Bogolyubov and Mitropolsky's averaging method. (Based on E. Perevedentsev's lectures read at Novisibirsk State University)

Eduard Pozdeyev

January 21, 2008

1 Hill's equation with perturbations

Hill's equation with a perturbation:

$$x'' + K(s)x = f(x, s) \tag{1}$$

This equation can be reduced to the oscillator equation with the time-dependent force using the new variables:

$$y = \frac{x}{w},\ d\psi = \frac{ds}{\nu w^2} \tag{2}$$

The Hill's equation (1) can be written in this new variables as

$$\ddot{y} + \nu^2 y = \nu^2 w^3 f(w(s(\psi))) \cdot y, s(\psi)), \tag{3}$$

where dots mean diffirentiation with respect to ψ. This equation is equvalent to the oscillator equation:

$$\ddot{x} + \omega^2 x = f(x, t) \tag{4}$$

If the force is zero, the solution of this equation is

$$x = A \cdot \cos(\omega t + \phi), \tag{5}$$

where A and ϕ are constant. For the inhomgenious equation, the amplitude and the phase will not be constants anymore. To solve this equation, let's change vriables to the amplitude and phase. Because the new A and ϕ are not known we have to define them. First, we want the $\dot{x}$ to look the same as it was not perturbed:

$$\dot{x} = -\omega A \cdot \sin(\omega t + \phi) \tag{6}$$

This gives us one equation defining the amplitude and the phase:

$$\dot{x} = -\omega A \cdot \sin(\omega t + \phi) + \dot{A}\cos(...) + \dot{\phi} A \sin(...) \tag{7}$$

Therefore:

$$\dot{A}\cos(...) + \dot{\phi} A \sin(...) = 0 \tag{8}$$

The other equation is obtained by substituting $x = A\cos(\omega t + \phi)$ into (4):

$$\ddot{x} = -\omega^2 A\cos(...) - \omega \dot{A}\sin(...) - \omega A\dot{\phi}\cos(...) = -\omega^2 x + f(x, t) \tag{9}$$

Combining the last equaqtion with (7), we obtain exact equations for the phase and amplitude:

$$\dot{A} = -\frac{f}{\omega}\sin(\omega t + \phi) \tag{10}$$

$$\dot{\phi} = -\frac{f}{\omega A}\cos(\omega t + \phi) \tag{11}$$

These amplitude and phase contain fast oscillatory and slow drift/oscillation terms. Let's concentrate only on the slow terms. This can be achived by averaging the right hand side part over one pertiod of oscillations of unperturbed oscillations frequency ω with the A and ϕ on the right-hand side asumed to be constant during the averaging:

$$\dot{A}_s = \overline{-\frac{f}{\omega}\sin(\omega t + \phi)} \tag{12}$$

$$\dot{\phi}_s = \overline{-\frac{f}{\omega A}\cos(\omega t + \phi)}, \tag{13}$$

where the line means averaging. Because we will be interested only in slow oscillations, I'll drop the s subscript from now on!!!

Returning to our equation of the particle motion (3), we rwrite it as:

$$\dot{A} = \overline{-\nu w^3 F\sin(\nu\psi + \phi)} \tag{14}$$

$$A\dot{\phi} = \overline{-\nu w^3 F\cos(\nu\psi + \phi)}, \tag{15}$$

or in with the derivative with respect to s:

$$A' = \frac{dA}{ds} = \overline{-wF\sin(\nu\psi(s) + \phi)} \tag{16}$$

$$A\phi' = A\frac{d\phi}{ds} = \overline{-wF\cos(\nu\psi(s) + \phi)}. \tag{17}$$

2 Time independent perturbations

2.1 Changed rigidity of an oscillator

$$f = -\delta k x \tag{18}$$

$$\dot{A} = -\frac{\delta k}{\omega}\overline{A\cos(\omega t + \phi)\sin(\omega t + \phi)} = 0 \tag{19}$$

$$\dot{\phi} = \overline{-\frac{\delta k}{\omega A}\cos^2(\omega t + \phi)} = \frac{\delta k}{2\omega} \tag{20}$$

This is the first order error to the oscillator frequency:

$$\ddot{x} + (\omega^2 + \delta k)x = 0 \tag{21}$$

Therefore:

$$\Omega = \sqrt{\omega^2 + \delta k} \approx \omega + \frac{\delta k}{2\omega} \tag{22}$$

2.2 Friction force

$$f = -2\alpha\dot{x} \tag{23}$$

$$\dot{A} = -2\alpha A\overline{\sin^2(\omega t + \phi)} = -\alpha A \tag{24}$$

$$\dot{\phi} = -2\alpha\overline{\sin(...)\cos(...)} = 0 \tag{25}$$

Exact solution yields the same result in the first order:

$$x = Ae^{-\alpha t}\cos(\sqrt{\omega^2 - \alpha^2}t + \phi) \tag{26}$$

2.3 Nonlinear oscillator

Nonlinearities in accelerators arise from:

- Magnetic field imperfections of linear elements
- Second order field nonlinearity (sextupole) is used to compensate energy dependent focusing effects
- Third order field nonlinearity (octupoles) to introduce betatron frequency spread in the beam to compencate coherent collective instabilities

Nonlinear forces are described by the hamiltonian terms $x^m \cdot y^n$ with $m + n > 2$ or force terms with $m+n > 1$ in the Hill equation. Nonlinear force can cause significant beam quality and life time degradation. Let's assume the force law:

$$f = -\omega^2\beta_m x^m \tag{27}$$

$$\dot{A} \sim \overline{\cos^m(...)\sin(...)} = 0 \tag{28}$$

$$\dot{\phi} = \omega\beta_m A^{m-1}\overline{\cos^{m+1}(...)} \tag{29}$$

The last equation is equal to 0 if m is even, and not equal to 0 if m is odd.

