerating voltage.

$$Z_{sh} = \frac{V_{peak}^2}{2 P_{rms}}$$

The shunt impedance Z_{sh} relates the peak voltage across the gap to the rms RF power suppled from the power source.

The shunt impedance is directly related to the Q of the cavity. $Q = \frac{\omega U}{P}$

The value Z_{sh}/Q_0 is independent of the wall resistivity of the cavity (assuming that it is constant over the entire cavity inner surface). Z_{sh}/Q_0 is entirely dependent on the geometry of the cavity.

The subscript "0" of Q_0 refers to the quality factor of the cavity that does not have any external circuit elements. Since the cavity must have an RF source connected to it, the actual Q is a function of the intrinsic cavity Q_0 and the external circuit elements, such as the RF coupler and source. We will cover coupled systems later.

Skin Depth

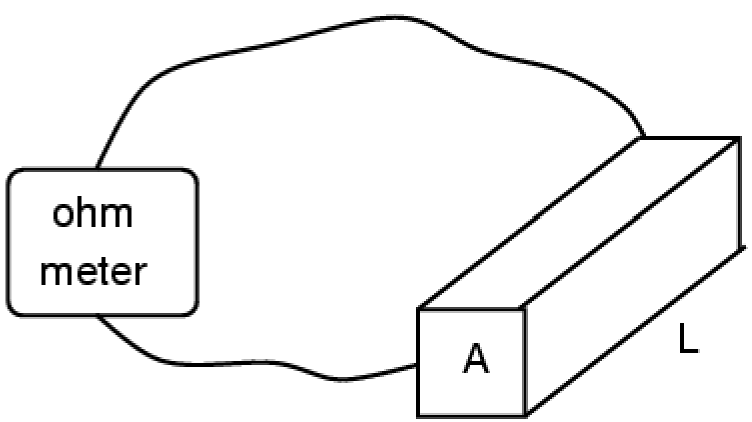
High frequency (RF) current tends to flow along the surface, and not in the bulk of conductors. The apparent RF resistivity of a conductor is higher than the DC resistivity. The current flow through a conductor with finite resistance results in the generation of heat.

The 1/e decay depth $\boldsymbol{\delta}$ of RF current flowing in a conductor is

$$\delta = \sqrt{\frac{\lambda}{\pi \mu_0 c \sigma}} = \sqrt{\frac{2}{\mu_0 \omega \sigma}} = \sqrt{\frac{1}{\pi \mu_0 f \sigma}}$$

Where λ is the wavelength of the RF, σ is the bulk (DC) conductivity of the conductor.

The bulk DC resistivity $\sigma = 1/\rho$ of a conductor can be measured on a sample of cross-sectional area *A* and length *L.* The ohmmeter measures a resistance *R*.



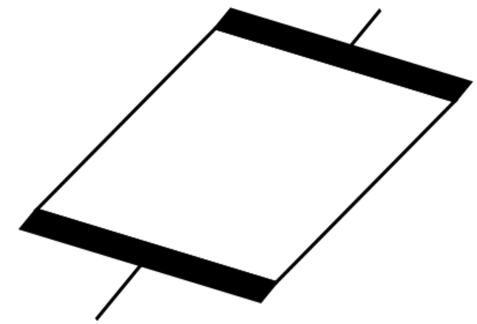
$$\rho = R\frac{A}{L}, \qquad R = \frac{\rho L}{A}$$

Conductivity $\sigma = 1/\rho$ [ohm^{-1} meter^{-1}]

Skin Effect

Since the RF current flow is confined to the surface of the conductor, not its bulk, the resistance may be expressed as **resistance/square.**

The sheet resistivity (resistance/square) is



$$R_{sq} = \frac{1}{\delta \sigma}$$

Where δ is the skin depth and $\boldsymbol{\sigma}$ is the bulk conductivity of the material.

For copper at room-temperature, $\rho = 1/\sigma = 1.724\times10^{-8}$ ohm-meter.

At a frequency of 200 MHz, $\delta = 4.7\times10^{-6}$ meter, $R_{sq} = 0.0037$ ohms/square.

Note that R_{sq} scales as $f^{1/2}$.

