USPAS

MICROWAVE MEASUREMENTS COURSE

UC SAN DIEGO

JANUARY 22-26, 1996

MICROWAVE NETWORK ANALYSIS AND MEASUREMENT TECHNIQUES

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LECTURE OUTLINE:

- Introduction / Motivation / Accelerator Applications
- TEM Transmission Lines / Coax
- Waveguides (TE) / TE_{10} rectangular guide
- Low-loss conductors / Coax
- Terminated TEM Lines
- Microwave circuit theory / network representations
- Examples

INTRODUCTION/MOTIVATION/APPLICATIONS

- Lecture/Lab covers basic microwave network analysis and measurement techniques with emphasis on particle accelerator applications.

- Microwave/RF devices play major roles in all charged particle accelerators. Some examples are, RF cavities, pickups and kickers, beam position monitor systems, and coupled-bunch feedback systems. Microwave/RF effects also play an integral roll in beam dynamics. Basic microwave measurement techniques are an essential skill for the accelerator engineer/scientist.

- Transmission line concepts are of fundamental importance in microwave theory and techniques. Many microwave devices in accelerators, and in general, have transmission line foundations. Examples include: filters, matching networks, pickups, and kickers.

- Because of their importance, transmission lines and some basic devices derived from them will be used to illustrate basic microwave circuit analysis and measurement techniques.

- Part 1 of lecture provides an overview of basic microwave transmission line and network theory. Part 2 illustrates finer points through selected case studies. Lab emphasizes an understanding of physical concepts through frequency domain and time domain measurements.

TRANSMISSION LINES AND WAVEGUIDES

- Transmission lines and waveguides are used to transport electromagnetic energy from one point to another without radiation taking place.

- The structures treated here are characterized by having axial uniformity, i.e., their cross-section and electrical properties do not vary along the axis (chosen to be z).

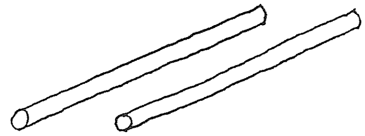
- Transmission lines:

  1) Consist of two or more parallel conductors.

  2) Support TEM (transverse electromagnetic waves) as the primary mode of propagation. TEM waves have no field components in the direction of propagation (axial) i.e. $\vec{E}_z = \vec{H}_z = 0$

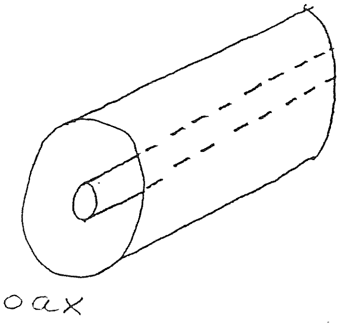
  3) Fields may be derived from a solution to Laplace equation (electrostatic solution).

  4) Examples:



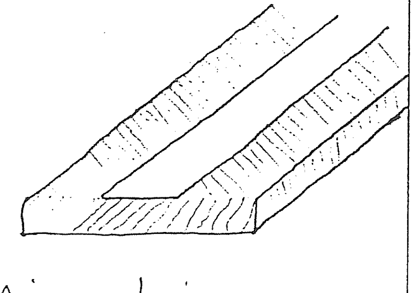
Twin lead

(Radio, TV etc.)



Coax

DC - Microwave



Microstrip

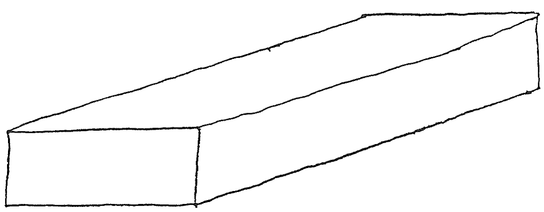
RF and Microwave printed circuit boards.

- Cylindrical conducting waveguides:

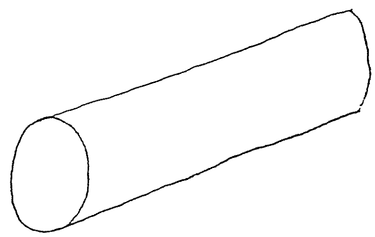
  1) Consist of empty or dielectric filled conducting tubes.

  2) Support TE (transverse electric $- \vec{E}_z = 0$) and TM (transverse magnetic $- \vec{H}_z = 0$) modes of propagation.

  3) Examples:



Rectangular waveguide



Circular waveguide

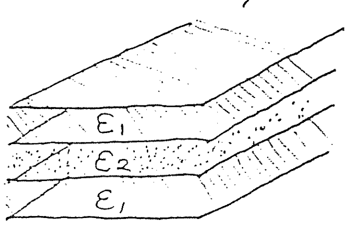
500 MHz − 225 GHz
High power handling capacity for lower frequency (larger) guides.

- Dielectric waveguides:

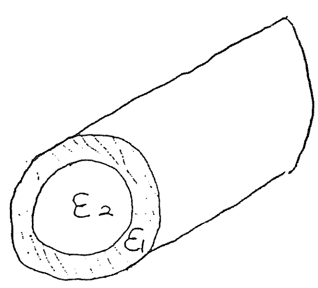
  1) Consist of layers of dielectrics, of different ϵ to confine propagating waves.

  2) Support TE, TM and mixed (HE, EH) modes with $\vec{E}_z \neq 0$ and $\vec{H}_z \neq 0$.

  3) Examples:

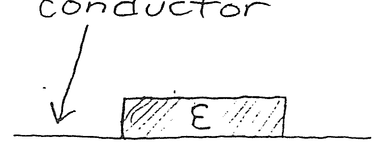


Slab guide (optical)



Fiber-optic (optical)

conductor



Surface guide millimeter wave − optical.

TRANSVERSE ELECTROMAGNETIC (TEM) TRANSMISSION LINES

- For reference, start with frequency domain Maxwell and vector Helmholtz equations in homogeneous, isotropic media, ϵ, μ, with no sources:

$$\vec{\nabla} \times \vec{E} = -j\omega\mu \vec{H} \quad (1) \qquad \vec{\nabla} \cdot \vec{E} = 0 \quad (3)$$

$$\vec{\nabla} \times \vec{H} = j\omega\epsilon \vec{E} \quad (2) \qquad \vec{\nabla} \cdot \vec{H} = 0 \quad (4)$$

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \quad (5)$$

$$\nabla^2 \vec{H} + k^2 \vec{H} = 0 \quad (6)$$

where: $k = \omega\sqrt{\mu\epsilon}$ (complex wave number)

- For axially uniform lines, assume that waves propagate in the axial ($\hat{z}$) direction with propagation constant β. TEM solutions ($E_z = H_z = 0$) to (5) and (6) may then be written as:

$$\vec{E}(x) = \vec{E}_\perp(x_\perp)\, e^{\pm j\beta z} \quad (7)$$

$$\vec{H}(x) = \vec{H}_\perp(x_\perp)\, e^{\pm j\beta z} \quad (8)$$

where: x indicates general 3-dimensional coordinates and $x_\perp$ indicates coordinates or vector components that are transverse to the direction of propagation, $\hat{z}$.

- With no loss in generality, take the case of propagation in the $+\hat{z}$ direction so that the z dependence of the fields is $e^{-j\beta z}$. In this case:

$$\vec{\nabla} = \vec{\nabla}_\perp - j\beta\hat{z}$$

Substitute the assumed TEM solutions, (7) and (8) into the curl equation (1):

$$\underbrace{\vec{\nabla}_\perp \times \vec{E}_\perp(x_\perp)}_{\hat{z}} - \underbrace{j\beta\hat{z} \times \vec{E}_\perp(x_\perp)}_{\perp} = \underbrace{-j\omega\mu\vec{H}_\perp(x_\perp)}_{\perp}$$

Equating components gives:

$$\vec{\nabla}_\perp \times \vec{E}_\perp(x_\perp) = 0 \qquad (9)$$

and:

$$\vec{H}_\perp = \frac{\beta}{\omega\mu}\hat{z} \times \vec{E}_\perp(x_\perp) \qquad (10)$$

- From (9), $\vec{E}_\perp$ is curl free (no magnetic flux normal to the transverse plane). Thefore:

$$\vec{E}_\perp(x_\perp) = -\nabla_\perp\phi(x_\perp)$$

where: $\phi(x_\perp)$ is a two dimensional scaler potential.

From (3), ϕ satisfies the Laplace equation:

$$\nabla_\perp^2\phi(x_\perp) = 0$$

An immediate consequence is that structures which guide TEM waves must have two or more conductors capable of having potential differences between them.

Now, $\vec{E}_\perp$ must satisfy the Helmholtz equation (5). Using $\nabla^2 = \nabla_\perp^2 - \beta^2$ gives:

$$\nabla_\perp^2 \vec{E}_\perp + (k^2 - \beta^2)\vec{E}_\perp = 0$$

or

$$\nabla_\perp \left[\nabla_\perp^2 \phi + (k^2 - \beta^2)\phi \right] = 0$$

therefore, the propagation $\beta = k$ as for plane waves in media μ, ϵ.

The magnetic field may be found from the electric field using (10)

$$\vec{H}_\perp = \frac{\beta}{\omega\mu} \hat{z} \times \vec{E}_\perp = \frac{k}{\omega\mu} \hat{z} \times \vec{E}_\perp = \sqrt{\frac{\epsilon}{\mu}}\, \hat{z} \times \vec{E}_\perp$$

Let $Y = \sqrt{\frac{\epsilon}{\mu}} \equiv 1/Z \equiv$ wave admittance

if $\epsilon = \varepsilon_0$, $\mu = \mu_0$ then $Z = 377\Omega$ (impedance of free space).

- Summary for TEM modes

1) Find scaler potential $\phi(x_\perp)$ which satisfies:

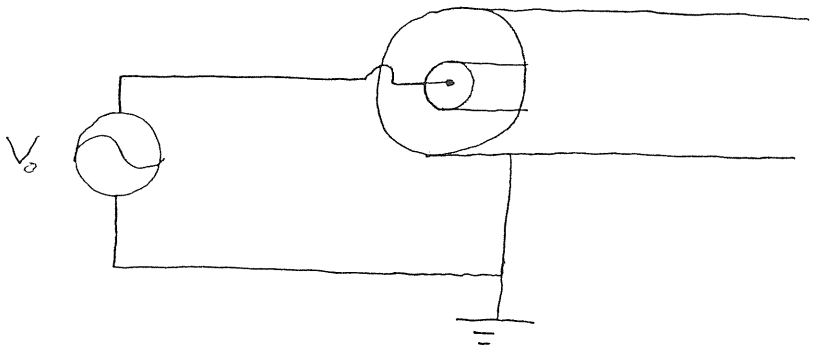
$$\nabla_\perp^2 \phi(x_\perp) = 0$$

subject to boundary conditions of the guiding structure.

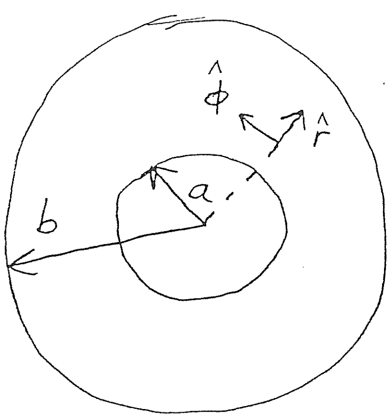
2) $$\vec{E} = -\nabla_\perp \phi(x_\perp)\, e^{\mp jkz}$$

3) $$\vec{H} = \pm Y \hat{z} \times \vec{E}\, e^{\mp jkz}$$

Coaxial TEM transmission line:



line is infinite or terminated so that only forward waves exist.