2.3.1 Cubic nonlinearity far from resonance, m=3

The third order field nonlinearity (octupoles) are used to introduce betatron frequency spread in the beam to compencate coherent collective instabilities.

$$\overline{\cos^4(...)} = \frac{(e^{i\psi} + e^{-i\psi})^4}{16} = \frac{3}{8} \tag{30}$$

Therefore,

$$\delta\omega = \dot{\phi} = \frac{3}{b}\omega\beta_3 A^2 \tag{31}$$

In a real-world accelerator, the right-hand side of the Hill's equations with the cubic nonlinearity is

$$f(x,s) = \frac{B'''x^3}{6B\rho}. \tag{32}$$

$$A\phi' = \frac{d\phi}{ds} = \overline{\frac{wB'''}{6B\rho}x^3\cos(\nu\psi + \phi)} \tag{33}$$

$$\delta\nu = \frac{\delta\mu}{2\pi} = \frac{A^2}{12\pi B\rho}\int w^4 B'''\overline{\cos^4(...)} = \frac{A^2}{32\pi}\int \frac{B'''\beta^2}{B\rho}ds. \tag{34}$$

,where $B\rho$ is the magnetic rigidity.

3 Time dependent perturbations: resonances

Far from resonances, imperfections introduce the frequency shift but do not affect the aplitude in the first order. Let's study resosnances.

$$f(x,s) = x^m \cdot g(s) = y^m w^m g(s(\psi)) = y^m \sum_q |f_q| \cos(q\psi + \alpha_q) \tag{35}$$

$$f_q = \nu \int w^{m+3} g(s(\psi)) e^{iq\psi} \frac{d\psi}{2\pi} = \frac{1}{2\pi} \int g(s) w^{m+1} e^{iq\psi(s)} ds, \tag{36}$$

where $\nu w^2 d\psi = ds$.
Let's consider a specific harmonics of the perturbation. Thus,

$$f \to y^m |f_q| cos(q\psi + \alpha_q). \tag{37}$$

By a proper choice of the initial s we can set $\alpha_q = 0$.

$$\frac{dA}{d\psi} = -f_q A^m \overline{\cos(q\psi) \cos^m(\nu\psi + \phi) \sin(\nu\psi + \phi)} \tag{38}$$

$$A\frac{d\phi}{d\psi} = -f_q A^{m-1} \overline{\cos q\psi \cos^{m+1}(\nu\psi + \phi)}, \tag{39}$$

3.1 Driven oscillator, m=0

$$f = h\cos(q\psi) \tag{40}$$

$$\dot{A} = -f_q \overline{\cos(q\psi)\sin(\nu\psi + \phi)} = -\frac{f_q}{2}\overline{(\sin(\nu + q)\psi + \phi) + \sin((\nu - q)\psi + \phi))}. \tag{41}$$

$$\dot{\phi} = -\frac{f_q}{A}\overline{\cos(q\psi)\cos(\nu\psi + \phi)} = -\frac{f_q}{2A}\overline{(\cos(\nu + q)\psi + \phi) + \cos((\nu - q)\psi + \phi))}. \tag{42}$$

If $\nu - q = \delta$ is small, then, the external force is in resonance with the beam. Then, the first term in the last equation avergaes out while the second term stays because it changes slowly:

$$\dot{A} = -\frac{f_q}{2}\sin((\nu - q)\psi + \phi) = -\frac{f_q}{2}\sin(\delta\psi + \phi) \tag{43}$$

$$\dot{\phi} = -\frac{f_q}{2A}\cos((\nu - q)\psi + \phi) = -\frac{f_q}{2A}\cos(\delta\psi + \phi). \tag{44}$$

Let's introduce new variables:

$$I = A^2 \tag{45}$$

$$\Phi = \delta\psi + \phi. \tag{46}$$

This choice of the phase is equivalent to a transition into a frame "rorating" with the frequency q instead of ν and removes any dependence on "time", ψ, in the equations of motion. We will also add a cubic nonlinearity non-resosnant term to the phase equation. All this yields a pair of Hamiltonian equations

$$\dot{I} = -f_q\sqrt{I}\sin(\Phi) = \frac{\partial H}{\partial \Phi} \tag{47}$$

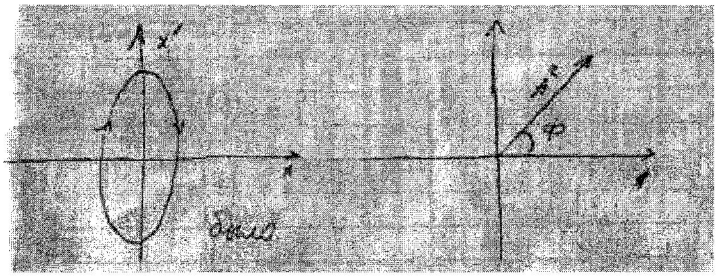

Figure 1: Phase space plane with new variables.

$$\dot{\Phi} = -\frac{f_q}{2\sqrt{I}}\cos(\Phi) + \delta + BI = -\frac{\partial H}{\partial I} \tag{48}$$

,where B is the cubic nonlinearity, with the Hamiltonian function

$$H(I,\Phi) = \frac{B}{2}I^2 + \delta I - F_q\sqrt{I}\cos\Phi = const \tag{49}$$

Because the Hamiltonian function does not explicitely depend on time, it is constant. Therefore, particles will move on curves defined by the equation

$$H = const \tag{50}$$

in the $I\cos(\Phi) - I\sin(\Phi)$ plane.