Power Dissipation on Cavity Walls

We have calculated the RF magnetic field distribution within the cavity. The current density [amps/meter] on the wall is numerically the same as the magnetic field H [amps/meter] at the wall.

The power dissipation over an area element dA of the wall is

$$P_{diss} = \frac{R_{square}}{2} \int_{walls} H_{wall}^2 \; dA$$

using the peak value of the magnetic field H_{wall} and the average (thermal) value of the power P_{diss}. The lumped circuit analogy for a DC current I is

$$P_{diss} = R \; I^2$$

The quality factor Q_0 is purely geometric and is

$$\frac{Q_0}{R_{sq}} = \frac{1}{R_{sq}} \frac{\omega U}{P} = \frac{\omega \frac{1}{4} \int_{cavity} \left(\epsilon_0 E^2 + \mu_0 H^2 \right) dV}{\frac{1}{2} \int_{walls} H^2 \; dA}$$

Kilpatrick Criterion

High surface electric fields in a cavity can lead to **electron emission** and **sparking**. Experiments carried out 50 years ago led to an empirical criterion of the safe surface field limit in a cavity, above which electrical breakdown was probable. The cavities were provided with oil-pumped vacuum systems, unlike the clean organics-free vacuum system we use today.

Kilpatrick established his formula for a safe surface field as a function of RF frequency. The equation is in implicit form:

$$f\,[\,MHz\,] \;=\; 1.64\,E^{2}\,e^{-8.5/E}$$

Freq [MHz]	E [MV/m]
200	14.7
400	19.4
1300	32.1

where the electric field E is expressed in units of MV/m.

It is known that this formula significantly underestimates the sparking limit at frequencies lower than 200 MHz. At higher frequencies, the criterion scales approximately as $f^{\,0.4}$.

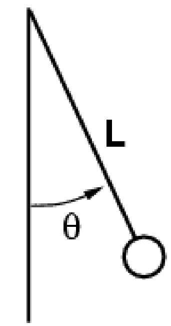
Today, with much cleaner vacuum systems, cavities are operated at much higher surface electric field levels. Fields in accelerators are still sometimes expressed in units of kilpatrick. Now, better physics models are known to predict the surface breakdown field, but kilpatrick is here to stay.

Groups of Cavities – Coupled Oscillators

As an illustration of the modes of coupled oscillators, we'll consider two pendula.

The second-order differential equation for the oscillation angle θ of a pendulum is

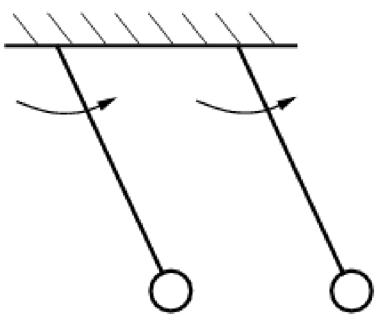
$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\theta = 0$$



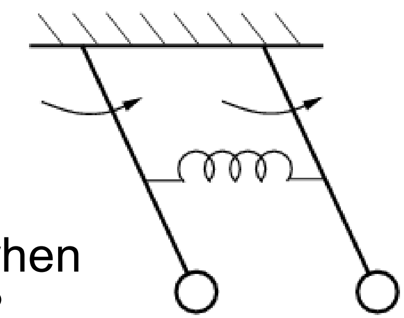
This is a nonlinear equation, but for small angles, $sin\theta \sim \theta$

which is solved by $\theta(t) = \theta_0 cos\omega t, \qquad \omega^2 = \frac{g}{L}$

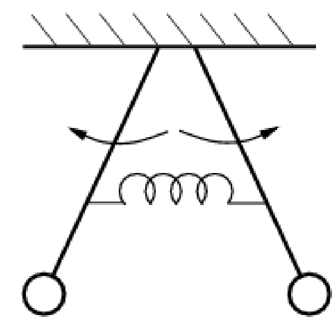
Now, consider 2 identical pendula swinging with the same phase and amplitude



If we place a spring between the two pendula, what does it do to the frequency when they are in phase?



What if they are out of phase?