Need to find $\phi(x_\perp) = \phi(r)$ with boundary conditions:

$$\left.\begin{aligned} \phi(r=a) &= V_o \\ \phi(r=b) &= 0 \end{aligned}\right\}$$

1) $\nabla_\perp^2 \phi = 0$ or $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = 0$

Simple electrostatic problem:

$$\phi = C_1 \ln r + C_2$$

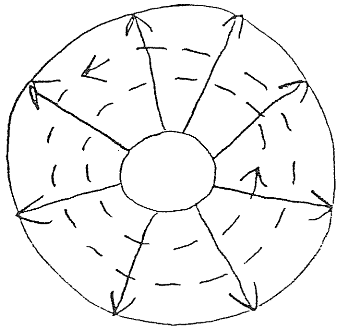
Applying boundary conditions gives

$$\phi(r) = V_o \frac{\ln(r/b)}{\ln(a/b)}$$

To find fields:

$$\vec{E} = -\nabla_\perp \phi \, e^{-jkz} = \frac{V_o}{\ln(b/a)} \frac{\hat{r}}{r} e^{-jkz}$$

$$\vec{H} = \frac{1}{Z} \hat{z} \times \vec{E} = \frac{V_o}{Z \ln(b/a)} \frac{\hat{\phi}}{r} e^{-jkz}$$



—— E

- - - H

propagation out of page.

The voltage between conductors is obviously V_o. To find the current on the conductors, consider:

The surface current density on the inner conductor:

$$\vec{J}_{sa} = \hat{n} \times \vec{H}(r=a) = \frac{V_o}{Z \ln(b/a)} \frac{1}{a} e^{-jkz} \hat{z} \quad \text{amp/m}$$

The total current carried by the conductor is:

$$I_o = \int_0^{2\pi} \vec{J}_s \, a \, d\phi = \frac{2\pi V_o}{Z \ln(b/a)} e^{-jkz}$$

Similarly, the current on the outer conductor is $-I_o$.

The characteristic impedance is:

$$\boxed{Z_C = \frac{V_o}{I_o} = \frac{Z \ln b/a}{2\pi}} = 60 \ln b/a \quad \text{(for an air filled line)}.$$

The power propagated down the line is:

$$P = \frac{1}{2} \mathrm{Re} \int_a^b \int_0^{2\pi} \vec{E} \times \vec{H}^* \cdot \hat{z}\, r\, dr\, d\phi$$

$$= \frac{1}{2} \frac{V_o^2}{Z[\ln b/a]^2} \int_0^b \int_0^{2\pi} \frac{d\phi\, dr}{r}$$

$$\boxed{P = \frac{\pi V_o^2}{Z \ln b/a}}$$

Note that this power is also found from:

$$P = \frac{1}{2} V I^* = \frac{1}{2} V_o I_o = \frac{\pi V_o^2}{Z \ln (b/a)}$$

$$= \frac{V_o^2}{2 Z_c} = \frac{1}{2} Z_c I_o^2$$

Transmission lines support true voltage and current waves.

TRANSVERSE ELECTRIC (TE) WAVEGUIDE MODES

- TE waveguides are another common transmission technique for electromagnetic signals. In this case:

$$E_z = 0 \quad , \quad H_z \neq 0$$

So TE solutions to (1) – (6) can be written as:

$$\vec{E}(x) = \vec{E}_\perp(x_\perp)\, e^{\pm j\beta z} \tag{11}$$

$$\vec{H}(x) = \vec{H}_\perp(x_\perp)\, e^{\pm j\beta z} + \vec{H}_z(x_\perp)\, e^{\pm j\beta z} \tag{12}$$

By substituting (11) and (12) into (1) and (2) and equating components, the following relations are obtained:

$$j\omega\mu\, \hat{z} \times \vec{H}_\perp = -j\beta \vec{E}_\perp \tag{13}$$

$$-j\beta \vec{H}_\perp + j\omega\epsilon\, \hat{z} \times \vec{E}_\perp = \nabla_\perp H_z \tag{14}$$

From (13) and (14), it is clear that all field components can be obtained from H_z. Solving (13) and (14) simultaneously gives:

$$\vec{H}_\perp = \frac{-j\beta}{K_c^2} \nabla_\perp H_z \tag{15}$$

$$\vec{E}_\perp = -Z_h\, \hat{z} \times \vec{H}_\perp \tag{16}$$

where: $k_c^2 = k^2 - \beta^2$ cutoff wave number

$Z_h = \frac{k}{\beta} Z$ TE wave impedance

H_z is found by solving the Helmholtz equation:

$$\nabla_\perp^2 H_z + k_c^2 H_z = 0 \qquad (17)$$

This is an eigenvalue problem in the transverse coordinates. When solved subject to the boundary conditions of the waveguide, a spectrum of solutions with eigenvalues $k_{c\lambda}^2$, $\lambda = 1, 2 \ldots$ are obtained. These solutions are called modes of the waveguide.

- Note that the propagation constant of a given waveguide mode is given by:

$$\beta_\lambda = \sqrt{k^2 - k_{c\lambda}^2} = \sqrt{\mu\epsilon}\,\sqrt{\omega^2 - \omega_{c\lambda}^2} \qquad (18)$$

Therefore when:

$\omega = \omega_{c\lambda}$ No propagation takes place

$\omega > \omega_{c\lambda}$ $\cos(\omega t \pm \beta z)$ propagation takes place (propagating modes)

$\omega < \omega_{c\lambda}$ $\cos\omega t\, e^{-\beta z}$ – fields decay exponentially in the direction of propagation (evanescent modes)

Because β is a function of ω, waveguide modes are <u>dispersive</u>, i.e. different frequency components of a signal travel with different phase and group velocities resulting in signal distortion.

The phase velocity is given by:

$$V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}\left[\frac{1}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}\right] \quad (19)$$

The group velocity (velocity with which energy and information travels) is given by:

$$V_g = \left(\frac{\partial\beta}{\partial\omega}\right)^{-1} = \frac{1}{\sqrt{\mu\epsilon}}\left[\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}\right] \quad (20)$$

Note that:

$$V_p V_g = V^2 \quad (21)$$

$V = \frac{1}{\sqrt{\mu\epsilon}}$ velocity of light in medium μ, ϵ. (22)

- Summary for TE waves:

1) Find H_z as a solution to the Helmholtz equation:

$$\nabla_\perp^2 H_z + k_c^2 H_z = 0$$

$$k_c^2 = k^2 - \beta^2$$

subject to the boundary conditions of the guiding structure.

2) $H_\perp = \frac{-j\beta}{k_c^2} \nabla_\perp H_z$

3) $E_\perp = -Z_h \hat{z} \times H_\perp$

TE MODES IN RECTANGULAR WAVEGUIDE

$E_z = 0$

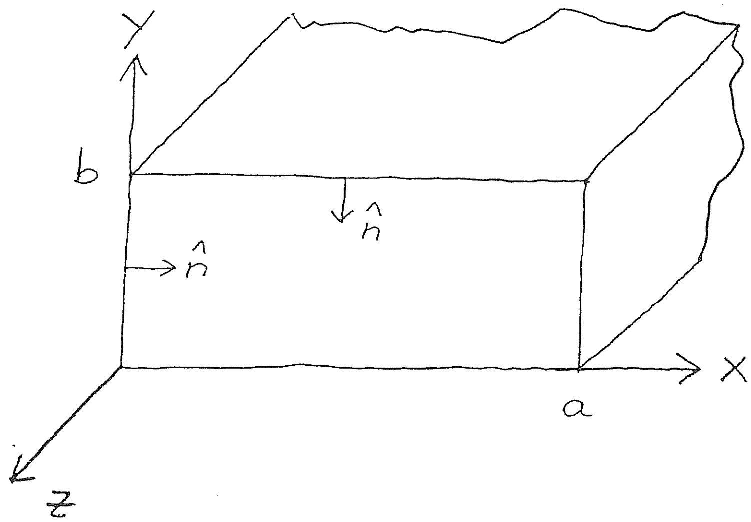
$$\nabla_\perp^2 H_z + K_c^2 H_z = 0 \qquad (1)'$$

$$K_c^2 = K^2 - \beta^2 \qquad (2)'$$

$$\vec{H}_\perp = \frac{-j\beta}{K_c^2} \nabla_\perp H_z \qquad (3)'$$

$$\vec{E}_\perp = -Z_h \hat{z} \times \vec{H}_\perp \qquad (4)'$$

$$Z_h = \frac{K}{\beta} Z \qquad (5)'$$



Must solve: $\nabla_\perp^2 H_z(x,y) + K_c^2 H_z(x,y) = 0$

Subject to $\hat{n} \cdot \vec{H} = 0$ on conducting walls.

From $(3)'$ we have:

$$\hat{n} \cdot \vec{H}_\perp = \frac{-j\beta}{K_c^2} \hat{n} \cdot \nabla_\perp H_z = 0$$

So the boundary condition becomes:

$$\hat{n} \cdot \nabla_\perp H_z = 0 \qquad \text{on conducting surface}$$

Or:

$$\boxed{\begin{array}{lll} \dfrac{\partial H_z}{\partial x} = 0 & & x = 0,\ x = a \\[2ex] \dfrac{\partial H_z}{\partial y} = 0 & & y = 0 \quad y = b \end{array}}$$

Solve the Helmholtz equation for H_z by separation of variables:

Let $H_z(x,y) = f(x)\,g(y)$

so: $$\frac{\partial^2 f(x)g(y)}{\partial x^2} + \frac{\partial^2 f(x)g(y)}{\partial y^2} + k_c^2 f(x)g(y) = 0$$

divide by $f(x)g(y)$ to get:

$$\underbrace{\frac{1}{f}\frac{d^2 f}{dx^2}}_{\text{function of } x} + \underbrace{\frac{1}{g}\frac{d^2 g}{dy^2}}_{\text{function of } y} + \underbrace{k_c^2}_{\text{constant}} = 0$$

For the above to hold for all x and y, the functions of x, and y only must each equal constants.

Let: $\dfrac{1}{f}\dfrac{d^2 f}{dx^2} = -k_x^2$ and $\dfrac{1}{g}\dfrac{d^2 g}{dy^2} = -k_y^2$

As a consequence:

$$\boxed{k_x^2 + k_y^2 = k_c^2 \qquad (6)'}$$

And:

$$H_z(x,y) = (A_1 \cos k_x x + A_2 \sin k_x x)(B_1 \cos k_y y + B_2 \sin k_y y)$$

Applying the boundary condition:

$$\frac{\partial H_z}{\partial x} = 0 \qquad x = 0,\ x = a$$

gives: $-A_1 k_x \sin k_x x + A_2 k_x \cos k_x x = 0, \quad x = 0, a$

$\therefore \quad A_2 = 0$ and $\boxed{k_x = \frac{n\pi}{a}}$

Similarly, the boundary condition:

$$\frac{\partial H_z}{\partial y} = 0 \qquad y = 0, b$$

gives $B_2 = 0$ and

$$\boxed{k_y = \frac{m\pi}{b}}$$

So the final result is:

$$H_z = A_{nm} \cos\frac{n\pi}{a}x \cos\frac{m\pi}{b}y\, e^{\mp j\beta_{nm}z} \qquad (7)'$$

Equations (3)' and (4)' give the remaining field components:

$$H_x = \pm j \frac{\beta_{nm}}{k_{c\,nm}^2} A_{nm} \frac{n\pi}{a} \sin\frac{n\pi}{a}x \cos\frac{m\pi}{b}y\, e^{\mp j\beta_{nm}z} \qquad (8)'$$

$$H_y = \pm j \frac{\beta_{nm}}{k_{c\,nm}^2} A_{nm} \frac{m\pi}{b} \cos\frac{n\pi}{a}x \sin\frac{m\pi}{b}y\, e^{\mp j\beta_{nm}z} \qquad (9)'$$

$$E_x = Z_{h\,nm} A_{nm}\, j \frac{\beta_{nm}}{k_{c\,nm}^2} \frac{m\pi}{b} \cos\frac{n\pi}{a}x \sin\frac{m\pi}{b}y\, e^{\mp j\beta_{nm}z} \qquad (10)'$$

$$E_y = -Z_{h\,nm} A_{nm}\, j \frac{\beta_{nm}}{k_{c\,nm}^2} \frac{n\pi}{a} \sin\frac{n\pi}{a}x \cos\frac{m\pi}{b}y\, e^{\mp j\beta_{nm}z} \qquad (11)'$$

A_{nm} is an arbitrary excitation level of H_z for the n,m th mode.