To study the system let's find fixed points on the (I,Φ)-phase diagram. These points are in resonance with the driving force and are given by

$$\frac{d\Phi}{d\psi} = \frac{\partial H}{\partial I}, \frac{dI}{d\psi} = -\frac{\partial H}{\partial \Phi} \tag{51}$$

These points define separatrices on the phase space plane. Particles move around those points (in opposite directions), if the fixed points are stable. The unstable points separate the stable areas. The stable points for this case are :

$$\Phi = 0, \pi : \; BI^{3/2} + \delta I^{1/2} \pm \frac{f_q}{2} = 0 \tag{52}$$

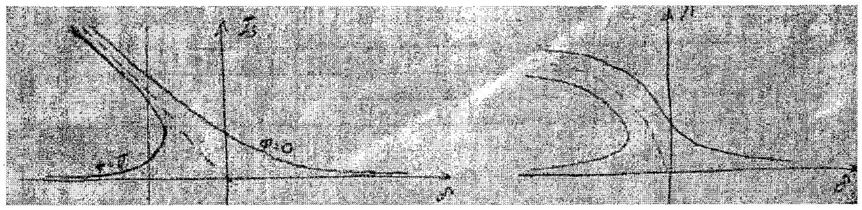

Figure 2: Fixed points as functions of δ for m=0.

They can be also approximately obtained graphically as shown below.

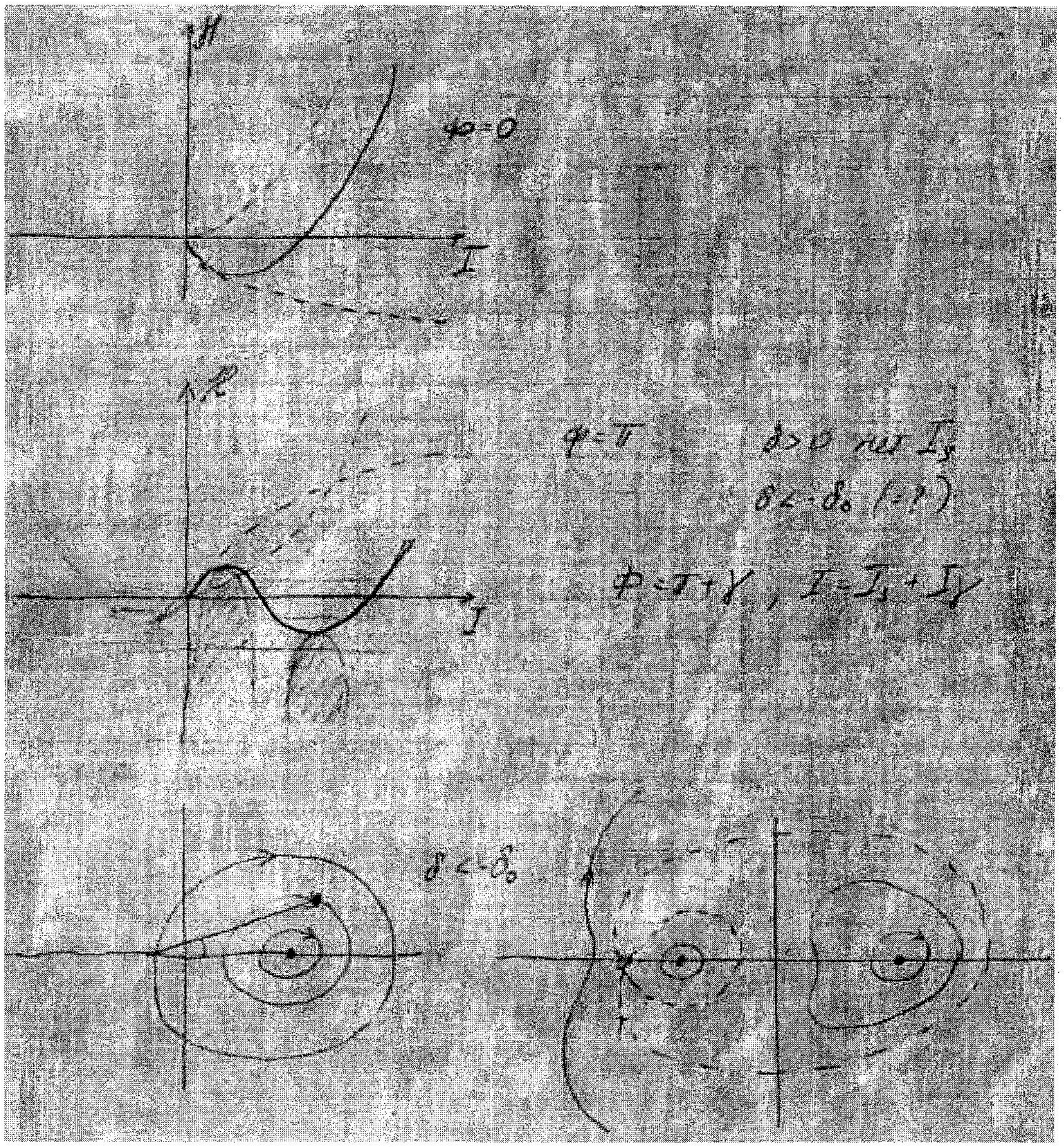

Figure 3: Hamiltonian cross sections for $\Phi = 0, \pi$ and its function levels for m=0.