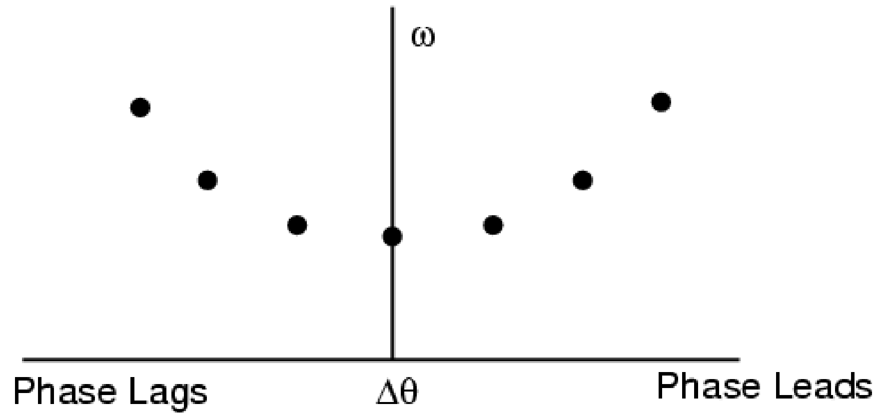


The spring stores no energy in the first case, but it does in the second, increasing the frequency of oscillation. These two cases are known as the **normal modes**.

Coupled Cavity Modes

Coupled accelerator cavities also exhibit this type of behavior. The normal modes are characterized by the phase difference between each cavity (oscillator) and its nearest neighbors, and the frequency of each mode.

A set of modes and their frequencies can be represented by a **dispersion plot**



This is not the same as the mode spectrum for a single cavity. A chain of coupled pillbox cavities, for example, will exhibit this type of a mode spectrum, all operating in the TM_{010} mode.

Impedance

Impedance is a measure of the ratio of the voltage across a circuit element to the current flowing through the circuit element. It is a generalization of **resistance R**.

$$R = \frac{V}{I}$$

This is adequate for DC circuits, but for RF, the voltage and current may not be in phase. Impedance includes the in-phase and quadrature phase (90^0) components. Impedance is expressed as a complex quantity, a sum of *R* and *X*, with *X* representing the quadrature component.

$$Z = R + jX, \qquad j = \sqrt{-1}$$

Frequently, in electrical circuit nomenclature, *j* instead of *i* is used for the imaginary part.

R is the **resistive**, or in-phase component
X is the **reactive**, or quadrature phase component

The impedance of an inductor *L* and capacitor *C* are, with the $j = \sqrt{-1}$ explicit:

$$X_L = j\omega L, \qquad X_C = \frac{1}{j\omega C}$$

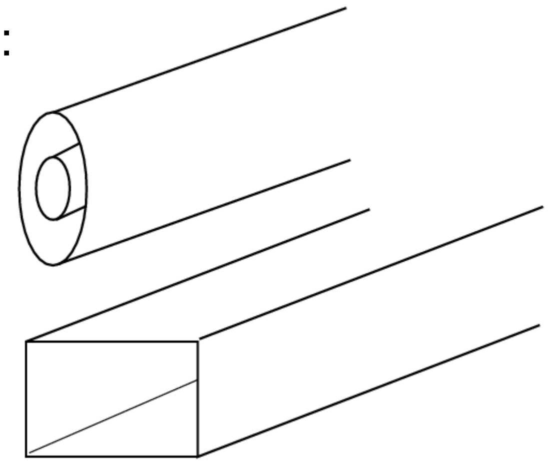
RF Transmission Lines

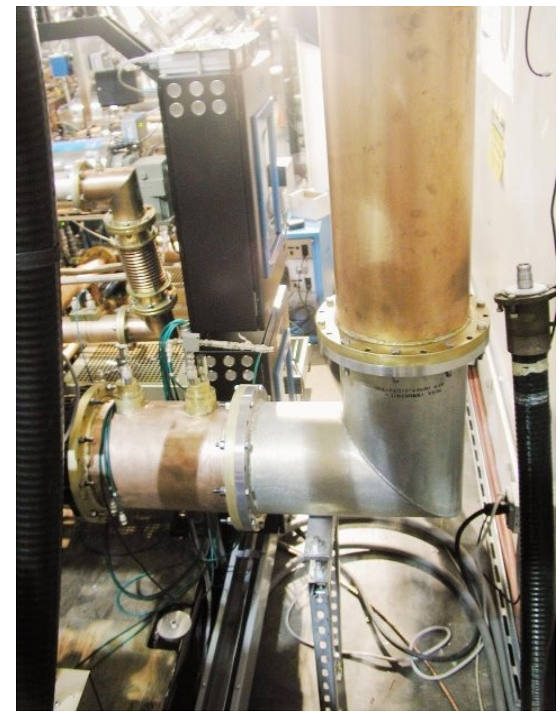
Transmission lines transmit RF power from one point to another with minimum loss and external radiation of energy.