The cutoff wave number for the nmth mode is:

$$K_{cnm} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \qquad (12)'$$

and is a function only of the waveguide dimensions.

The propagation constant for the nm^{th} mode is found from (2):

$$\beta_{nm} = \sqrt{K^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2} \qquad (13)'$$

The wavelength in the guide is

$$\lambda_{g_{nm}} = \frac{2\pi}{\beta_{nm}} \qquad (14)'$$

The cutoff frequency (below which β is imaginary and no energy propagates) is:

$$f_{cnm} = \frac{c}{2\pi}\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \quad \text{for } \mu_r = \varepsilon_r = 1 \qquad (15)'$$

Typical waveguides have $a = 2b$ in which case:

$$f_{cnm} = \frac{c}{2a}\sqrt{n^2 + 4m^2}$$

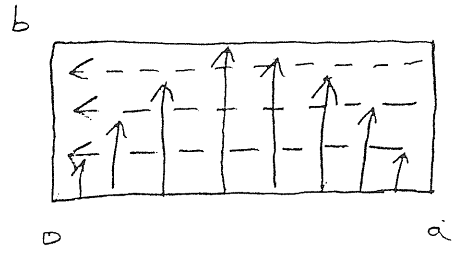
The lowest mode $n=1$, $m=0$ or TE_{10} has:

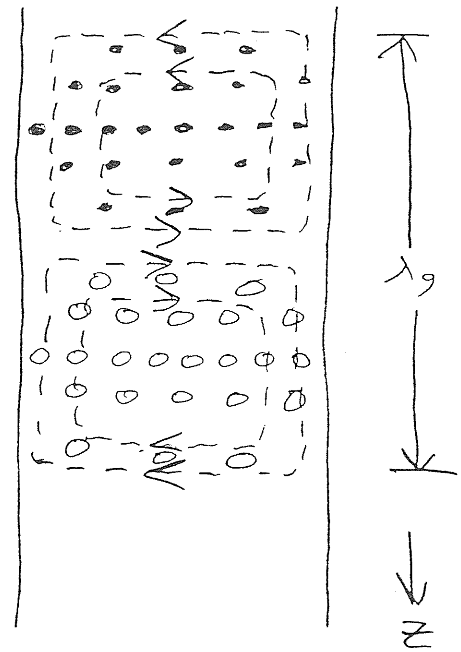
$$f_{c10} = \frac{c}{2a}$$

The next mode, TE_{01} has $f_{c01} = \dfrac{c}{a}$

Therefore, the TE_{10} waveguide has a single mode operating range of 1 octave.

TE_{10} Mode:





$$H_z = A\cos\frac{\pi x}{a}\, e^{-j\beta z}$$

$$H_x = \frac{j\beta}{K_c} A\sin\frac{\pi x}{a}\, e^{-j\beta z}$$

$$E_y = -jAZ_h\frac{\beta}{K_c}\sin\frac{\pi x}{a}\, e^{-j\beta z}$$

TE_{10} solution is independent of y

$K_c = \pi/a \qquad \beta = \sqrt{K^2 - \left(\frac{\pi}{a}\right)^2}$

Decomposition of TE_{10} mode into two plane waves:

Look at: $E_y \propto \sin\frac{\pi x}{a} e^{-j\beta z}$

or; $E_y \propto \frac{1}{2}\left[e^{j\frac{\pi x}{a} - j\beta z} - e^{-j\frac{\pi x}{a} - j\beta z}\right]$

remember: $\left(\frac{\pi}{a}\right)^2 + \beta^2 = k_o^2$

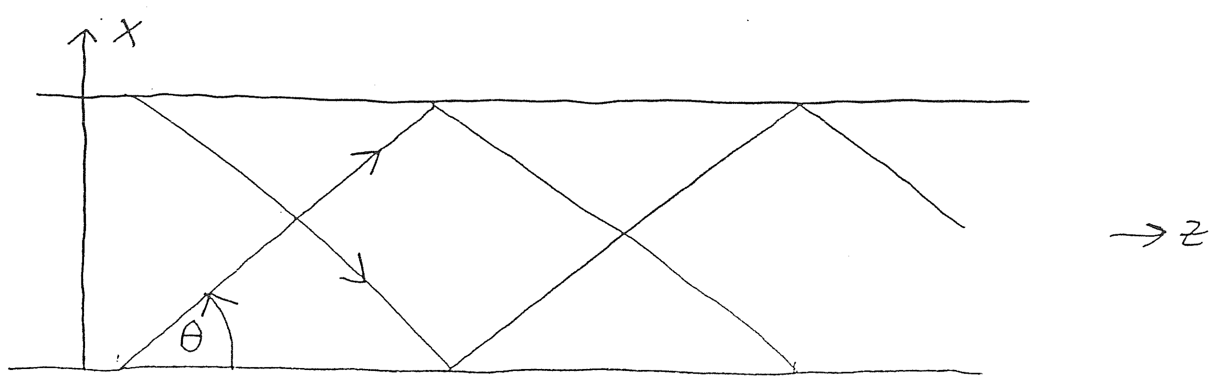
let: $\frac{\pi}{a} = k_o \sin\theta$, $\beta = k_o \cos\theta$

The above relation remains true and:

$$\theta = \tan^{-1}\frac{\pi}{a\beta} = \tan^{-1}\frac{k_c}{\beta}$$

Also $E_y \propto \frac{1}{2}\left[e^{-jk_o(z\cos\theta - x\sin\theta)} - e^{-jk_o(z\cos\theta + x\sin\theta)}\right]$

Which is the sum of two plane waves propagating at angles of $\pm\theta$ with respect to the $\hat{z}$ axis.



The phase velocity in the $\hat{z}$ direction is clearly:

$$K_0 \cos\theta = \frac{\omega}{V_P} \quad \text{so} \quad V_P = \frac{c}{\cos\theta} = \frac{\beta c}{K_0}$$

The velocity at which energy propagates along the axis is:

$$V_g = c\cos\theta = \frac{K_0}{\beta} c$$

As ω approachs cutoff the wave bounce back and forth in the x direction ($\theta \to 90^\circ$). No energy propagates and the phase velocity goes to ∞.

LOSSES IN GOOD CONDUCTORS

In conductors:

$$\vec{J} = \sigma \vec{E} \qquad (23)$$

In this case:

$$\vec{\nabla} \times \vec{H} = (\sigma + j\omega\epsilon)\vec{E} \qquad (24)$$

In good conductors, the conductivity, σ is very large so that the conduction current is large compared to the displacement current, $j\omega\epsilon\vec{E}$. So:

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} \qquad (25)$$

To find the differential equation for $\vec{H}$:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \sigma\, \vec{\nabla} \times \vec{E}$$

or:

$$\nabla(\underbrace{\vec{\nabla} \cdot \vec{H}}_{0}) - \nabla^2 \vec{H} = \overbrace{-j\omega\mu\sigma}^{\text{from (1)}} \vec{H}$$

so:

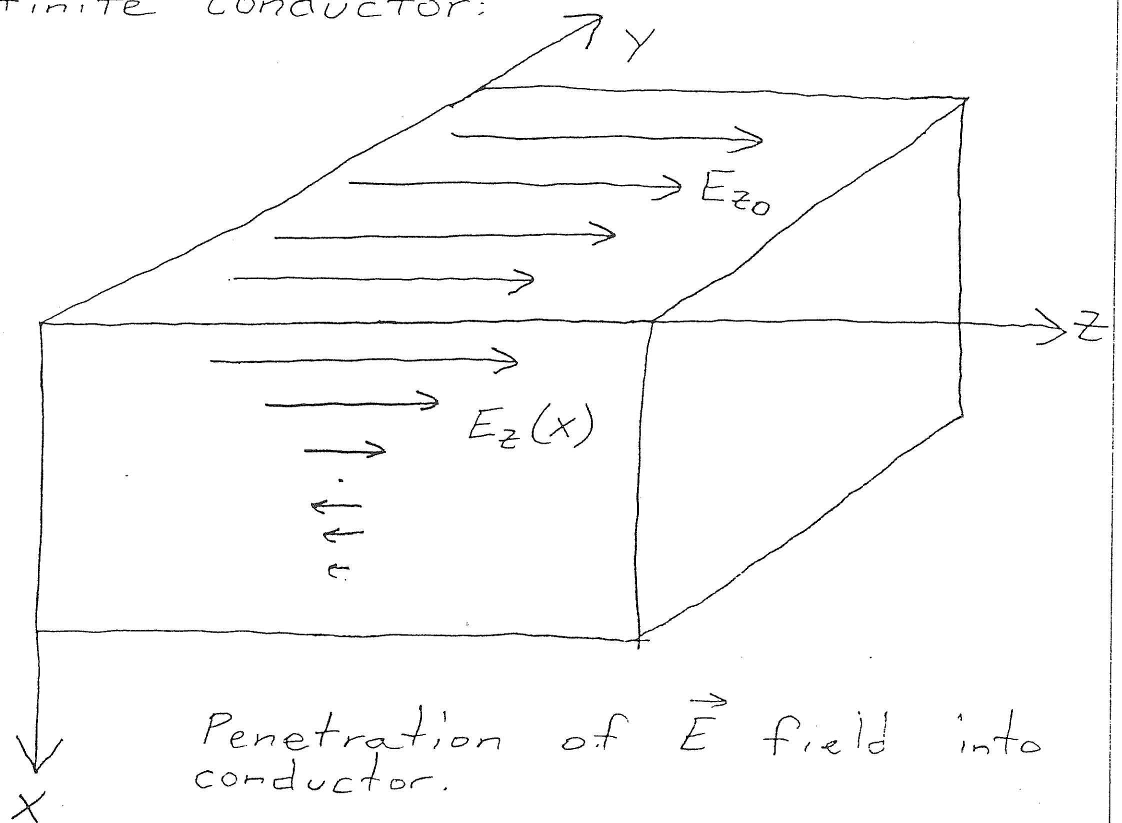
$$\nabla^2 \vec{H} = j\omega\mu\sigma \vec{H} \qquad (26)$$

Similarly:

$$\nabla^2 \vec{J} = j\omega\mu\sigma \vec{J} \qquad (27)$$

$$\nabla^2 \vec{E} = j\omega\mu\sigma \vec{E} \qquad (28)$$

Now look at field solutions in a semi-infinite conductor:



Penetration of $\vec{E}$ field into conductor.

Assume conductor infinite in the half space $x>0$ with $\hat{z}$ directed electric field at the x-z surface:

$$E_z(x=0) = E_{z0}$$

Equation (28) becomes: (for E_z uniform in y and z)

$$\frac{d^2 E_z}{dx^2} - j\omega\mu\sigma E_z = 0 \qquad (29)$$

With solution:

$$E_z(x) = E_{z0}\, e^{-\sqrt{j\omega\sigma\mu}\, x} = E_{z0}\, e^{-(1+j)/\delta\, x} \qquad (30)$$

where: $\delta = \dfrac{1}{\sqrt{\pi f \mu \sigma}}$ meters

δ = "skin depth", the depth in the conductor at which the fields attenuate to 1/e of their surface value.

Similarly,

$$J_z = J_{zo}\, e^{-\frac{(1+j)}{\delta}x} \qquad (31)$$

$$H_y = H_{yo}\, e^{-\frac{(1+j)}{\delta}x} \qquad (32)$$

The skin depth is very small compared to curvatures of radio and microwave structures. In addition, δ is small compared to radio and microwave wavelengths and field variations in structures so that the infinite depth, uniform field approximation is good.