3.2 1/3-resonance, m=2

$$\dot{A} = -f_q A^2 \overline{\cos(q\psi)\cos^2(\nu\psi + \phi)\sin(\nu\psi + \phi)}. \quad (53)$$

Using the relation $2\cos^2(\xi) = 1 + cos(2\xi)$, we see that there are two frequensies ν and 3ν. Let's chose the betatron frequency close to $q/3$:

$$\nu = \frac{q}{3} + \delta. \quad (54)$$

$$\dot{A} = -f_q A^2 \sin(3\delta\psi + 3\phi) \quad (55)$$

$$\dot{\phi} = -f_q A \overline{\cos(q\psi)\cos^3(\nu\psi + \phi))} = -f_q A \cos(3\delta\psi + 3\phi) \quad (56)$$

$$\dot{I} = -\frac{f_q}{4} I^{3/2} \sin(\Phi) \quad (57)$$

$$\dot{\Phi} = -\frac{3f_q}{8} I^{1/2} \cos\Phi + 3\delta + 3BI = -f_q A \cos(3\delta\psi + 3\phi) \quad (58)$$

Corresponding Hameltonian function is:

$$H = \frac{3}{2} BI^2 + 3\delta I - \frac{3}{8} F_q I^{3/2} \cos(\Phi) \quad (59)$$

The Hamiltonian function levels plotted below.

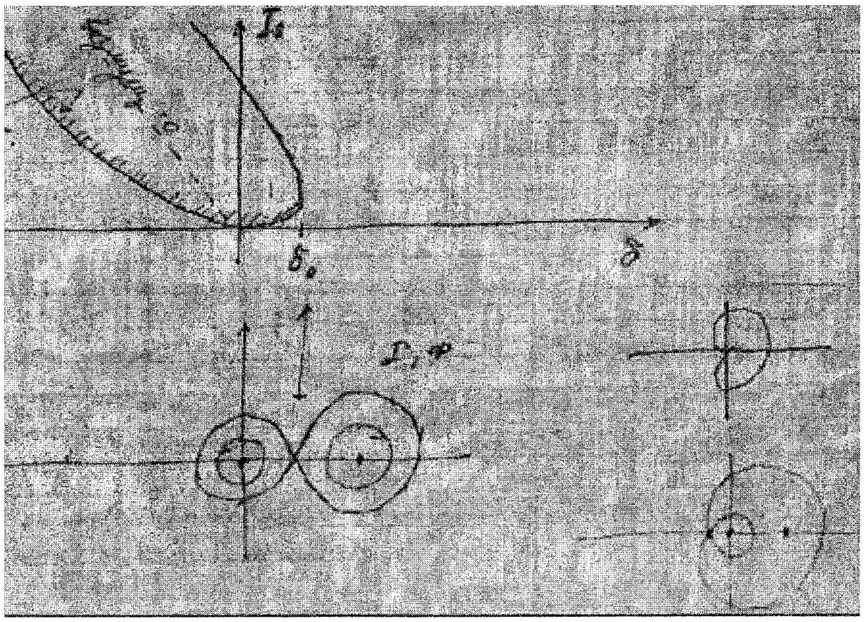

Figure 4: Fixed points as functions of δ for m=0.

4 Chromaticity compensation with sextupoles

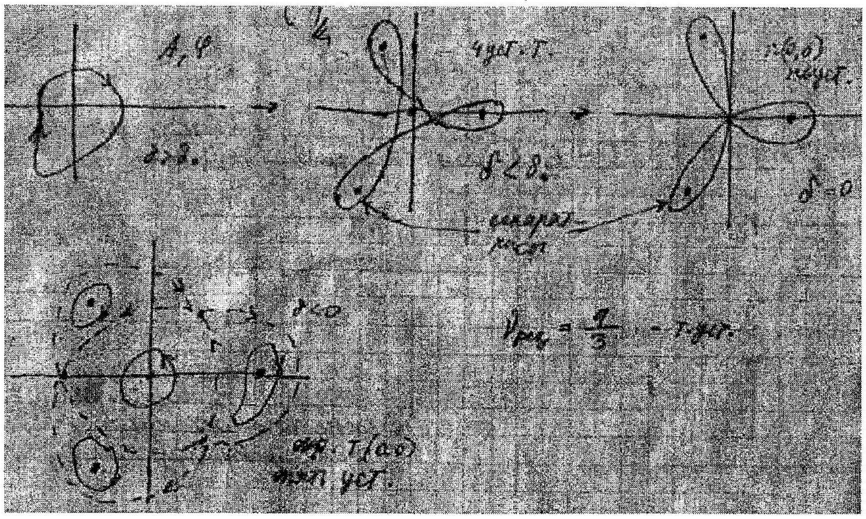

Figure 5: $A - \phi$ phase space plots for m=2.

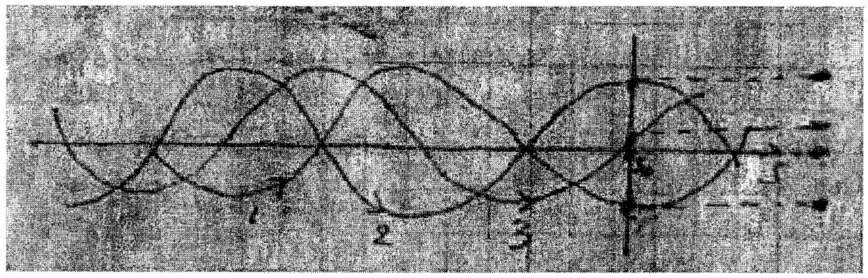

Figure 6: Trajectories on the 3rd resonance.

$$X'' + kX = 0$$

$$K = \left(\frac{dH_0}{dx}\right) \Big/ \left(H_0 r_0 \left(1+\frac{\Delta p}{p}\right)\right), \qquad H_z = H_0 + H'x + \frac{1}{2}H''x^2$$

$$K = \frac{H' + H'' X_p}{H_0 r_0 \left(1+\frac{\Delta p}{p}\right)} = \frac{H' + 2H''\frac{\Delta p}{p}}{r_0 H_0 \left(1+\frac{\Delta p}{p}\right)} = \underbrace{K_0}_{\frac{H'}{H_0 r_0}} + \underbrace{\frac{\Delta p}{p}\left(2\frac{H''}{H_0 r_0} - \frac{H'}{H_0 r_0}\right)}_{\Delta K}$$

$$\Delta\nu = \frac{\Delta\mu}{2\pi} = \frac{1}{2\pi}\oint \Delta K\, ds\, \overline{\cos^2\psi}\,)$$

$$\Delta\nu = -\frac{\Delta p}{p} \cdot \frac{1}{4\pi}\oint \frac{H' - 2H''}{H_0 r_0}\beta\, ds \qquad X_1 = 2\frac{\Delta p}{p}$$

Figure 7: Chromaticity compensation with quads.

Space charge effects in accelerators.

Eduard Pozdeyev, USPAS, Winter, 2008

January 23, 2008

1 Field of a relativistic bunch and transverse force cancellation

Note: in this section we temporally use x for the direction of the bunch motion.

Consider a bunch in its rest frame. The bunch electric field is the rest frame is $E' = (E'_x, E'_y, E'_z)$. The average magnetic field in the bunch rest frame is zero because the bunch is at rest: $B' = (0, 0, 0)$.

Transformations of the electric and magnetic fields from the bunch rest frame to the lab frame is

$$E_x = E'_x,\ B_x = 0 \tag{1}$$
$$E_y = \gamma E'_y,\ B_y = -\gamma\beta E'_z \tag{2}$$
$$E_z = \gamma E'_z,\ B_z = \gamma\beta E'_y \tag{3}$$

The field in the lab frame

$$\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp} = \vec{E}'_{\parallel} + \gamma\vec{E}'_{\perp} \tag{4}$$
$$\vec{B} = \gamma\vec{\beta} \times \vec{E}'_{\perp} = \vec{\beta} \times \vec{E}_{\perp} = \vec{\beta} \times \vec{E} \tag{5}$$

The Lorentz force is

$$\vec{F} = e\vec{E} + e\vec{\beta} \times \vec{B} = \tag{6}$$
$$e\vec{E} + e\vec{\beta} \times [\vec{\beta} \times \vec{E}] = \tag{7}$$
$$e\vec{E} + e(\vec{\beta}(\vec{\beta} \cdot \vec{E}) - \vec{E}\beta^2) \tag{8}$$

Because the direction of $\vec{\beta}$ and $\vec{E}_{\parallel}$ are the same by definition

$$\vec{\beta}(\vec{\beta} \cdot \vec{E}) = \vec{E}_{\parallel}\beta^2 \tag{9}$$

Thus, the Lorentz force becomes

$$\vec{F} = e\vec{E} + e(\vec{E}_{\parallel}\beta^2 - (\vec{E}_{\parallel} + \vec{E}_{\perp})\beta^2) = \tag{10}$$
$$e\vec{E} - e\vec{E}_{\perp}\beta^2 = \tag{11}$$
$$e(\vec{E}_{\parallel} + \vec{E}_{\perp}) - e\vec{E}_{\perp}\beta^2 = \tag{12}$$
$$e\vec{E}_{\parallel} + \frac{e\vec{E}_{\perp}}{\gamma^2} \tag{13}$$

Thus, the transverse component of the Lorentz force induced by the space charge is reduced by a factor of γ^2. The longitudinal component is the same as in the rest frame.

2 Transverse space charge (simple considerations)

Note: from now on we will use z for the direction of motion of the beam.

2.1 Round uniform beam

The transverse electric field of a uniformly charged round beam of the radius a is (indside the beam)

$$E_r = 2\pi\rho r \tag{14}$$

The field is linear through out the beam cross section and can be expressed via the longitudinal particle density $\lambda = dN/ds$:

$$E_r = \frac{2\lambda e r}{a^2} \tag{15}$$

Therefore, the transverse force for the beam with a relativistic factor of γ is

$$F_r = \frac{2\lambda e^2 r}{\gamma^2 a^2} \tag{16}$$

2.2 Round Gaussian beam

The electric field is no longer linear inside the Gaussian beam because the charge density changes with the radius. However, there is the linear component of the force that will affect betatron tunes. To calculate the linear component of the field, we can use the formula (14) because the charge density is almost uniform in the center of the Gaussian beam. The only thing that has to be adjusted is the charge density ρ.

Let's assume for now that the bunch has a Gaussian cross section and a uniform longitudinal charge distribution of length l. The normalized distribution of this bunch is

$$f(x, y, z) = \frac{1}{2\pi\sigma_x\sigma_y l} e^{-\left\{\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right\}} \tag{17}$$

Thus, the charge density of the bunch with the total number of particles N at $x, y = 0$ is

$$\rho = \frac{Ne}{2\pi\sigma_x\sigma_y l} = \frac{\lambda e}{2\pi\sigma_x\sigma_y} \tag{18}$$

For $\sigma_x = \sigma_y$, the formula becomes

$$\rho = \frac{\lambda e}{2\pi\sigma^2} \tag{19}$$

Therefore, using (14), we can conclude that the linear term of the electric field close to the beam center is

$$E_r(r \ll a) = 2\pi\rho r = \frac{\lambda r}{\sigma^2} \tag{20}$$

The corresponding transverse force is

$$F_r = \frac{\lambda e^2 r}{\gamma^2 \sigma^2} \tag{21}$$

2.3 Gaussian beam with $\sigma_x \neq \sigma_y$

Without derivation, the linear component of the electric field of the Gaussian beam is

$$\vec{F} = \frac{2e^2\lambda}{\gamma^2}\left[\frac{x\hat{e}_x}{\sigma_x(\sigma_x + \sigma_y)} + \frac{y\hat{e}_y}{\sigma_y(\sigma_x + \sigma_y)}\right] \tag{22}$$

2.4 Betatron tune depression due to space charge (Laslett tune shift)

The tune shift due to a thin quadrupole with the focusing length f is

$$\delta\nu = \frac{1}{4\pi}\frac{\hat{\beta}}{f}, \quad (23)$$

where we temporary use the $\wedge$ symbol for the beta-function to distinguish it from the velocity factor v/c. If focusing is distributed, the tune shift can be calculated as

$$\delta\nu = \frac{1}{4\pi}\oint \hat{\beta}\Delta K\, ds, \quad (24)$$

where ΔK is the additional focusing strength in the Hill's equation of motion $x'' + (K + \Delta K)x = 0$.

For a small δs,

$$\delta x' = \frac{\delta p_{\perp}}{p_0} = \frac{\int F\delta t}{p_0} = \frac{F\delta s}{pv} \quad (25)$$

$$\frac{\delta x'}{\delta s} = \frac{F}{pv} \quad (26)$$

As $\delta s \to 0$,

$$\frac{\delta x'}{\delta s} \to x'' \quad (27)$$

Thus, for the round Gaussian ($\sigma_x = \sigma_y$) beam,

$$\Delta K = \frac{\lambda e^2}{\gamma^2\sigma^2 pv} = \frac{\lambda r_0}{\gamma^3\beta^2\sigma^2}, \quad (28)$$

where r_0 is the particle classical radius, e^2/mc^2. Using 24 and the fact that $\sigma^2 = \hat{\beta\epsilon_n}/\beta\gamma$, we get

$$\delta\nu = \frac{1}{4\pi}\oint \hat{\beta}\frac{\lambda r_0 ds}{\gamma^3\beta^2\hat{\beta}(\epsilon_n/(\beta\gamma)} = \quad (29)$$

$$\frac{1}{4\pi}\frac{\lambda r_0 C}{\gamma^2\beta\epsilon_n} \quad (30)$$

For ions with the charge state Z/A, the tune shift is

$$\delta\nu = \frac{1}{4\pi}\frac{Z^2\lambda r_0 C}{A\gamma^2\beta\epsilon_n}. \quad (31)$$

The most interesting fact is that this tune shift in the first order does not depend on the machine specifics, like the beta function. The space charge tune shift is proportional to the machine length. Therefore, to have a small tune shift it is better to keep the machine size small. Therefore, low energy ion machines have to be limited in length to keep the space charge tune shift under control.

If the space charge field is nonlinear, the space charge force will provide a tune spread. Particles with small amplitudes will have the maximum tune shift while particles with large amplitudes will have a small tune shift. This will spread the beam foot print on the tune diagram and can cause overlapping with resonances. All particles will have same tune shift only if the space charge force is linear. Unfortunately, most distributions (all realistic?) produce a nonlinear force. (See the discussion of KV distribution below.)

The maximum tolerable tune spread/shift depends on a particular machine. However, it is obvious that a tune shift of the order of 0.25 can be dangerous because $q/4$ resonances, where q is integer, usually cause fast beam losses.

It is also worth noting that space charge effects are most dangerous in low-γ ion machines. In electron machines, the space charge becomes negligible above 10 MeV or so. Although, it depends on machine design and a required beam quality.

3 Longitudinal space charge effects

Longitudinally uniform beam with (almost) constant radius does not have the longitudinal field. Longitudinal charge density non-uniformities produce the field. In the case when the size of charge density perturbations is small comparatively to the diameter of the vacuum chamber, one can neglect image charges and use formulas presented in Section 1. This case is also briefly discussed below in Section 3.3. In the other case, when the length of charge density variations is much larger than the vacuum chamber size, it is necessary to include the effect of image charges and, what is important, image currents. This case is discussed in the following section.

3.1 Long wavelength approximation

We can calculate the longitudinal electric field using Faraday's law:

$$\oint \vec{dl} \cdot \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{dS} \cdot \vec{B}, \tag{32}$$

where $\vec{S}$ (capital S) is the area surrounded by the contour l. Consider the contour shown in Figure 1, where ds is much smaller than the length of the field perturbation.

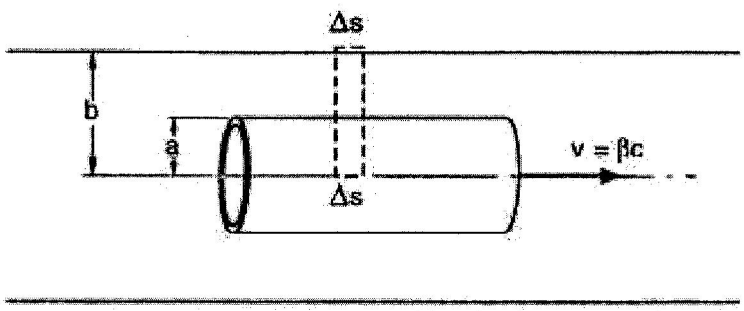

Figure 1: Integration contour to calculate the longitudinal electric field on the beam axis.

Here we assume that the beam has a uniform round (axisymetric) distribution. Then, Farday's law becomes

$$E_s \Delta s + 2e(\lambda(s + \Delta s - \beta ct) - \lambda(s - \beta ct)) \left(\int_0^a \frac{dr}{r} + \frac{1}{a^2}\int_a^b r dr \right) = \tag{33}$$

$$-\frac{2e\beta}{c}\frac{\partial \lambda(s - \beta ct)}{\partial t} \left(\int_0^a \frac{dr}{r} + \frac{1}{a^2}\int_a^b r dr \right) \Delta s \tag{34}$$

Taking into account that $\lambda(s + \Delta s - \beta ct) - \lambda(s - \beta ct) = \lambda'(s - \beta ct)$ and $\partial\lambda/\partial t = -\beta c \lambda'$, we obtain

$$E_s = -\frac{2e}{\gamma^2}\lambda' \left(\log\left(\frac{b}{a}\right) + \frac{1}{2} \right) \tag{35}$$

Technically speaking we had to calculate the longitudinal field at different radii, using the same procedure, and average it over the distribution of betatron amplitudes.

3.2 Synchrotron tune shift

Synchrotron motion equations with the longitudinal space charge are

$$\frac{dE}{dn} = eV\sin(h\omega_0\tau) + eV_{sc} \tag{36}$$

$$\frac{d\tau}{dn} = T\eta\frac{\Delta E}{\beta^2 E}, \tag{37}$$

where n is the turn number, T is the revolution time equal to $2\pi/\omega$. V is the RF voltage per turn, h is the RF harmonic, and V_{sc} is the voltage per turn due the space charge electric field. Introducing new variables $\phi = \omega_0\tau$ and $\Delta\gamma = \Delta E/E_0$, where E_0 is the particle rest energy (mass), we can rewrite our system as

$$\frac{d\gamma}{dn} = \frac{eV\sin(h\phi)}{E_0} + \frac{eV_{sc}}{E_0} \tag{38}$$

$$\frac{d\phi}{dn} = \frac{2\pi\eta}{\gamma\beta^2 E}\Delta\gamma, \tag{39}$$

The stable phase of oscillations depends on the sign of η. Here we choose $\eta < 0$, meaning that beam energy low (and the space charge is strong). For $\eta < 0$, the stable phase is 0. For small oscillations, $sin(h\phi) \approx h\phi$. Using this and our system of equations, we can write a second order equation for ϕ as

$$\frac{d^2\phi}{dn^2} + \frac{2\pi h|\eta|eV}{\gamma\beta^2 E_0}\phi = -\frac{2\pi|\eta|eV_{sc}}{\gamma\beta^2 E_0} \tag{40}$$

From the last equation, we can see that the unperturbed synchrotron frequency is

$$4\pi^2\nu_{s0}^2 = \frac{2\pi h|\eta|eV}{\gamma\beta^2 E_0} \Rightarrow \tag{41}$$

$$\Rightarrow \nu_{s0} = \sqrt{\frac{h|\eta|eV}{2\pi\gamma\beta^2 E_0}} \tag{42}$$

To simplify calculations, let's assume that the beam consists of bunches with a parabolic profile with the total length equal to $2\hat{z}$. This longitudinal particle density is given by

$$\lambda = \frac{3N}{4\hat{z}^3}(-z^2 + \hat{z}^2), \tag{43}$$

where N is the number of particles per bunch. Thus,

$$\lambda' = -\frac{3N}{2\hat{z}^3}z = \frac{3N}{2\hat{z}^3}R\phi, \tag{44}$$

where R is the average machine radius. Thus, the space charge voltage per turn is

$$V_{sc} = -\frac{6\pi R^2 eN}{\gamma^2\hat{z}^3}\left(\log\left(\frac{b}{a}\right) + \frac{1}{2}\right)\phi \tag{45}$$

Plugging the last equation into (40) we obtain the synchrotron frequency with the space charge

$$\nu_s = \nu_{s0} + \frac{3N\eta r_0 R^2}{\gamma^3\beta^2\hat{z}^3}\left(\log\left(\frac{b}{a}\right) + \frac{1}{2}\right) \tag{46}$$

Thus, the synchrotron tune shift is

$$\delta\nu_s = \frac{3N\eta r_0 R^2}{2\nu_{s0}\gamma^3\beta^2\hat{z}^3}\left(\log\left(\frac{b}{a}\right) + \frac{1}{2}\right) \tag{47}$$

This synchrotron tune shift has the same sign as η. The space charge force is focusing or defocusing depending on whether a machine is operated above or below the transition. One the contrary, the transverse space charge is always defocusing.

3.3 Short wavelength approximation in free space

Equation 35 is valid only if the characteristic length of the charge density variation is much larger than the diameter of the vacuum pipe divided by γ. This condition can be written as

$$\frac{b\lambda'}{\gamma\lambda} \ll 1. \tag{48}$$

The factor of gamma appears from two facts (see also discussion about calculating the longitudinal space charge field below):

- the longitudinal bunch size is gamma times larger in the bunch rest frame
- the longitudinal electric field is the same in the bunch rest frame and the laboratory frame.

When the characteristic length of the charge density variation multiplied by γ is much smaller than the vacuum pipe diameter, the effect of image charges can be neglected. Because any charge density can represented as a combination of sine and cosine waves, it is customary to calculate the electric field induced by a sine wave. This field will be a cosine wave with some coefficient which is frequently referred to as the space charge impedance. The SC impedance is a function of frequency. Then, if the response to a sine wave is known, the field of an arbitrary distribution can be calculated as a convolution of the charge density spectrum with the impedance. Without derivation, the space charge impedance in the free space is given by:

$$|Z(\omega)_{\|}| = \frac{cZ_0}{\omega a^2}\left[1 - \frac{\omega a}{\gamma c} \cdot K_1\left(\frac{\omega a}{\gamma c}\right)\right], \tag{49}$$

(Z. Huang, SLAC-PUB-9788, Z. Huang and T. Shaftan, in Proc. of FEL 2003, also find references to earlier work in their papers) where K_1 is the modified Bessel function of the second order and Z_0 is the impedance of free space.

3.4 Note on space charge field calculations/simulations

Do not just assume that the longitudinal space charge field reduces as $1/\gamma^2$. This assumption works reasonably well only if the bunch is very thin and long, its gamma is high, and it does not not have fast variations of the longitudinal charge density. More accurately, this assumption works only if the characteristic length of density variations in the rest frame is much larger than the diameter of the beam pipe.

In principle, the space charge field can be calculated directly in the laboratory frame for some simple cases. For more complicated cases, it is more convenient to calculate the electro-static problem in the beam rest frame and transform the field in the lab frame. The procedure is approximately as follows:

- Transform the beam charge distribution from the lab frame to the beam rest frame using Lorentz' transformation for the coordinates. The transformation of the charge density causes to bunch to extend in the longitudinal direction by a factor of gamma. Thus, this transformation causes the bunch density to reduce by the same factor of gamma. Additionally, the effect of image charges becomes more pronounced because the bunch is longer and all the field variations are smoother.
- Calculate or simulate the electrostatic field in the rest frame. Analytic calculation of the field with image charges becomes impractically difficult for almost any real bunch distribution. Resolve to simulation codes like POISSON, OPERA/Tosca, etc.
- Transform the field back to the lab frame using transformations for the electro magnetic field. Transforming fields back to the lab frame, transform also the coordinates according to the Lorentz transformations. For example, the longitudinal electric field is transformed according to

  $$E_z(z = z'/\gamma) = E'_z \tag{50}$$

  (Note that we used x for the direction of the beam motion in Section 1)

4 Beam envelope equations with the space charge

4.1 Kapchinsky-Vladimirsky (KV) distribution

$$f(x, x', y, p') = f_0 \delta \left(\frac{A_x}{\epsilon_x} + \frac{A_y}{\epsilon_y} - 1 \right) \tag{51}$$

$$A_x = \gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 \tag{52}$$

$$A_y = \gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2 \tag{53}$$

The main attribute of the KV distribution is that any two-dimensional projection results in a uniform particle density. In $x - y$ plane:

$$f(x, y) = \frac{e\lambda}{\pi ab} H \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right), \tag{54}$$

where $H(u) = 1$ if $u > 0$ and 0 if $u < 0$. This is a uniform distribution within an ellipse with semi-axis a and b. The parameters $\epsilon_{x,y}$ are the horizontal and vertical beam emittances at the edge of the envelopes.

Because the charge density in x-y plane is uniform, the resulting space charge field is linear. If the KV beam is in a channel or a ring that consists of linear field elements, the net force produced by the beam and the external focusing is linear. The quantities Ax and Ay are integrals of motion in the linear field if the motion is stable. Because the distribution depends only on integrals of motion, it does not change in time.

Lorentz force is:

$$\vec{F} = \frac{4e^2\lambda}{\gamma^2} \left[\frac{x\hat{e}_x}{a(a+b)} + \frac{y\hat{e}_y}{b(a+b)} \right] \tag{55}$$

Plugging this into the eqaution of motion, we obtain the equations of motion

$$x'' + \left[K_x - \frac{\xi}{a(a+b)} \right] x = 0 \tag{56}$$

$$y'' + \left[K_y - \frac{\xi}{b(a+b)} \right] y = 0 \tag{57}$$

$$\xi = \frac{4r_0\lambda}{\beta^2\gamma^3} \tag{58}$$

Again, using parameterization $x = a(s)\cos(\phi(s))$ with the phase given by $\phi' = \epsilon/a(s)^2$ we obtain the equations describing the beam envelopes

$$a'' + K_x a = \frac{\epsilon_x^2}{a^3} + \frac{\xi}{(a+b)} \tag{59}$$

$$b'' + K_y b = \frac{\epsilon_y^2}{b^3} + \frac{\xi}{(a+b)} \tag{60}$$

4.2 Equation for the second moments with the space charge

Consider a one dimensional problem with the equations of motion

$$x' = p \tag{61}$$

$$p' = -K_x x + f_x, \tag{62}$$

where f_x is related to the force F_x as $f_x = F_x/pv$. Let the second moments of the beam be designated as $\langle x^2 \rangle$, $\langle xp \rangle$, and $\langle p2 \rangle$, where $\langle\rangle$ means taking an average over the beam distribution. The derivatives of the second moments are

$$\langle x^2 \rangle' = 2\langle xx' \rangle = 2\langle xp \rangle \tag{63}$$

$$\langle xp \rangle' = \langle x'p + xp' \rangle = \langle p^2 \rangle - K_x\langle x^2 \rangle + \langle xf_x \rangle \tag{64}$$

$$\langle p^2 \rangle' = 2\langle pp' \rangle = -2K_x\langle xp \rangle + 2\langle pf_x \rangle. \tag{65}$$

Combining (63 and 64), yields

$$\langle x^2 \rangle'' = 2\langle p^2 \rangle - 2K_x\langle x^2 \rangle + 2\langle xf_x \rangle \tag{66}$$

If we define the rms emittance as

$$\epsilon_x^2 = \langle x^2 \rangle\langle p^2 \rangle - \langle xp \rangle^2, \tag{67}$$

we can experss $\langle p^2 \rangle$ in terms of emittance as

$$\langle p^2 \rangle = \frac{\epsilon_x^2}{\langle x^2 \rangle} + \frac{\langle xp \rangle^2}{\langle x^2 \rangle} = \frac{\epsilon_x^2}{\langle x^2 \rangle} + \frac{\langle x^2 \rangle'^2}{4\langle x^2 \rangle} \tag{68}$$

Plugging the last equation into (66) and rewriting everything via the rms beam size $\sigma_x = \sqrt{(}\langle x^2 \rangle)$ we obtain

$$\sigma_x'' + K_x\sigma_x = \frac{\epsilon_x^2}{\sigma_x^3} + \frac{\langle xf_x \rangle}{\sigma_x} \tag{69}$$

Similarly, for the vertical plane,

$$\sigma_y'' + K_y\sigma_y = \frac{\epsilon_y^2}{\sigma_y^3} + \frac{\langle yf_y \rangle}{\sigma_y} \tag{70}$$

In general, the forces F_x and F_y, are nonlinear in x and y, and the quantities $\langle xf_x \rangle$, etc, involve moments higher than the second moments.