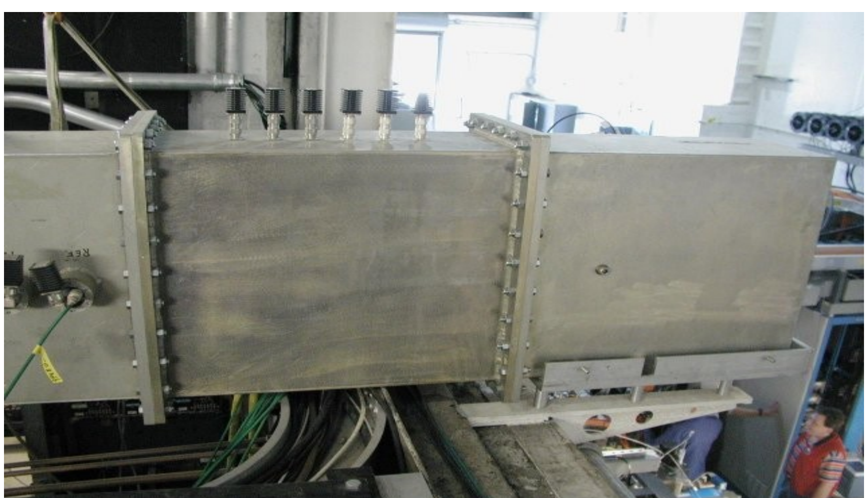
Two common types are:

Coaxial Cable

Waveguide



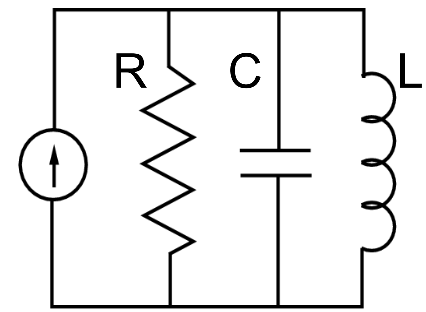




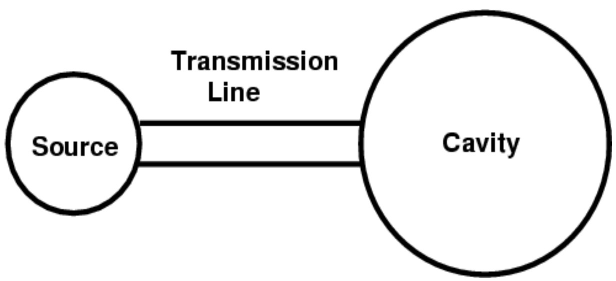
Coaxial cable (or hardline) is used for frequencies up to about 400 MHz and down to direct current, and waveguide at higher frequencies, where the loss is less than coax.

Loaded Cavity Q

Our idealized cavity equivalent circuit model shunts a resistance across a tuned circuit. Power is fed from a current source, which has infinite internal impedance.



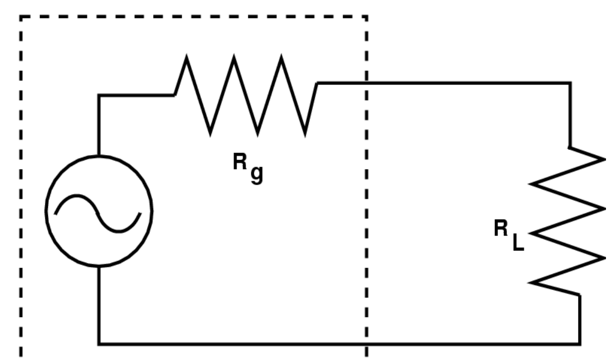
A real power source has a **source impedance**. The impedance of the power source interacts with the resonant cavity and modifies its bandwidth.



The source transmit power to the cavity, and the cavity returns power to the source.

For optimum power transfer, the source impedance equals the cavity impedance.

Consider a generator with internal impedance R_g driving a load of impedance R_L. Maximize the power dissipated in the load, and find that $R_g = R_L$. (Clearly R_L = 0 or infinity dissipates no power, so R_L must be some value in between.)



Loaded Cavity Q

An unloaded cavity (no drive loop) has an unloaded *Q*: $Q_0 = \frac{\omega U}{P}$

where *U* is the stored energy in the cavity and *P* is the power loss in the cavity. Coupling to the power source places an additional equivalent shunt resistance across the cavity. The Q of the total circuit, Q_{Loaded} is

$$\frac{1}{Q_{Loaded}} = \frac{1}{Q_0} + \frac{1}{Q_{external}}$$

This is the same equation for resistors in parallel. Q_{external} represents the source impedance of the power generator. The most effective power transfer from the source to the cavity is when $Q_{external} = Q_0$

We define a coupling factor β (yes, still another β) as the ratio of the cavity Q to the external Q.

$$\beta = \frac{Q_0}{Q_{external}}$$

$\beta < 1$ cavity is undercoupled
$\beta = 1$ cavity is critically coupled
$\beta > 1$ cavity is overcoupled

Cavity-Amplifier Interaction

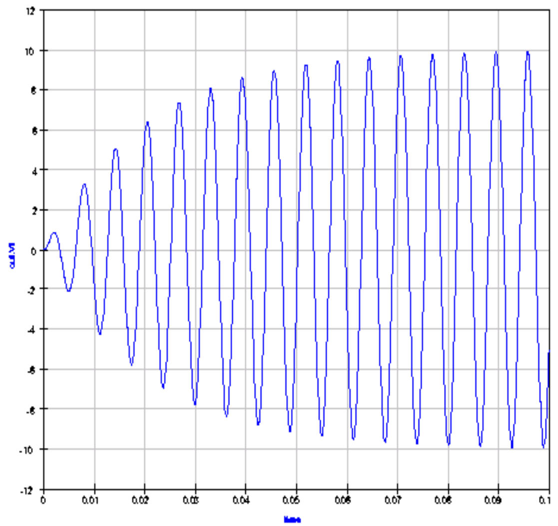
Does power actually from from the cavity back to the power source?

Yes. The power in the cavity can do work in the amplifier, dissipating power in the components of the amplifier, and accelerating the electrons in the power amplifier tube in the amplifier itself, increasing the plate dissipation of the power amplifier tube.

If the amplifier is suddenly turned off, stored energy in the cavity will flow back to the amplifier and be dissipated.

Cavity FIlling Time

When power is applied to an empty resonant cavity, the fields build up in time. The filling time, t_{fill}, is the time for the energy stored in the cavity with loaded $Q = Q_{loaded}$ to build to $1/e$ of its saturation point.



$$t_{fill} = \frac{Q_{Loaded}}{\omega}$$

What kind of load to the generator does the cavity present?

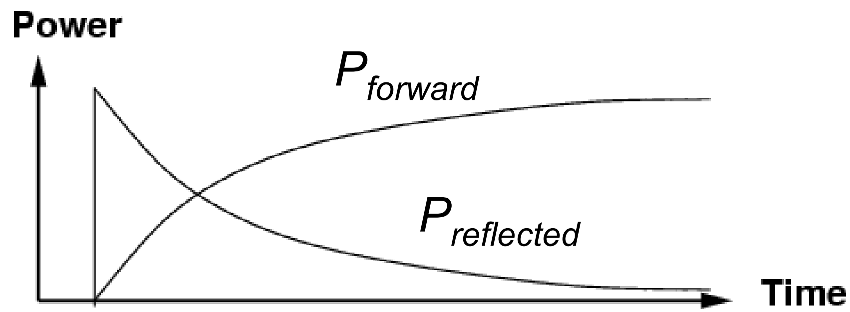
When the cavity is at full gradient, the voltage induced in the drive loop $V = \dot{B} A$ matches the impedance of the generator and transmission line and no power is reflected.

But at t=0, no field is present at the drive loop, and it appears as a short-circuit to the generator and power is reflected.

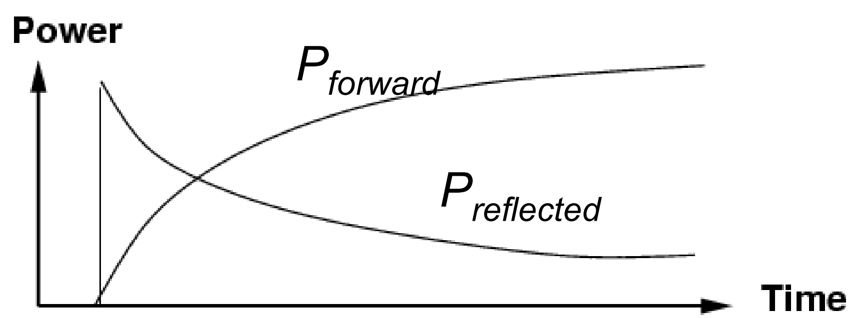
There exists a transient during the filling process that power is reflected from the cavity until the cavity is filled. If the cavity impedance is matched to the power source, the reflected power will asymptotically approach zero.

Reflected Power During Cavity Fill

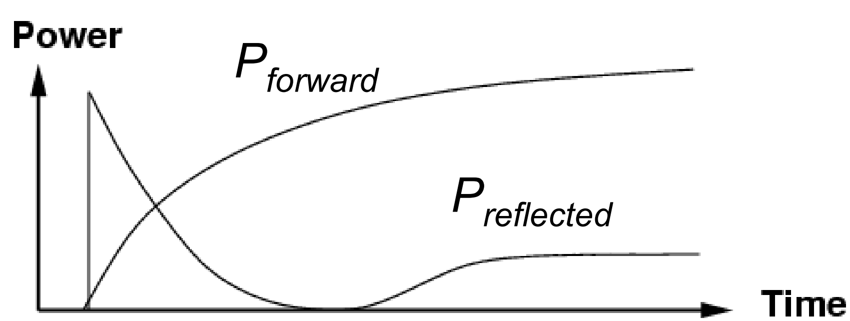
$\beta = 1$. Cavity fills and reflected power goes to zero



$\beta < 1$. Reflected power never goes to zero.



$\beta > 1$. At some point the cavity fields reflect enough power that the transmission line is matched, but the cavity continues to fill.



The Circulator – A Magic Device

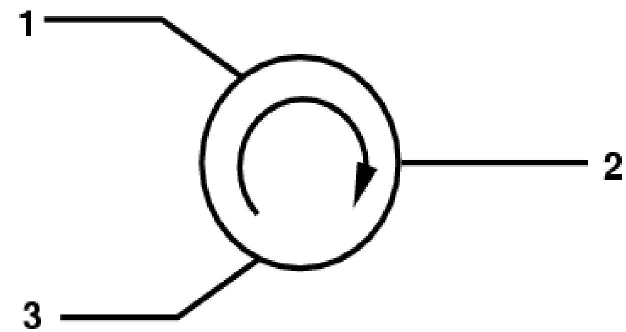
The devices we have studied so far are **reciprocal**, that is we can analyze them with the power going both in the forward or reverse direction consistently.

There is a device for which this does not apply: the circulator. It has the following characteristic:

Power into port 1 goes to port 2

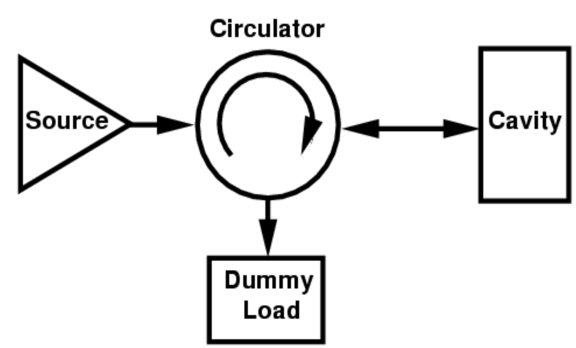
Power into port 2 goes to port 3

Power into port 3 goes to port 1



The circulator relies on the spin of electrons in particular materials aligned with a magnetic field. It is a **non-reciprocal** device.

The circulator can be used to isolate a power source from a load. Reflection from the load are shunted to a dummy load and the power source sees only a matched load.



The circulator is frequently used with power sources such as klystrons, which don't like mismatched loads.