The total current per unit width in y is:

$$J_{sz} = \int_0^\infty J_z\, dx = \frac{J_o\, \delta}{1+j} \quad \text{amp/m}$$

The electric field at the surface is:

$$E_{zo} = \frac{J_o}{\sigma}$$

The internal "surface" impedance of the conductor is defined as:

$$Z_s = \frac{E_{zo}}{J_{sz}} = \frac{1+j}{\sigma\delta} \qquad (33)$$

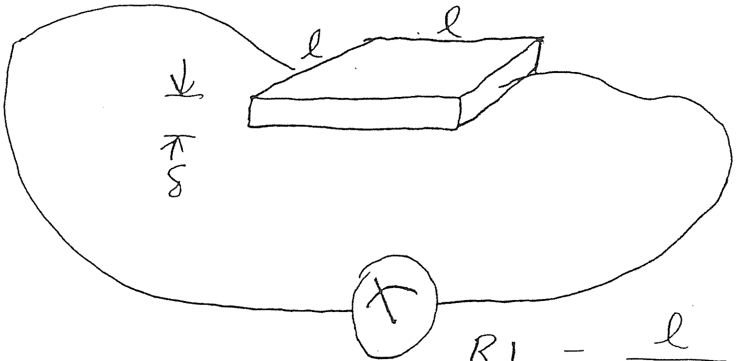
The surface impedance has equal resistive and inductive parts:

$$Z_s = R_s + j\omega L_s$$

where: $R_s = \dfrac{1}{\sigma\delta}$, $\omega L_s = \dfrac{1}{\sigma\delta}$

The surface resistance, R_s, is what causes power loss in the conductor.

Note that the surface resistance is equal to the D.C. resistance of a square conductor of width δ:

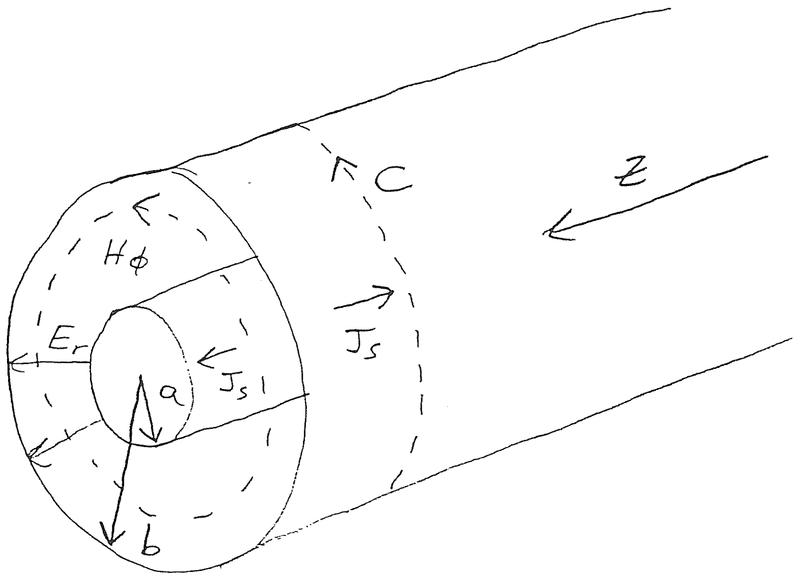


$$R_{dc} = \frac{l}{\sigma \delta l} = \frac{1}{\sigma \delta} \ \Omega$$

R_s is sometimes quoted in units of "ohms per square".

Note that $\vec{E}$ in the $\hat{z}$ direction and $\vec{H}$ in the $-\hat{y}$ direction means that power flows into the conductor ($\hat{x}$) direction, where it is dissipated as heat.

- Return to coaxial transmission line example and evaluate the effect of finite conductivity conductors.



For good conductors, use perturbation method which assumes that the current flowing internal to the conductors per unit length in azimuth equals the surface current calculated for the perfect conductor case. This is a good assumption because the skin depth is so small.

If the conductors have a surface impedance, Z_s, the surface current will give rise to a tangential electric field:

$$\vec{E}_t = Z_s \vec{J}_s = Z_s \hat{n} \times \vec{H}_t$$

The power lost into the conductor is:

$$P_\ell = \frac{-1}{2} Re \oint_S \vec{E}_t \times \vec{H}_t^* \cdot \hat{n} \, ds$$

or $P_\ell' = \frac{-1}{2} Re \oint_C \vec{E}_t \times \vec{H}_t^* \cdot \hat{n} \, dl$ is the power lost per unit length of cable.

So:

$$P_e' = \frac{-1}{2} \mathrm{Re} \oint_C [Z_s \hat{n} \times \vec{H}_t] \times [H_t]^* \cdot \hat{n} \, dl$$

$$= \frac{1}{2} \mathrm{Re} \oint_C Z_s |H_t|^2 \, dl$$

$$= \frac{R_s}{2} \oint_C |H_t|^2 dl \quad \text{or} \quad \frac{R_s}{2} \oint_C |J_s|^2 dl$$

For the coax line:

On conductor a: $|J_{sa}|^2 = \frac{V_o^2}{Z^2 \ln^2(b/a)} \frac{1}{a^2}$

On conductor b: $|J_{sb}|^2 = \frac{V_o^2}{Z^2 \ln^2(b/a)} \frac{1}{b^2}$

So: $$P_e' = \frac{R_s V_o^2}{2 Z^2 \ln^2(b/a)} \left[\int_0^{2\pi} \frac{1}{a^2} a \, d\phi + \int_0^{2\pi} \frac{1}{b^2} b \, d\phi \right]$$

$$\therefore \quad \boxed{P_e' = \frac{\pi R_s V_o^2}{Z^2 \ln^2(b/a)} \left(\frac{b+a}{ab} \right) \qquad (34)}$$

Power dissipated per unit length of coaxial cable.

- It is useful to examine how voltage waves and power propagate along a lossy TEM line.

The rate of decrease of power along the line must be proportional to the power along the line:

$$-\frac{\partial P(z)}{\partial z} = 2\alpha P(z) = P_\ell' \qquad (35)$$

with solution:

$$P(z) = P_0 e^{-2\alpha z} \qquad (36)$$

From (35), the attenuation constant, α, is given by:

$$\alpha = \frac{P_\ell'}{2P(z)} \qquad (37)$$

If V_0 is the voltage on the line at some point, z, the power at z is:

$$P = \frac{V_0^2}{2Z_c} \qquad (38)$$

To find α for a coaxial line, combine (34), (37) and (38) to get:

$$\alpha = \frac{R_s\left(\frac{b+a}{ab}\right)}{4\pi Z_c} \qquad (39)$$

If power propagates as $e^{-2\alpha z}$ then voltage must go as:

$$V(z) \rightarrow e^{-\alpha z}$$

To find the voltage transfer function for a long cable with conductor losses, use (39) to get:

$$H(\omega) = \frac{V(z)}{V(z=0)} = e^{-(\alpha + j\beta)z} \tag{40}$$

To include the inductive portion of the surface impedance, replace the surface resistance, R_s, in (39) with the surface impedance, Z_s, given by (33). Using the definition of skin depth, δ, one obtains:

$$H(\omega) = e^{-[\psi\sqrt{\omega} + j(\psi\sqrt{\omega} + \beta)]z} \tag{41}$$

where: $\psi = \frac{1}{4\pi Z_c}\sqrt{\frac{\mu_0}{2\sigma}}\left(\frac{a+b}{ab}\right)$

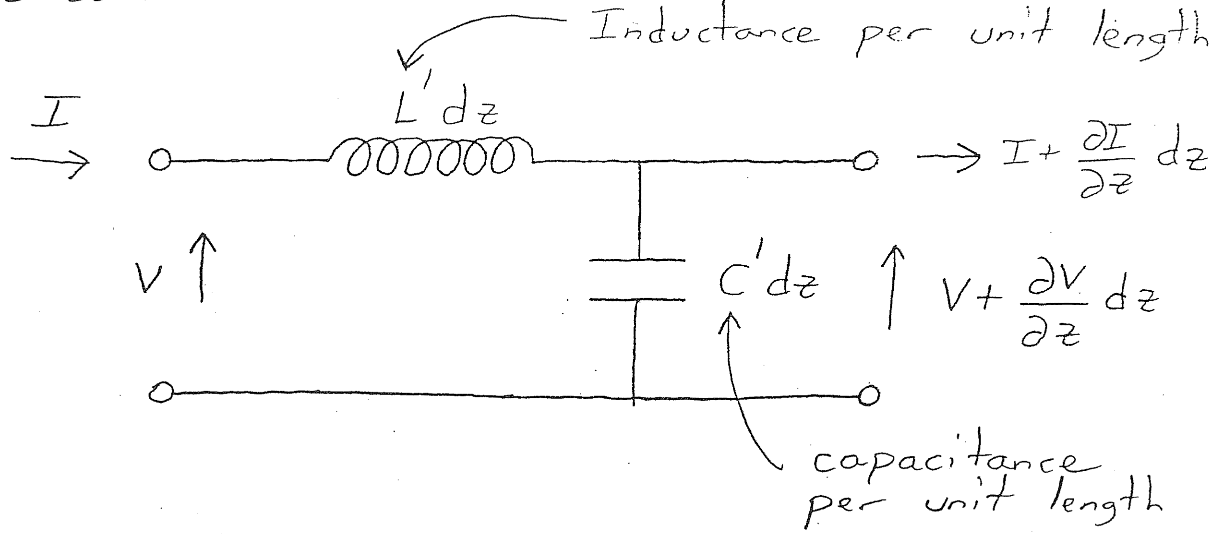
Thus, a long cable with conductor losses is dispersive as well as lossy because of the $\sqrt{\omega}$ dependence of the conductor surface impedances.

TERMINATED TEM LINES

- TEM transmission lines serve as basic interconnects between microwave devices and circuit elements. Therefore it is important to understand the effects of terminations on TEM lines.

- Before going on to terminated lines, the distributed element view of TEM lines is briefly discussed in order to introduce the concepts of inductance and capacitance per unit length

  A differential length of lossless TEM line is represented by the following equivalent circuit:



Using standard circuit analysis gives:

$$\frac{\partial V}{\partial z} = -j\omega L' I \qquad (42)$$

$$\frac{\partial I}{\partial z} = -j\omega C' V \qquad (43)$$

Or: $$\frac{\partial^2 V}{\partial z^2} = -j\omega L' \frac{\partial I}{\partial z} = -\omega^2 L' C' V \qquad (44)$$

$$\frac{\partial^2 I}{\partial z^2} = -j\omega C' \frac{\partial V}{\partial z} = -\omega^2 L' C' I \qquad (45)$$

Equation (44) has solution:

$$V(z) = V^{+} e^{-j\omega\sqrt{L'C'}\,z} + V^{-} e^{j\omega\sqrt{L'C'}\,z} \tag{46}$$

These are forward and backward propagating waves.

Using (42), the solution for $I(z)$ can be obtained:

$$I(z) = \sqrt{\frac{C'}{L'}}\, V^{+} e^{-j\omega\sqrt{L'C'}\,z} - \sqrt{\frac{C'}{L'}}\, V^{-} e^{+j\omega\sqrt{L'C'}\,z} \tag{47}$$

From (46) and (47), the following relations are obtained:

$$Z_c = \sqrt{\frac{L'}{C'}} \tag{48}$$

$$v = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\mu\epsilon}} \tag{49}$$

$$C' = \frac{\sqrt{\mu\epsilon}}{Z_c} \tag{50}$$

$$L' = Z_c\sqrt{\mu\epsilon} \tag{51}$$

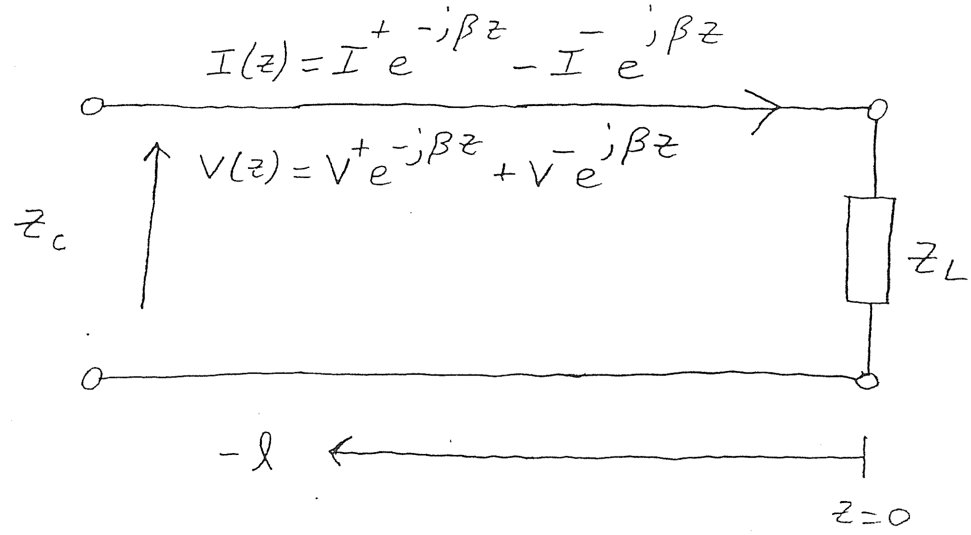
Expressions (50) and (51) can be easily verified with field analysis.

For a lossy line define R':

$$P' = \frac{1}{2} R' I_o^2 = 2\alpha \frac{1}{2} Z_o I_o^2$$

so: $\quad R' = 2\alpha Z_c$

- Consider a TEM transmission line terminated in an arbitrary load impedance, Z_L, located at $z=0$ along the line:



At $z=0$,

$$\frac{V(0)}{I(0)} = \frac{V^+ + V^-}{I^+ - I^-} = Z_L \qquad (52)$$

or

$$Z_L = Z_c\left(\frac{V^+ + V^-}{V^+ - V^-}\right) = Z_c\left(\frac{1+\Gamma_o}{1-\Gamma_o}\right) \qquad (53)$$

where: $\Gamma_o = \frac{V^-}{V^+}$ $\Rrightarrow$ Reflection coefficient at the load.

From (53),

$$\Gamma_o = \frac{Z_L - Z_c}{Z_L + Z_c} \qquad (54)$$

Also, a transmission coefficient can be defined,

$$T_o = \frac{V^+ + V^-}{V^+} = 1 + \Gamma_o \qquad (55)$$

- The power delivered to the load is given by:

$$P = \frac{1}{2} Re\left(V_L I_L^*\right) = \frac{|V^+|^2}{2Z_c}\left(1 - |\Gamma_0|^2\right) \qquad (56)$$

This is the incident power minus the reflected power.

- The total voltage at any point on the line ($z<0$) is:

$$V(z) = V^+ e^{-j\beta z} + \Gamma_0 V^+ e^{j\beta z}$$

with magnitude:

$$|V(z)| = |V^+|\left|1 + \Gamma_0 e^{j2\beta z}\right|$$

if $\Gamma_0 = \rho e^{j\theta}$, then

$$|V(z)| = |V^+|\left|1 + \rho e^{j(\theta + 2\beta z)}\right| \qquad (57)$$

Thus $V(z)$ has:

Maxima $\rightarrow$ $\theta + 2\beta z = 2\pi n$ $\quad [1+\rho]$

Minima $\rightarrow$ $\theta + 2\beta z = \pi + 2\pi n$ $\quad [1-\rho]$

Define standing wave ratio:

$$VSWR = \frac{1+\rho}{1-\rho} = \frac{V_{max}}{V_{min}} \qquad (58)$$

- The reflection coefficient at any point $z=-\ell$ from the load is:

$$\Gamma(\ell) = \frac{V^- e^{-j\beta\ell}}{V^+ e^{j\beta\ell}} = \Gamma_o e^{-j2\beta\ell} \qquad (59)$$

- The impedance seen looking toward the load at $z=-\ell$ is:

$$Z_{in} = \frac{1+\Gamma(\ell)}{1-\Gamma(\ell)} = \frac{1+\Gamma_o e^{-j2\beta\ell}}{1-\Gamma_o e^{-j2\beta\ell}} \qquad (60)$$

Using (54) and (60) the impedance transformation formula is obtained:

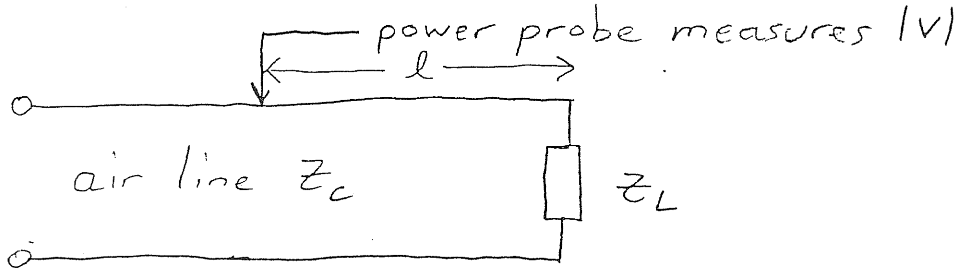
$$Z_{in}(\ell) = Z_c \left[\frac{Z_L + jZ_c \tan\beta\ell}{Z_c + jZ_L \tan\beta\ell} \right] \qquad (61)$$

For a lossy line, simply replace $j\beta$ with $\alpha + j\beta$ to obtain:

$$Z_{in}(\ell) = Z_c \left[\frac{Z_L + Z_c \tanh(\alpha\ell + j\beta\ell)}{Z_c + Z_L \tanh(\alpha\ell + j\beta\ell)} \right] \qquad (62)$$

- Techniques for measuring reflection coefficient.

Before network analyzers, slotted air lines were used to measure VSWR:



From (50), V_{max}/V_{min} gives $|\Gamma_o|$ and the location of of maxima or minima give $\theta = \angle \Gamma_o$.

This technique is tedious and can only be performed for one frequency at a time.

Now microwave network analyzers are used almost exclusively to make broad band swept reflection measurements. These instruments directly measure amplitude and phase of reflected signals as well as transmitted signals for 2-port devices.

Reflection coefficient vs. frequency is commonly displayed in two formats:

1) Magnitude (log or linear) and phase

2) Polar / Smith chart

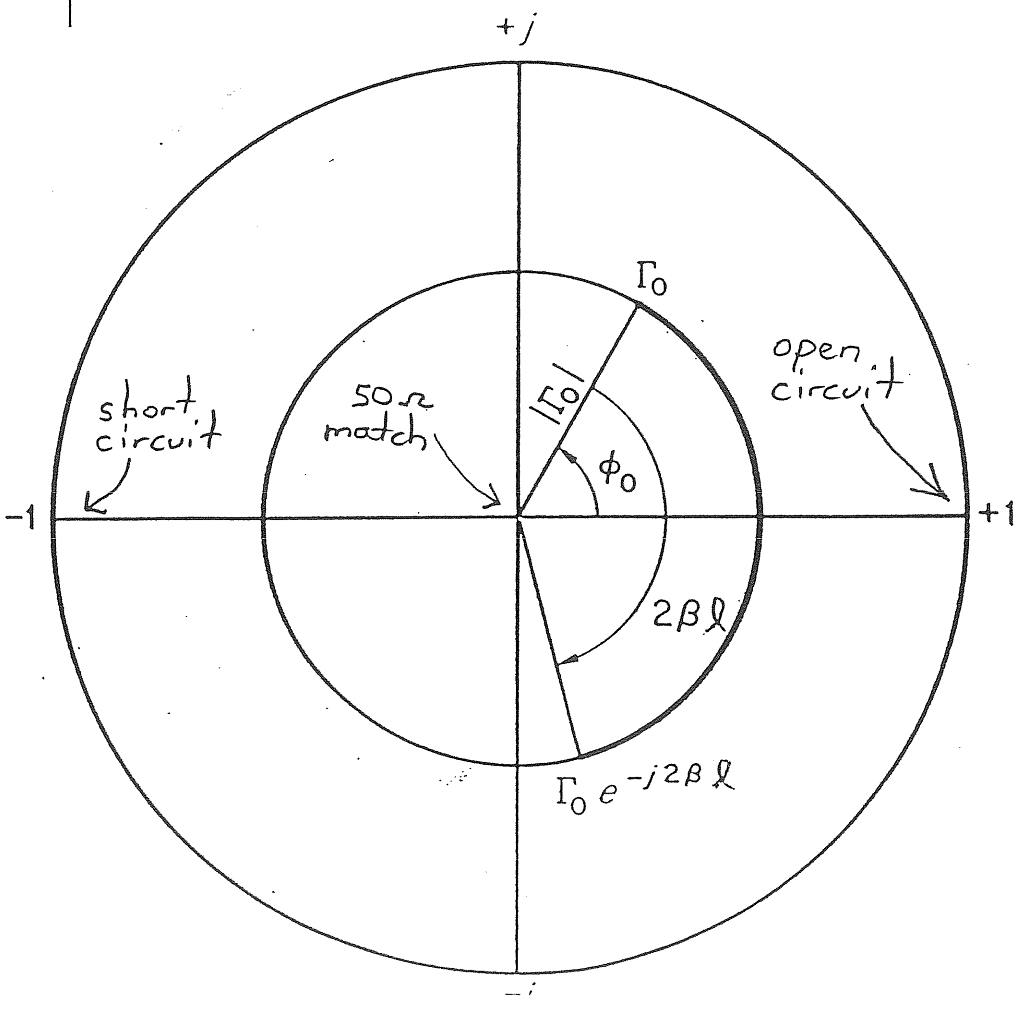
- Magnitude and phase format:

$$\Gamma_o(\omega) = |\Gamma_o(\omega)|\, e^{j\phi_o(\omega)} \left(e^{-j2\beta\ell}\right)$$ ← line length

Log mag	$\longrightarrow$	$20\log|\Gamma_o(\omega)|$	dB
Phase	$\longrightarrow$	$\phi_o(\omega)\ (e^{-j2\beta\ell})$	degrees
Linear mag	$\rightarrow$	$|\Gamma_o(\omega)|$	U or mU "units ≤ 1"

- Polar format:

Mag and phase plotted on unit circle in the complex plane.



Note:

Go around circle once for $\ell = \frac{\lambda}{2}$.

$\lambda/2 \rightarrow 2\pi$

because of "down and back" effect

- Smith chart format

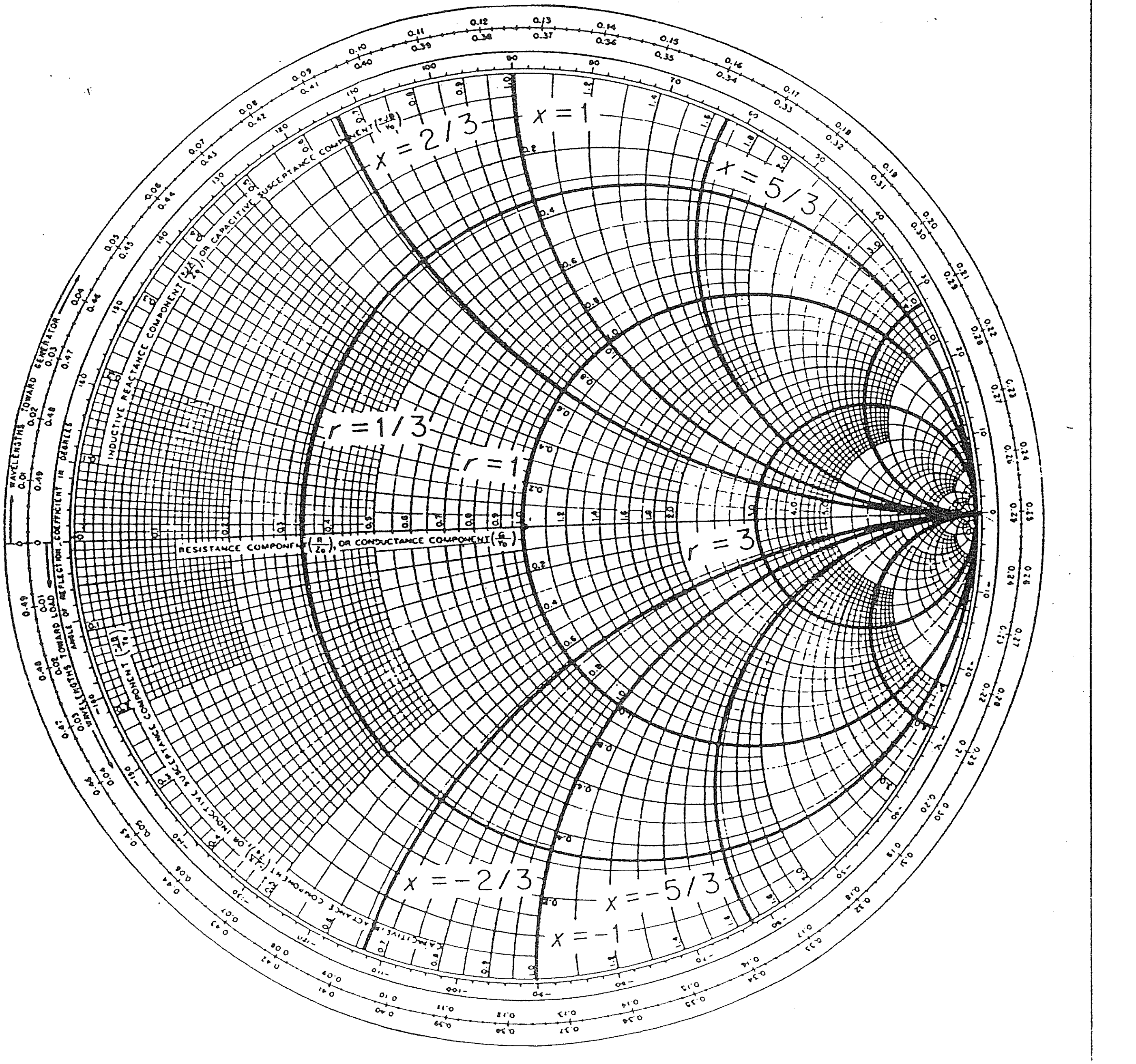
Locus of constant resistance and reactance of <u>normalized</u> load impedance on polar reflection plane.