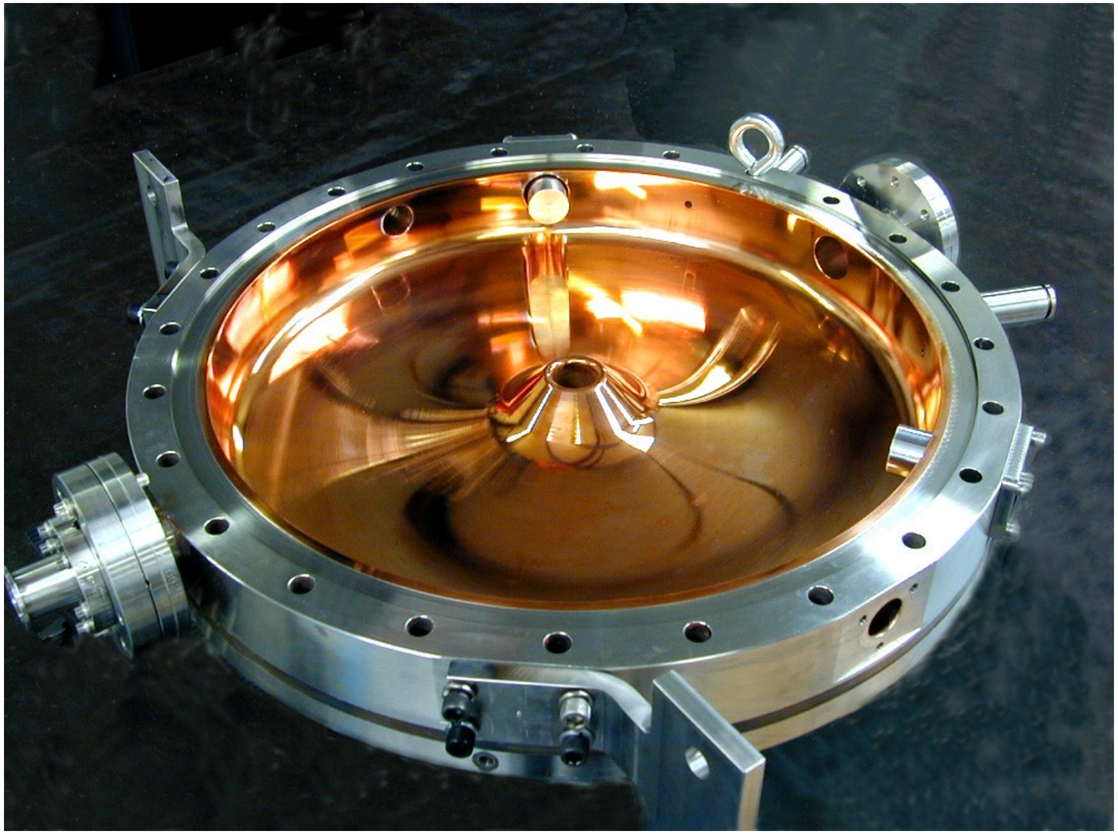
Rebuncher Cavity

We will use 2-D electromagnetics codes such as SUPEFISH to calculate the parameters of a rebuncher cavity.

A rebuncher (or a buncher) cavity is typically a pillbox cavity operating in the TM_{010} mode with the beam traveling along its axis in the region of maximum longitudinal E field. The cavity frequency is usually the same as the bunch or the linac frequency with a precise phase relationship to the bunches passing through it.

The rebuncher phase passes through zero at the center of the bunch, for no net energy change of the bunch (otherwise it would be an accelerating cavity).

This is one-half of the SNS rebuncher, showing the beam aperture and tuning pistons.



Rebuncher Cavity Optimization

The rebuncher cavity must resonate at the desired frequency. Further optimization is need in the following areas:

Shunt Impedance	minimize required RF power
Length	to fit into a crowded transport system
Multipactoring	to ease conditioning
Peak surface fields	to minimize tendency of sparking
Atmospheric pressure	to minimize barometric frequency changes
Tuner configuration	maximize effectiveness, minimize RF loss
Vacuum	configure vacuum ports, ultimate pressure
Thermal control	minimize frequency shift
Profile	ease of construction, minimize peak fields

and many others