$$\overline{Z}_L = \frac{Z_L}{Z_C} = R_n + jX_n$$

Circles give Γ for constant R_n, $-\infty \le X_n \le \infty$

Arcs give Γ for constant X_n, $-\infty \le R_n \le \infty$

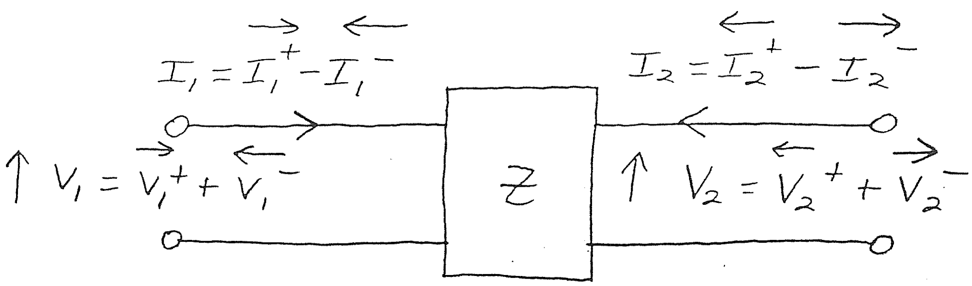
Upper half plane → inductive, lower half plane → capacitive



MICROWAVE NETWORK REPRESENTATIONS

- The relationships between input and output voltages and currents for a general N-port microwave network can be represented in many ways. The most common input/output matrices for microwave networks are the impedance, voltage/current transmission, scattering, and wave-amplitude transmission matrices. Each of these representations is discussed below for the common case of a two-port network. Generalization to N-port networks is straight forward.

- Impedance [Z] matrix:



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

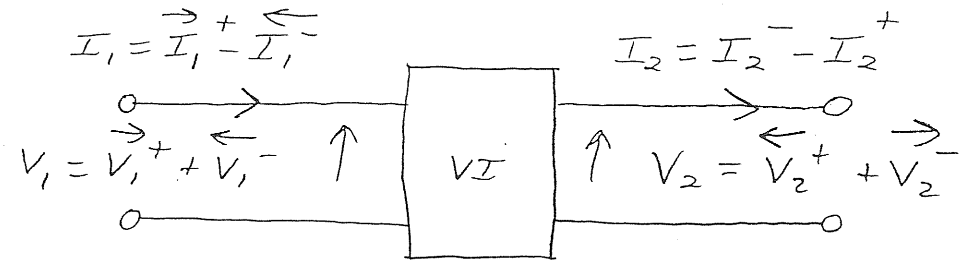
<u>Properties</u>

1) [Z] is symmetric if Z is reciprocal

2) [Z] is imaginary if Z is lossless

3) The admittance matrix $[Y] = [Z]^{-1}$

- Voltage/current [VI] transmission matrix:

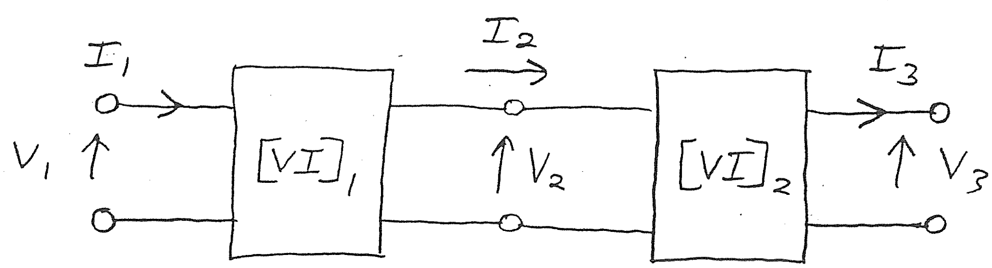


$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Properties

1) $|VI| = 1$ for reciprocal network

2) Output V, I of one network serves as input V, I of another network. Allows cascading of [VI] matrices.

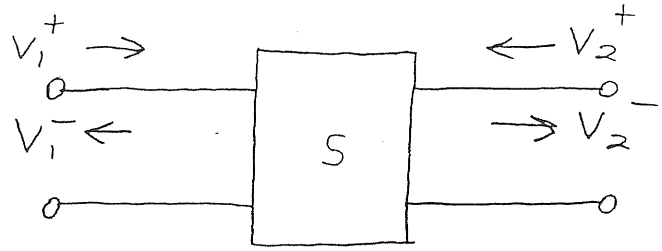


$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$

3) Relationship to [Z] matrix:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} Z_{11}/Z_{12} & (Z_{11}Z_{22} - Z_{12}^2)/Z_{12} \\ 1/Z_{12} & Z_{22}/Z_{12} \end{bmatrix}$$

- Scattering [S] matrix:



$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}$$

Properties

1) [S] is symmetric if the network is reciprocal.

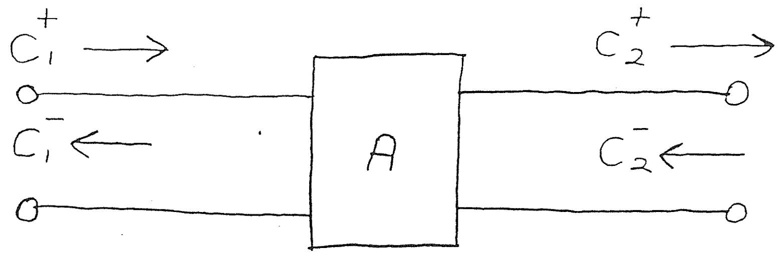
2) For a lossless network, conservation of power gives:

$$|S_{11}| = |S_{22}|$$

and: $|S_{12}| = \sqrt{1 - |S_{11}|^2}$

3) Forward, reflected and transmitted voltages are easily measured for microwave networks. Therefore, [S] matrix is easily measured. This is in contrast to [Z] and [VI] matrices which require measurement of total voltage and current at each port.

- Wave amplitude transmission [A] matrix:

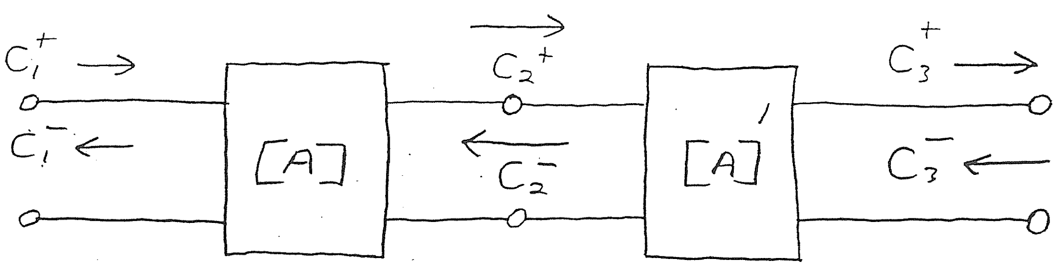


$$\begin{bmatrix} C_1^+ \\ C_1^- \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

<u>Properties</u>

1) $|A| = 1$ for reciprocal networks

2) [A] matrices can be cascaded:



$$\begin{bmatrix} C_1^+ \\ C_1^- \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}' & A_{12}' \\ A_{21}' & A_{22}' \end{bmatrix} \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix}$$

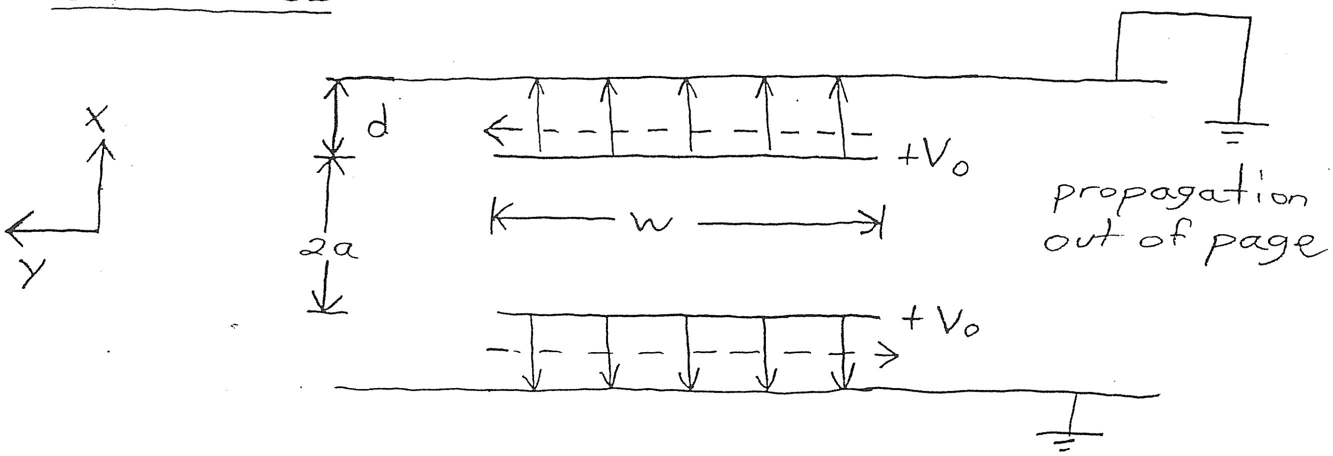
3) Relationship to [S] matrix:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1/S_{12} & -S_{22}/S_{12} \\ S_{11}/S_{12} & (S_{12}^2 - S_{11}S_{22})/S_{12} \end{bmatrix}$$

EXAMPLE 1

- Impedance of stripline electrodes in sum and difference (longitudinal and transverse) mode:

Sum mode



Assume $d \ll w$ so that fringe field effects are small. Assume electrodes in vacuum.

In sum mode TEM fields exist only in space d behind electrodes. From symmetry, the characteristic impedances of top and bottom electrodes are equal. Examine top electrode:

$$\vec{E} = \frac{V_o}{d} e^{-jk_o z} \hat{x}$$

$$\therefore \quad \vec{H} = \frac{1}{Z_o} \hat{z} \times \vec{E} = \frac{V_o}{d Z_o} e^{-jk_o z} \hat{y}$$

where: $Z_o = 377\,\Omega$

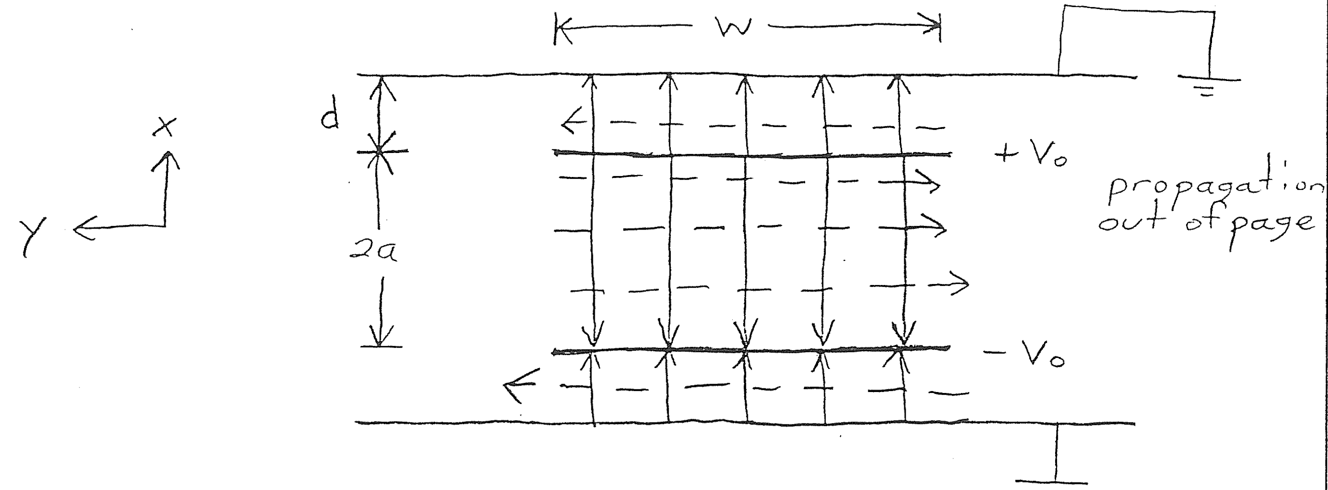
The current on the electrode is:

$$|I_o| = |(\hat{n} \times \vec{H})\, w| = \left| \frac{V_o w}{d Z_o} \hat{z} \right|$$

So:

$$\boxed{Z_{C\Sigma} = \frac{V_o}{I_o} = Z_o \frac{d}{w}}$$

Difference mode



Again, from symmetry, the characteristic impedances of the top and bottom electrodes are equal.

For the top electrode:

In the "d" region:

$$\vec{E} = \frac{V_o}{d} e^{-jk_o z} \hat{x}$$

$$\vec{H} = \frac{V_o}{Z_o d} e^{-jk_o z} \hat{y}$$

so I_o on the top of the electrode is:

$$I_{o_{top}} = \frac{V_o w}{d Z_o}$$

Similarly, looking in the "2a" region gives

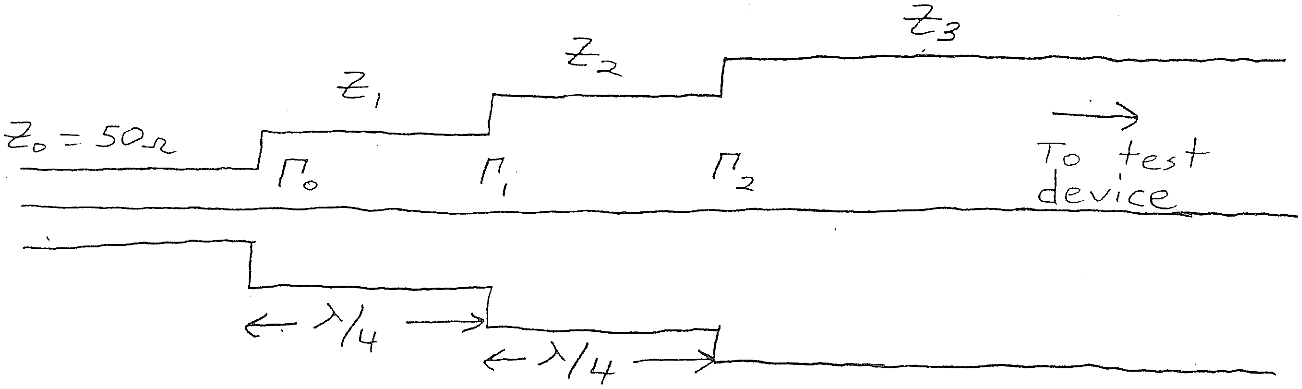
$$I_{o_{bottom}} = \frac{V_o w}{2a Z_o}$$

And:

$$\boxed{Z_c = \frac{V_o}{I_{o_{top}} + I_{o_{bottom}}} = \frac{Z_o}{w}\left(\frac{1}{d} + \frac{1}{2a}\right)}$$

EXAMPLE 2

- The 2-section quarter wave transformer is useful for matching a high impedance line to 50Ω for pickup and beam impedance measurements:



Assume that the transitions are gradual so that $|\Gamma_n| << 1$ and:

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \approx \frac{1}{2} \ln \frac{Z_{n+1}}{Z_n}$$

If $|\Gamma_n| << 1$, consider only 1st order reflections, so that the input reflection coefficient becomes:

$$\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta}$$

where: $\theta = \beta \ell = \frac{\pi}{2} \frac{f}{f_0}$, $f_0 = \lambda/4$ frequency

We require $\Gamma(\theta) = 0$ at $f = f_0$. Also, require that $\Gamma(\theta)$ be flat about $f_0 \pm \Delta f$.

$\therefore$

$$\Gamma\left(\frac{\pi}{2}\right) = 0$$

$$\left.\frac{d\Gamma}{d\theta}\right|_{\pi/2} = 0$$

These two conditions give:

$$\Gamma_0 - \Gamma_1 + \Gamma_2 = 0$$

$$j2\Gamma_1 - j4\Gamma_2 = 0$$

Or: $\Gamma_1 = 2\Gamma_2$

$\Gamma_0 = \Gamma_2$

Which in turn become:

$$\ln \frac{Z_2}{Z_1} = 2 \ln \frac{Z_3}{Z_2}$$

$$\ln \frac{Z_1}{Z_0} = \ln \frac{Z_3}{Z_2}$$

Solving for Z_1, Z_2 gives:

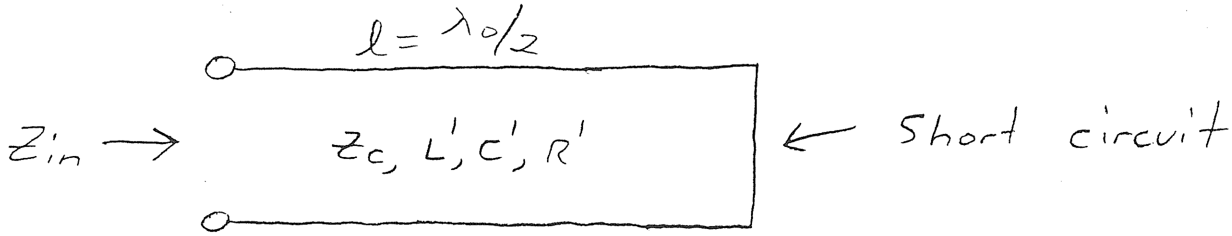
$$Z_1 = Z_0^{3/4} Z_3^{1/4} \quad , \quad Z_2 = Z_0^{1/4} Z_3^{3/4}$$

Which are the required impedances for the two matching sections.

The frequency characteristics of the match can be investigated by finding Γ_n and evaluating $\Gamma(\theta)$.

EXAMPLE 3

- $\lambda/2$ short circuited TEM transmission line resonator:



The input impedance is, from (62):

$$Z_{in} = Z_c \tanh(\alpha l + j\beta l)$$

$$= Z_c \left[\frac{\sinh 2\alpha l + j \sin 2\beta l}{\cosh 2\alpha l + \cos 2\beta l} \right]$$

For low loss, $\alpha l \ll 1$. In addition:

Let $l = \frac{\lambda_0}{2}$ at $f = f_0$

for f near f_0, $f = f_0 + \Delta f$,

$$\beta l = \frac{2\pi f l}{c} = \frac{2\pi (f_0 + \Delta f)\lambda_0}{2c} = \pi + \pi \frac{\Delta f}{f_0}$$

Thus for $\alpha l \ll 1$, and $\Delta f / f_0 \ll 1$,

$$Z_{in} = Z_c \left[\frac{\cancel{2}\alpha l + j \cancel{2}\pi \frac{\Delta f}{f_0}}{\cancel{1+1}} \right]$$

$$Z_{in} = Z_c \left[\alpha l + j\pi \frac{\Delta \omega}{\omega_0} \right]$$

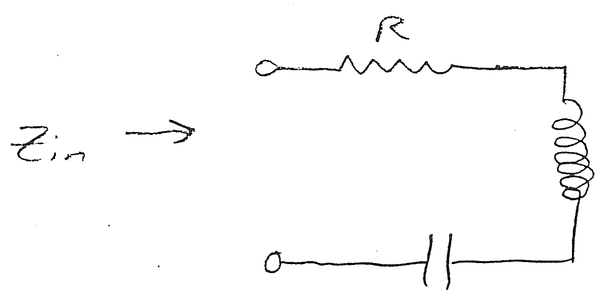
Now: $Z_c \alpha \ell = \frac{1}{2} R' \ell$

and: $\beta_o \ell = \frac{\omega_o \ell}{c} = \omega_o \sqrt{LC}\, \ell = \pi$

so $\frac{\pi}{\omega_o} = \ell \sqrt{LC}$ $\qquad \omega_o = \frac{\pi}{\ell\sqrt{LC}}$

and $\boxed{Z_{in} = \frac{1}{2} R' \ell + L' \ell \Delta \omega}$

Compare to series R, L, C circuit:



$$Z_{in} = R + j\omega L + \frac{1}{j\omega C} = R + j\omega L \left(1 - \frac{1}{\omega^2 LC}\right)$$

Let: $\omega_o^2 = 1/LC$

$$Z_{in} = R + j\omega L \left(\frac{\omega^2 - \omega_o^2}{\omega^2}\right) \approx \boxed{R + j\, 2 L \Delta \omega}$$

Thus the circuits are equivalent with

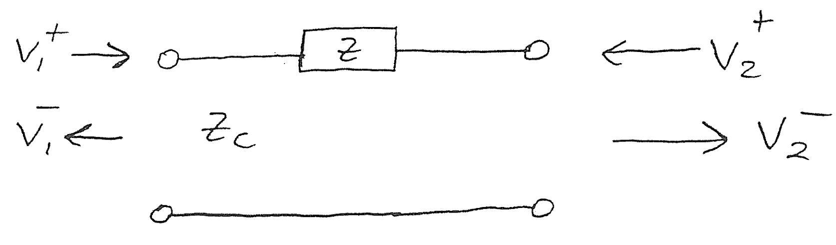
$\frac{1}{2} R' \ell = R$, and $\frac{1}{2} L' \ell = L$

Also: $Q = \frac{\omega_o L}{R} = \frac{\beta_o}{2\alpha}$

EXAMPLE 4

- [S] and [A] matrices for a series impedance:

1) [S]



$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}$$

Assume port 2 is terminated in Z_c so that $V_2^+ = 0$, then:

$$S_{11} = \frac{V_1^-}{V_1^+} = \Gamma_o = \frac{(Z+Z_c)-Z_c}{(Z+Z_c)+Z_c} = \frac{Z}{Z+2Z_c}$$

From symmetry $S_{11} = S_{22}$

Also:

$$V_2^- = (I_1^+ - I_1^-)Z_c = (V_1^+ - V_1^-) = V_1^+(1-S_{11})$$

So:

$$S_{21} = S_{12} = 1 - S_{11} = \frac{2Z_c}{Z+2Z_c}$$

By direct analysis [A] may be obtained.

Or

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1/S_{12} & -S_{22}/S_{12} \\ S_{11}/S_{12} & (S_{12}^2 - S_{11}S_{22})/S_{12} \end{bmatrix}$$

USPAS Microwave Measurements Homework

Answers

Reflection of 25Ω Shunt

① $$\Gamma_o = \frac{Z_L - Z_c}{Z_L + Z_c} = \frac{25-50}{25+50} = -\frac{1}{3}$$

$$|\Gamma_o| = 20 \log \frac{1}{3} = -9.5 \text{ dB}$$

$$\angle \Gamma_o = 180^\circ$$

② $$\Gamma(\omega) = \Gamma_o e^{-j2\frac{\omega}{c}l_e}$$

$$|\Gamma(\omega)| = |\Gamma_o| \qquad \angle \Gamma(\omega) = -\frac{2\omega l_e}{c} + \pi$$

$$l_e = l_{physical} \frac{c}{v} = l_{physical} \sqrt{\varepsilon_r}$$

③ $$Z_{in} = Z_c \left[\frac{Z_L + jZ_c \tan \beta_o l_e}{Z_c + jZ_L \tan \beta_o l_e} \right] \approx Z_c \left[\frac{Z_L + jZ_c \beta_o l_e}{Z_c + jZ_L \beta_o l_e} \right]$$

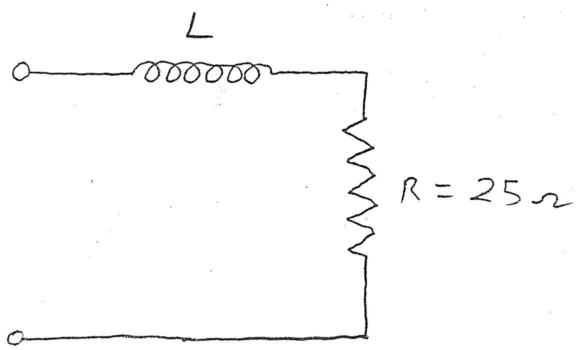
$$= \frac{Z_L + jZ_c \beta_o l_e}{1 + j\frac{Z_L}{Z_c}\beta_o l_e} \approx (Z_L + jZ_c\beta_o l_e)\left(1 - j\frac{Z_L}{Z_c}\beta_o l_e\right)$$

$$\approx Z_L + jZ_c\beta_o l_e - j\frac{Z_L^2}{Z_c}\beta_o l_e$$

$$= Z_L + j\omega\left[\frac{l_e}{c} Z_c \left(1 - \frac{Z_L^2}{Z_c^2}\right)\right]$$

Note: $\beta_o = \frac{\omega}{c}$

④



$\frac{15\times10^{-3}}{3\times10^{8}}$ $\quad 5\times10^{-11}\cdot 37.5 \quad 1.85\times10^{-11} \quad 1.85\ nH$

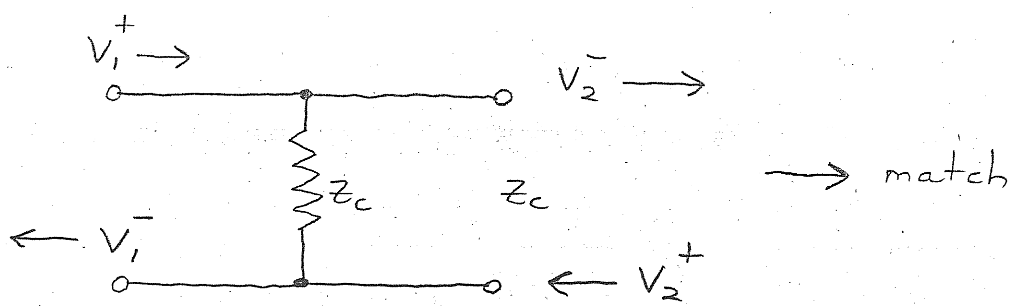
$$L = \frac{l_e}{c} Z_c\left(1 - \frac{Z_L^2}{Z_c^2}\right) = l_e L'\left(1 - \frac{Z_L^2}{Z_c^2}\right)$$

$50\cdot\frac{3}{4}$

⑤ No. 50Ω terminations matched to line.

Transmission for Shunt Elements

①



S-parameters

$$S_{11} = S_{22} = \frac{V_1^-}{V_1^+} = \frac{Z_c - \frac{Z_c}{2}}{Z_c + \frac{Z_c}{2}} = -\frac{1}{3}$$

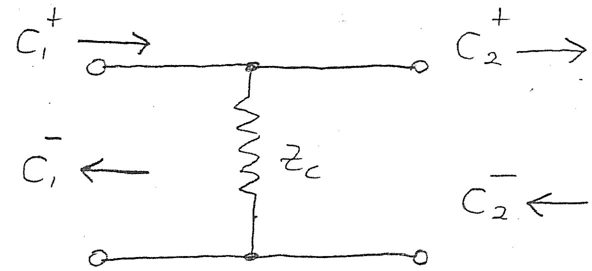
$$\frac{\frac{Z_c}{2} - Z_c}{\frac{Z_c}{2} + Z_c} = -\frac{1}{3}$$

$$S_{21} = S_{12} = \frac{V_2^-}{V_1^+} = \frac{V_1^+ + V_1^-}{V_1^+} = 1 + S_{11} = \frac{2}{3}$$

$\therefore$

$$[S] = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Wave amplitude



Z_c

$$\begin{bmatrix} C_1^+ \\ C_1^- \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

Drive side 1, match side 2:

$$C_1^+ = a_{11} C_2^+ \quad \therefore \quad a_{11} = C_1^+ / C_2^+ = \frac{1}{S_{21}} = 3/2$$

$$C_1^- = a_{21} C_2^+ \quad \therefore \quad a_{21} = C_1^- / C_2^+ = S_{11}/S_{21} = -1/2$$

Drive side 2, match side 1: $(C_1^+ = 0)$

$$a_{11} C_2^+ = -a_{12} C_2^-$$

so $$a_{12} = -\frac{1}{S_{21}} \frac{C_2^+}{C_2^-} = -\frac{S_{22}}{S_{21}} = -\left(\frac{-\frac{1}{3}}{\frac{2}{3}} \right) = \frac{1}{2}$$

and:
$$C_1^- = a_{21} C_2^+ + a_{22} C_2^-$$

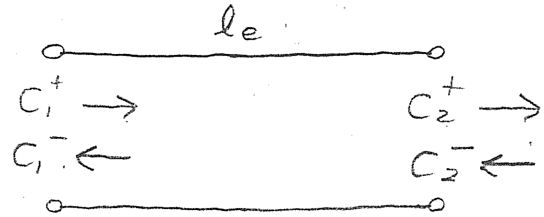
so $$a_{22} = \frac{C_1^-}{C_2^-} - a_{21} \frac{C_2^+}{C_2^-} = S_{12} - \frac{S_{11} S_{22}}{S_{21}} = 1/2$$

② $$P_o = P_{ref} + P_{trans} + P_{shunt}$$

$$1 = \underset{(S_{11}^2)}{\frac{1}{9}} + \underset{(S_{21})^2}{\frac{4}{9}} + P_{shunt}$$

$P_{shunt} = 4/9$, not a lossless network

3



$$\begin{bmatrix} C_1^+ \\ C_1^- \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

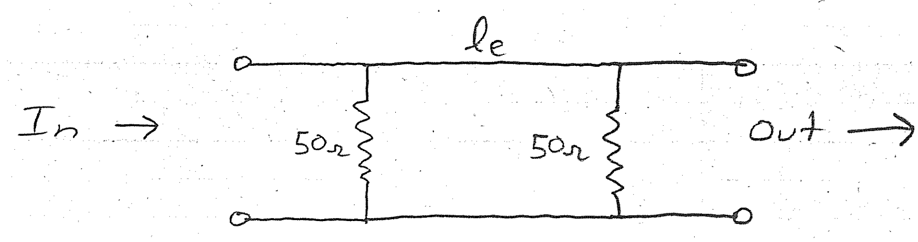
Drive 1 match 2, Drive 2 match 1 gives:

$$[A_T] = \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix} \qquad \theta = \frac{\omega l_e}{c}$$

4

$$[A_{STS}] = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 9e^{j\theta} - e^{-j\theta} & 3e^{j\theta} + e^{-j\theta} \\ -3e^{j\theta} - e^{-j\theta} & -e^{j\theta} + e^{-j\theta} \end{bmatrix}$$



5 $S_{21} = 1/a_{11}$

$$|S_{21}|^2 = \frac{16}{|a_{11}|^2} = \frac{16}{82 - 18\cos 2\theta}$$

$$|S_{21}| = \frac{4}{\sqrt{82 - 18\cos 2\theta}}$$

6 $$20 \log \frac{4}{\sqrt{82 \pm 18}} = -6\,dB,\ -8\,dB$$

7 $$S_{21} = \frac{4}{9e^{j\theta} - e^{-j\theta}} = 1/ \left\{ 9e^{-j\theta} - e^{j\theta} \right\}$$

$$= 1/\left\{ 9 - e^{j2\theta} \right\} e^{-j\theta} \leftarrow \text{linear phase } \frac{\omega l_e}{c}$$

$$\angle_\Delta = \tan^{-1}\left[\frac{-\sin 2\theta}{9 - \cos 2\theta}\right] \approx \tan^{-1}\left[\frac{-\sin 2\theta}{9}\right]$$

$$\approx \frac{-\sin 2\theta}{9}$$

Long Cable

1 $$|H(\omega)| = e^{-\psi z\sqrt{\omega}}$$

at $\omega_o = \omega_{3dB}$, $\frac{1}{\sqrt{2}} = e^{-\psi z\sqrt{\omega_o}}$

or $\psi z = \frac{\ln\sqrt{2}}{\sqrt{\omega_o}}$

Rise time is 10% - 90%

From tables (remember erfc = 1 - erf)

$$\frac{\psi z}{\sqrt{2t_{10}}} = 1.16 \quad \text{and} \quad \frac{\psi z}{\sqrt{2t_{90}}} = .09$$

$$t_{90} - t_{10} = \frac{\psi^2 z^2}{2}\left(\frac{1}{.09^2} - \frac{1}{1.16^2}\right) = 61.36\,\psi^2 z^2 = \frac{1.17}{f_o}$$

where: $f_o = \omega_o/2\pi$

2 In time domain, $e^{-j\beta z}$ is a time delay equal to $f \to 0$ delay of cable. Apart from this,

$$H(\omega) = e^{-(1+j)\psi z \sqrt{\omega}}$$

$$s = j\omega$$

$$H(s) = e^{-(1+j)\psi z \sqrt{-js}} = e^{-(1+j)\psi z \frac{(1-j)}{\sqrt{2}}\sqrt{s}}$$

$$= e^{-\sqrt{2}\psi z \sqrt{s}}$$

The inverse Laplace transform of $H(s)$ gives the impulse response. We want the step response which is the integral of the impulse response or:

$$h_s(t) = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right] = \operatorname{erfc}\left(\frac{\psi z}{\sqrt{2t}}\right)$$

by given formula.

Including the $e^{-j\beta z}$ delay gives:

$$h_s(t) = \begin{cases} 0 & 0 \le t \le \tau \\ \operatorname{erfc}\left(\frac{\psi z}{\sqrt{2(t-\tau)}}\right) & t > \tau \end{cases}$$

Two - Tap Notch Filter

① $$h_\Sigma(t) = \frac{\delta(t)}{2} + \frac{\delta(t-\tau)}{2}$$

$$h_\Delta(t) = \frac{\delta(t)}{2} - \frac{\delta(t-\tau)}{2}$$

By Fourier transform:

$$H_\Sigma(\omega) = \frac{1}{2}\left[1 + e^{-j\omega\tau}\right] = e^{-j\frac{\omega\tau}{2}} \cos\frac{\omega\tau}{2}$$

$$H_\Delta(\omega) = \frac{1}{2}\left[1 - e^{-j\omega\tau}\right] = je^{-j\frac{\omega\tau}{2}} \sin\frac{\omega\tau}{2}$$

② By inspection of impulse response and integrating: