USPAS, Winter 2008

Accelerator Physics

V.N.Litvinenko, E.Pozdeyev, T.Satogata

Selected solutions for Home-works and Midterm Exam

Accelerator Physics: Homework 1 Solutions

1 Basic relativity

(a) (3 points) In one dimension the work done by a force F acting through a distance dl is $dE = F\,dl$. Show directly that increasing the Lorentz factor of a particle of mass m by $\Delta\gamma$ changes the particle's energy by

$$\Delta E = \Delta\gamma\, mc^2 \tag{1.1}$$

where the rest energy of the particle is $E_0 = mc^2$. From this it follows that $E = \gamma E_0$. Use this to show that

$$E^2 = p^2c^2 + m^2c^4 \tag{1.2}$$

Solution: Restricting ourselves to one dimension without loss of generality:

$$\frac{\mathrm{d}E}{\mathrm{d}l} = F = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{1.3}$$

where

$$p = mc\beta\gamma \qquad \gamma = (1-\beta^2)^{-1/2} \qquad v = \frac{\mathrm{d}l}{\mathrm{d}t} = \beta c \tag{1.4}$$

Then we can write

$$\frac{\mathrm{d}E}{\mathrm{d}\gamma} = \frac{\mathrm{d}E}{\mathrm{d}l}\cdot\frac{\mathrm{d}l}{\mathrm{d}\gamma} = \frac{\mathrm{d}p}{\mathrm{d}t}\cdot\frac{\mathrm{d}l}{\mathrm{d}\gamma} = \beta c\frac{\mathrm{d}p}{\mathrm{d}\gamma} = mc^2\cdot\beta\frac{\mathrm{d}(\beta\gamma)}{\mathrm{d}\gamma} \tag{1.5}$$

and using

$$\frac{\mathrm{d}\beta}{\mathrm{d}\gamma} = \frac{1}{\beta\gamma^3} \tag{1.6}$$

then

$$\frac{\mathrm{d}E}{\mathrm{d}\gamma} = mc^2\left(\beta^2 + \frac{1}{\gamma^2}\right) = mc^2 \tag{1.7}$$

From this $E = \gamma E_0$ follows from integration since mc^2 is constant. To demonstrate the latter part of this problem,

$$E = \gamma E_0 \rightarrow E^2 = \gamma^2 E_0^2 = \gamma^2 m^2 c^4$$

But

$$\gamma^2 = \frac{1}{1-\beta^2} = \frac{1-\beta^2+\beta^2}{1-\beta^2} = 1 + \frac{\beta^2}{1-\beta^2} = 1 + \beta^2\gamma^2$$

We also know that relativistic momentum is $p = \beta\gamma mc$, so we have

$$E^2 = (1+\beta^2\gamma^2)m^2c^4 = m^2c^4 + p^2c^2$$

(b) (3 points) Show that an infinitesimal increase in energy dE is related to the infinitesimal increase in momentum dp by

$$\frac{dE}{E} = \beta^2 \frac{dp}{p} \tag{1.8}$$

where $\beta \equiv v/c$.

Solution: Using that the differential of the Lorentz invariant, $\mathrm{d}(E^2 - p^2c^2)$, is zero,

$$E\mathrm{d}E = c^2 p\mathrm{d}p$$

Dividing this equation by E^2 and using $E = cp/\beta$ yields:

$$\frac{\mathrm{d}E}{E} = \beta^2 \frac{\mathrm{d}p}{p}$$

(c) (4 points) A unit charge particle of momentum p travels through a constant magnetic field B, and is bent in a circular arc of radius ρ. Show that

$$B\rho\ [\mathrm{T-m}] = 3.3357\ p\ [\mathrm{GeV/c}] \tag{1.9}$$

Solution: Here the bending force from the magnetic field must provide the centripetal force to keep the particle moving in a circle, so

$$F_{\mathrm{B}} = qvB = F_{\mathrm{cent}} = mv^2/\rho \rightarrow qB\rho = mv = p \rightarrow B\rho = p/q$$

Here we have to be careful about units. SI units for $B\rho$ in [T-m] are [kg m s^{-2} A^{-1}]; these match the units for p/q if we express p/q in SI units. However, we are expressing p/q in odd units of [GeV/c]. The unit charge cancels, and we need to divide by c in SI units to find

$$B\rho[\mathrm{T-m}] = 3.3357 \times 10^{-9} p[\mathrm{eV/c}] = 3.336 p[\mathrm{GeV/c}]$$

2 RHIC energy and current

Gold ions $^{197}\mathrm{Au}^{+77}$ (A=197, Z=79) are injected into the Brookhaven Alternating Gradient Synchrotron (AGS) with a kinetic energy of 100 MeV/u (i. e. MeV per nucleon). **(NOTE: This was accidentally typo'd in the homework handout as 100 GeV/u.)** The AGS accelerates protons up to a kinetic energy of 22.9 GeV for injection into Relativistic Heavy Ion Collider (RHIC). The circumference of the AGS is 807.1 m, and the rest mass of a gold ($^{197}\mathrm{Au}^{+77}$) ion is 183.434 GeV/c^2.

(a) (4 points) What is the velocity of the injected gold ions?
Solution:

$$\begin{aligned}
\gamma_{\mathrm{inj}} &= 1 + \frac{K}{Mc^2} = 1 + \frac{197 * 0.1\ \mathrm{GeV}}{183.434\ \mathrm{GeV}} = 1.1704 \\
\beta_{\mathrm{inj}} &= \sqrt{1 - \gamma_{\mathrm{inj}}^{-2}} = 0.4296 \\
v_{\mathrm{inj}} &= \beta_{\mathrm{inj}} c = 1.2879 \times 10^8\ \frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

(b) (3 points) What is the corresponding kinetic energy for $^{197}\mathrm{Au}^{+77}$ ions extracted from the AGS for RHIC?
Solution:

Assuming the same extraction field in the AGS as for protons, i.e., the same rigidity:

$$
\begin{aligned}
p_{\mathrm{p}} &= \gamma\beta m_{\mathrm{p}}c = \sqrt{\left(1+\frac{K_{\mathrm{p}}}{m_{\mathrm{p}}c^2}\right)^2 - 1}\; m_{\mathrm{p}}c = 23.820 \text{ GeV/c} \\
&\quad (=79.455 \text{ T m}) \\
p_{\mathrm{gold}} &= 77 * p_{\mathrm{p}} = 1.8341 \text{ TeV/c} \\
(\gamma\beta)_{\mathrm{gold}} &= \frac{p_{\mathrm{gold}}}{Mc} = 9.9988 \\
K &= (\gamma-1)Mc^2 = \left[\sqrt{\left(\frac{p_{\mathrm{gold}}}{Mc}\right)^2+1} - 1\right] Mc^2 \\
&= 1.6598 \text{ TeV} \quad (\text{or } 8.4256 \text{ GeV/u})
\end{aligned}
$$

(c) (3 points) Why does the beam current increase although the circulating charge stays constant during acceleration?
Solution: Current is a measure of rate of charge passing a point per unit time, I=dQ/dt. As the particles accelerate, they pass a reference point more quickly, and the total beam current goes up.

3 Basic collision kinematics

(a) (3 points) Show that the total energy for a head-on collision of two particles, each with center of mass energy $\gamma_{\mathrm{cm}}mc^2$, is equal to the total energy of a fixed-target collision, where one particle is at rest and the other has energy $\gamma_{\mathrm{fixed}}mc^2$, where

$$\gamma_{\mathrm{fixed}} = 2\gamma_{\mathrm{cm}}^2 - 1 \tag{3.1}$$

Solution: It is most convenient to do this type of problem using Mandelstam variables, in particular the Mandelstam variable s:

$$s \equiv (p_1 + p_2)^2 \tag{3.2}$$

where $p_{1,2}$ are the Lorentz four-momenta of the interacting particles:

$$p \equiv (-E, p_x, p_y, p_z)^{\mathrm{T}} = (-\gamma mc^2, p_x, p_y, p_z)^{\mathrm{T}} \tag{3.3}$$

and the total center of mass energy is given by $\sqrt{s}$. It is easy to see that in the head-on collision scenario, the momenta are equal and opposite, so the total energy $\sqrt{s} = 2\gamma_{\mathrm{cm}}mc^2$. In the case where one particle carries all the momentum, s is somewhat more complicated:

$$s = p_1^2 + 2p_1p_2 + p_2^2 = m_1^2c^4 + 2p_1p_2 + m_2^2c^4 \tag{3.4}$$

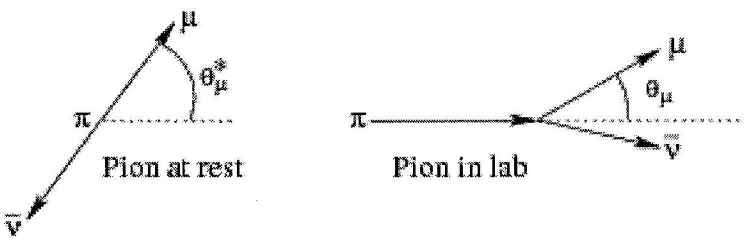
All momentum components of the rest particle are zero, so $p_1p_2 = \gamma_{\mathrm{fixed}}mc^2$ and

$$s = mc^2(2mc^2 + 2\gamma_{\mathrm{fixed}}mc^2) = 2m^2c^4(1+\gamma_{\mathrm{fixed}}) \tag{3.5}$$

Comparison to the colliding beam $s = 4\gamma_{\mathrm{cm}}^2m^2c^4$ immediately gives $\gamma_{\mathrm{fixed}} = 2\gamma_{\mathrm{cm}}^2 - 1$. The Mandelstam variable s is used so often at accelerator facilities that we commonly refer to the center of mass energy available in collisions at each facility by saying, for example, "$\sqrt{s}$=250 GeV", which is the center of mass energy for top-energy polarized proton collisions at RHIC.

Consider a charged pion decaying into a muon plus an antineutrino:

$$\pi^- \to \mu^- + \bar{\nu}_\mu \tag{3.6}$$



Use $M_{\pi^\pm} = 140\text{MeV/c}^2$, $m_\mu = 106\text{MeV/c}^2$, and $m_{\bar{\nu}} = 0$.

(b) (3 points) In the rest system of the pion, what are the energies and momenta of the muon and antineutrino?

Solution: This is simply cranking through the kinematics before plugging in numbers:

$$\begin{aligned}
0 &= \vec{p}^*_\mu + \vec{p}^*_\nu \\
m_\pi c^2 &= E^*_\mu + E^*_\nu \\
E^*_\nu &= p^*_\nu c = p^*_\mu c = \sqrt{(E^*_\mu)^2 - (m^*_\mu c^2)^2} \\
(m_\pi c^2 - E^*_\mu)^2 &= E^{*2}_\mu - (m^*_\mu c^2)^2 \\
E^{*2}_\mu - 2E^*_\mu m_\pi c^2 + (m_\pi c^2)^2 &= E^{*2}_\mu - (m^*_\mu c^2)^2 \\
2E^*_\mu m_\pi c^2 &= (m^2_\pi + m^2_\mu)c^4 \\
E^*_\mu &= \frac{m^2_\pi + m^2_\mu}{2m_\pi} c^2 = 110.1 \text{ MeV} \\
p^*_\nu &= -p^*_\mu = E^*_\nu = m_\pi c - E^*_\mu = 29.87 \text{ MeV/c}
\end{aligned}$$

(c) (4 points) For a moving pion with total energy $U_\pi = \gamma M_\pi c^2$ find an expression for the direction, θ_μ of the muon relative to the pion in the lab in terms of the angle θ^*_μ in the in the pion's rest system.

Solution: Assume the pion is moving along the z-axis, and that the collision takes place in the xz-plane. (A rotation about the z-axis will not make a difference.) In the pion's rest system, the muon will have the proper 4-velocity:

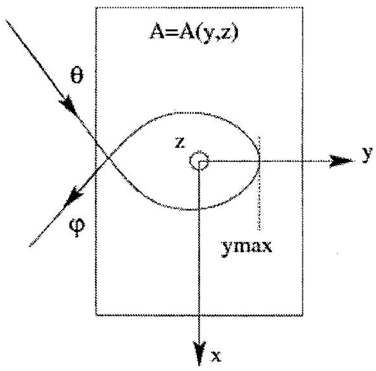
$$\mathbf{u}^* = \frac{\mathbf{p}^*_\pi}{m_\pi c} = \begin{pmatrix} E^*_\mu/c \\ p^*_\mu \sin\theta^*_\mu \\ 0 \\ p^*_\mu \cos\theta^*_\mu \end{pmatrix} \frac{1}{m_\pi c}$$

Boosting to the lab system we find

$$\begin{aligned}
\mathbf{u} &= \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E_\mu^*/c \\ p_\mu^* \sin\theta_\mu^* \\ 0 \\ p_\mu^* \cos\theta_\mu^* \end{pmatrix} \frac{1}{m_\pi c} \\
\text{with } \beta &= \sqrt{1-\gamma^{-2}} \\
u_x &= \frac{p_\mu^* \sin\theta_\mu^*}{m_\pi c} \\
u_z &= \gamma \frac{\beta E_\mu^* + p_\mu^* c \cos\theta_\mu^*}{m_\pi c^2} \\
\tan\theta_\mu &= \frac{u_x}{u_z} = \frac{1}{\gamma} \frac{p_\mu^* c \sin\theta_\mu^*}{\beta E_\mu^* + p_\mu^* c \cos\theta_\mu^*} \\
&= \frac{1}{\gamma} \frac{\sin\theta_\mu^*}{\beta(E_\mu^*/p_\mu^* c) + \cos\theta_\mu^*} \\
\theta_\mu &= \tan^{-1}\left(\frac{1}{\gamma} \frac{\sin\theta_\mu^*}{\beta(E_\mu^*/p_\mu^* c) + \cos\theta_\mu^*}\right) = \tan^{-1}\left(\frac{1}{\gamma} \frac{\sin\theta_\mu^*}{3.6868\beta + \cos\theta_\mu^*}\right)
\end{aligned}$$

You can also take other ratios of u_x, u_z, and u to find other trig functions of θ_μ, but the approach to find the tangent (or cotangent) is easiest since the u_x and u_z components immediately fall out of the Lorentz transformation.

4 Magnetic Mirror



(a) (2 points) An electron moves through a magnetic field with vector potential $\vec{A} = \vec{A}(y,z)$. Find an additional invariant of motion from the independence of $\vec{A}$ from x. Write an explicit expression for p_x using this invariant.

Solution: The Hamiltonian for particle motion in an a vector potential $\vec{A}(y,z)$ is

$$H = \sqrt{m^2c^2 + \left(\vec{p} - \frac{e}{c}\vec{A}(y,z)\right)^2}$$

Since this does not depend on x, the momentum p_x is invariant:

$$\dot{p}_x = \frac{-\partial H}{\partial x} = 0 \quad \rightarrow \quad p_x = \text{constant}$$

(b) (2 points) Consider a magnet with mid-plane symmetry, $\vec{H} = \hat{e}_z H(y)$ at $z = 0$, shown above, with $\vec{A} = \vec{A}(y, z)$ inside the magnet and $\vec{A} = 0$ outside the magnet. Consider an electron entering the magnet in the midplane $z = 0$ with mechanical momentum

$$\vec{p} = \hat{e}_x p_x + \hat{e}_y p_y = p(\hat{e}_x \cos\theta + \hat{e}_y \sin\theta) \tag{4.1}$$

which enters the magnet, turns around in the magnet, and comes back out.
Solution: This was a free two points; there was accidentally no question here!

(c) (2 points) Show that the trajectory of the electron remains in the $z = 0$ plane.
Solution: The Lorentz force is

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) = \frac{q\mu_0 \vec{v}}{c} \times \vec{H}$$

since $\vec{E} = 0$ here and $\vec{B} = \mu_0 \vec{H}$. The acceleration is thus always perpendicular to the magnetic field $\vec{H}$, and so is always in the $z = 0$ plane. Since the particle's initial velocity includes no $\hat{z}$ component, the particle always remains in the $z = 0$ plane.

(d) (2 points) Find the equation of the angle φ of the electron's exit trajectory relative to $\hat{e}_x$ direction.
Solution: It's almost trivial to see that $\varphi = \theta$ from symmetry, since $\vec{A}$ has no x dependence and the system is therefore reflection symmetric through the (y, z) plane. This is even easier to see knowing that Lorentz force from a pure magnetic field is conservative, since it always acts perpendicular to $\vec{p}$ and therefore does not change the particle's speed $|\vec{p}|$.

(e) (2 points) Find the equation defining the penetration depth $y_{\max}$ of the electron in the magnet in terms of $A(y, z = 0)$. You don't have to solve this equation generally, but do write down an equation that you could solve numerically for $y_{\max}$.
Solution: The Hamiltonian is conserved, so the Hamiltonian outside the magnet is equal to the Hamiltonian at $y_{\max}$. Note that at $y_{\max}$, the momentum only has one component, $\vec{p}(y_{\max}) = p_x \hat{x}$, so we have

$$p^2 = \left(\vec{p}(y_{\max}) - \frac{e}{c}\vec{A}(y_{\max}, z = 0)\right)^2$$

This can be solved for $y_{\max}$ if a form for $\vec{A}(y, z)$ is explicitly known.

5 The Lorentz Group

(a) (4 points) For the Lorentz boost

$$\vec{K} \equiv \hat{e}_x \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \hat{e}_y \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \hat{e}_z \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{5.1}$$

and rotation matrix

$$\vec{S} \equiv \hat{e}_x \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \hat{e}_y \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + \hat{e}_z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.2}$$

given in class, show that

$$(\vec{\epsilon}\,\vec{S})^3 = -\vec{\epsilon}\,\vec{S} \quad (\vec{\epsilon}\,\vec{K})^3 = \vec{\epsilon}\,\vec{K} \quad \text{for } \forall \vec{\epsilon} = \vec{\epsilon}^\star \quad \text{where } |\vec{\epsilon}| = 1 \tag{5.3}$$

and, more generally,

$$(\vec{a}\,\vec{S})^3 = -\vec{a}\,\vec{S}|\vec{a}|^2 \quad (\vec{a}\,\vec{K})^3 = \vec{a}\,\vec{K}|\vec{a}|^2 \quad \text{for } \forall \vec{a} = \vec{a}^\star \tag{5.4}$$

Solution: A standard solution is to write everything out and multiply, which certainly works, especially when assisted by Mathematica and the knowledge that the product of two antisymmetric matrices is symmetric, and the product of an antisymmetric matrix and a symmetric matrix is antisymmetric. Then, for example,

$$\vec{a}\vec{S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & a_2 \\ 0 & a_1 & 0 & -a_3 \\ 0 & -a_2 & a_3 & 0 \end{pmatrix} \rightarrow (\vec{a}\vec{S})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -a_1^2 - a_2^2 & a_2a_3 & a_1a_3 \\ 0 & a_2a_3 & -a_1^2 - a_3^2 & a_1a_2 \\ 0 & a_1a_3 & a_1a_2 & -a_2^2 - a_3^2 \end{pmatrix}$$

$$(\vec{a}\vec{S})^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_1(a_1^2+a_2^2+a_3^2) & -a_2(a_1^2+a_2^2+a_3^2) \\ 0 & -a_1(a_1^2+a_2^2+a_3^2) & 0 & a_3(a_1^2+a_2^2+a_3^2) \\ 0 & a_2(a_1^2+a_2^2+a_3^2) & -a_3(a_1^2+a_2^2+a_3^2) & 0 \end{pmatrix}$$
$$= -\vec{a}\,\vec{S}|\vec{a}|^2$$

An alternative solution: Rotate the coordinate system to an appropriate set of coordinates, for example:

$$M = \vec{a}\vec{S} = |\vec{a}|R^{-1}\,S_1\,R$$

This effectively aligns $\vec{a}$ along the S_1 axis. Diagonalize S_1: $S_1 = A^{-1}D_1A$ where D_1 is diagonal. Now $D_1^3 = -D_1$ (eigenvalues of S_1 are -1), so

$$M^3 = |\vec{a}|^3\left(B^{-1}D_1B\right)^3 = |\vec{a}|^3B^{-1}D_1^3B = -|\vec{a}|^3B^{-1}D_1B = -|\vec{a}|^2M$$

where $B = A\cdot R$. A similar derivation where $D_1^3 = D_1$ follows for the boost.

(b) (4 points) Use these results to show that

$$e^{\vec{\omega}\vec{S}} = I + \frac{\vec{\omega}\vec{S}}{|\vec{\omega}|}\sin|\vec{\omega}| - \frac{\left(\vec{\omega}\vec{S}\right)^2}{|\vec{\omega}|^2}\left(\cos|\vec{\omega}| - 1\right) \tag{5.5}$$

and

$$e^{\vec{\beta}\vec{K}} = I + \frac{\vec{\beta}\vec{K}}{|\vec{\beta}|}\sinh|\vec{\beta}| - \frac{\left(\vec{\beta}\vec{K}\right)^2}{|\vec{\beta}|^2}\left(\cosh|\vec{\beta}| - 1\right) \tag{5.6}$$

Solution:

(c) (2 points) Are $\vec{S}$ and/or $\vec{K}$ symplectic?
Solution: Both $\vec{S}$ and $\vec{K}$ have zero eigenvalues, so neither can have a unit determinant, and neither can be symplectic. No multiplicaion required!

Accelerator Physics: Homework 2 Solutions

1 Gaussian luminosity (Lee 1.7(b), p. 27)

The total counting rate of a physical interaction at a single collision point is given by $R = \mathcal{L}\sigma$, where σ is the cross-section of the interaction and the luminosity $\mathcal{L}$ (in units of cm^{-2} s^{-1}) is a measure of the interaction probability per unit area and time. When two accelerator bunches with relativistic velocities β collide head-on,

$$\mathcal{L} = 2fN_1N_2 \int \rho_1(x, y, s_1)\, \rho_2(x, y, s_2)\, dx\, dy\, ds\, d(\beta ct) \tag{1.1}$$

where $s_1 = s + \beta ct$, $s_2 = s - \beta ct$, f is the collision frequency, N_1 and N_2 are the number of particles in each bunch, and ρ_1 and ρ_2 are the normalized distribution functions for both bunches.

(a) (5 points) Using a Gaussian bunch distribution,

$$\rho(x, y, s) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_s} \exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{s^2}{2\sigma_s^2}\right] \tag{1.2}$$

where σ_x, σ_y, and σ_s are the rms horizontal, vertical, and longitudinal beam sizes, show that the luminosity for two bunches with identical distributions is

$$\mathcal{L} = \frac{fN_1N_2}{4\pi\sigma_x\sigma_y} \tag{1.3}$$

Solution: Direct substitution of the Gaussian bunch distributions into the luminosity integral gives

$$L = \frac{2fN_1N_2}{(2\pi)^3\sigma_x^2\sigma_y^2\sigma_s^2} \int \exp\left[-\frac{x^2}{\sigma_x^2} - \frac{y^2}{\sigma_y^2} - \frac{(s_1^2 + s_2^2)}{2\sigma_s^2}\right] dx\, dy\, ds\, d(\beta ct) \tag{1.4}$$

or, using $s_1 = s + \beta ct$ and $s_2 = s - \beta ct$,

$$L = \frac{fN_1N_2}{4\pi^3\sigma_x^2\sigma_y^2\sigma_s^2} \int \exp\left[-\frac{x^2}{\sigma_x^2}\right] dx \int \exp\left[-\frac{y^2}{\sigma_y^2}\right] dy \int \exp\left[-\frac{s^2}{\sigma_s^2}\right] ds \int \exp\left[-\frac{(\beta ct)^2}{\sigma_s^2}\right] d(\beta ct) \tag{1.5}$$

Each integral is a straightforward Gaussian integral of the form

$$\int \exp\left[-\frac{x^2}{\sigma_x^2}\right] dx = \sigma_x\sqrt{\pi} \tag{1.6}$$

and substitution gives the desired result

$$L = \frac{fN_1N2}{4\pi\sigma_x\sigma_y} \tag{1.7}$$

If you don't feel like looking up the integral of the Gaussian given in (1.6), it is easy to derive with an old trick. Call this integral I. Then

$$I^2 = \int\int \exp\left[\frac{-x^2 - y^2}{\sigma^2}\right] dx\, dy = \int\int \exp\left[\frac{-\rho^2}{\sigma^2}\right] \rho\, d\rho\, d\theta \tag{1.8}$$

where we have converted to polar coordinates. Taking $X \equiv \rho^2/\sigma^2$ and $dX = 2\rho\, d\rho/\sigma^2$, and integrating the angle coordinate over 2π, gives $I^2 = \sigma^2\pi$ and $I = \sigma\sqrt{\pi}$.

(b) (5 points) Show that if two bunches collide with a vertical offset of Δy, the luminosity is reduced by a factor of $\exp(-\Delta y^2/4\sigma_y^2)$.
Solution: Substituting $x \to x + b/2$ in ρ_1 and $x \to x - b/2$ in ρ_2 changes the dx Gaussian integral:

$$\begin{aligned} \int \exp\left[-\frac{x^2}{\sigma_x^2}\right] dx \;&\to\; \int \exp\left[-\frac{(x+b/2)^2 + (x-b/2)^2}{2\sigma_x^2}\right] dx \\ &\to\; \exp\left[-\frac{b^2}{4\sigma_x^2}\right] \int \exp\left[-\frac{x^2}{\sigma_x^2}\right] dx \end{aligned} \tag{1.9}$$

Hence luminosity is reduced by a factor of $\exp[-b^2/4\sigma_x^2]$. This quadratic dependence of luminosity on offset is used during "vernier scans" of luminosity vs. transverse beam position to measure the transverse beam sizes σ_x and σ_y.

2 EM similarity to Lorentz group

Consider an invariant equation of motion of a charged particle in a constant electromagnetic field:

$$mc\frac{du^i}{ds} = \frac{e}{c}F^i_k \cdot u^k \qquad \frac{d}{ds}[u] = D[u] \qquad [D] = \frac{e}{mc^2}F^i_k \tag{2.1}$$

where $[u]$ is a 4-vector, and which has the general solution

$$[u] = e^{Ds}[u_0] \tag{2.2}$$

(a) (4 points) Write matrix $[D]$. Identify the similarity of $[D]$ with the generator of the Lorentz group, and find the analogy between boost, rotations, and components of the electromagnetic field.
Solution: F^{ij} is given in explicit form in the lecture notes. The explicit form for $[D]$ is then:

$$[D] = \frac{e}{mc^2}F^{ij}g_{jk} = \frac{e}{mc^2}\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{2.3}$$

since we are working in a Euclidian metric $(1, -1, -1, -1)$ metric (Appendix equation A-21). The electric field components are equivalent to a boost, while the magnetic field components are equivalent to a rotation. This agrees with intuition, since electric field forces are along the direction of motion, while magnetic field forces are perpendicular to the direction of motion.

(b) (6 points) Write the explicit expression for $M = e^{Ds}$ in the case of a pure constant electric field ($B = 0$) and pure constant magnetic field ($E = 0$).

Solution: The case of a pure constant electric field is just like a boost, with the relation (apologies for the strange mixed notation):

$$[D] = \frac{e}{mc^2}\vec{E}\vec{K} \qquad [D] = \frac{e}{mc^2}\vec{B}\vec{S} \tag{2.4}$$

for the $B = 0$ and $E = 0$ cases respectively. Then

$$M(\mathrm{B}=0) = e^D = I + \frac{\vec{E}\vec{K}}{|\vec{E}|}\sinh\frac{e}{mc^2}|\vec{E}| - \frac{\left(\vec{E}\vec{K}\right)^2}{|\vec{E}|^2}\left(\cosh\frac{e}{mc^2}|\vec{E}| - 1\right) \tag{2.5}$$

Similarly for the pure magnetic field case, $[D] = (e/mc^2)\vec{B}\vec{S}$ and

$$M(\mathrm{E}=0) = e^D = I + \frac{\vec{B}\vec{S}}{|\vec{B}|}\sin\frac{e}{mc^2}|\vec{B}| + \frac{\left(\vec{B}\vec{S}\right)^2}{|\vec{B}|^2}\left(\cos\frac{e}{mc^2}|\vec{B}| - 1\right) \tag{2.6}$$

3 Cos-theta magnet

(NEED)(10 points) Show that current distributed in a thin cylindrical shell with a strength

$$I(\theta) = \frac{I_0}{n\pi}\cos(n\theta) \tag{3.1}$$

will produce a pure $2n$-multipole distribution inside the cylinder.
Solution: One can rescale $I_0' \equiv I_0/(n\pi)$, so this problem is equivalent to proving that

$$I(\theta) = I_0'\cos(n\theta) \tag{3.2}$$

produces a pure $2n$-multipole distribution inside the cylinder. In class we proved this for $n = 1$, where a $\cos(\theta)$ dependence of current gave a pure 2-multipole (dipole) field; there is no higher-order field component. One can extend that analysis, writing the general field from a single wire perpendicular to the complex plane as

$$B_y(x,z) + jB_x(x,y) = \frac{\mu_0 I}{2\pi(Z - Z_0)} \tag{3.3}$$

where $Z \equiv x + iy$ are coordinates in the complex plane and Z_0 is the coordinate of the wire. Integration of these current sources over the circumference of the cylinder for the given distribution then gives the desired result.

An elegant proof using a result called the Beth current sheet theorem was demonstrated by Richard Beth at Brookhaven in the late 1960s when considering how to build compact superconducting magnets as discussed in class. This result is shown in equations (8) and (22-24).

4 Quadrupole gradient, inductance (Lee 1.12, p. 29)

From Maxwell, $\nabla \times \vec{B} = 0$ in a current-free region, and the magnetic field can be derived from a magnetic potential Φ_m with $\vec{B} = -\nabla\Phi_m$.

(a) (2 points) For a quadrupole field with $B_z = Kx$, $B_x = Kz$, show that the magnetic potential is $\Phi_m = -Kxz + c$ where c is a constant. We can choose a gauge where $c = 0$.

Solution: The magnetic potential equation gives $B_x = -\partial\Phi_m/\partial x$ and $B_z = -\partial\Phi_m/\partial z$. We have been given $B_z = Kx$ and $B_x = Kz$ so we can integrate each to immediately find that

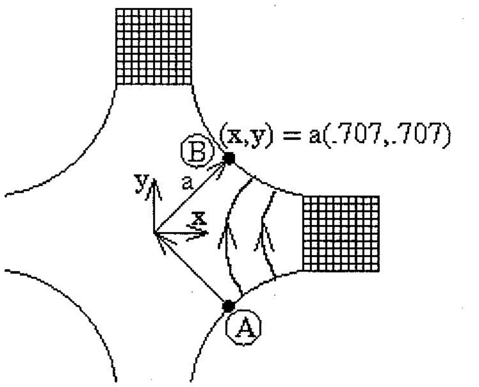
$$\Phi = -K\,xz + c \tag{4.1}$$

so the shape of an equipotential surface is a hyperbola.

(b) (5 points) Equipotential curves are therefore xz =constant. The pole shape of a quadrupole is therefore a hyperbolic curve described by $xz = a^2/2$ where a is the radius of the quadrupole. The magnitude of the field at the surface of the pole is $B_{\text{pole tip}} = Ka$. To avoid magnetic field saturation in the (typically iron) pole tip, the pole tip field in a quadrupole is normally designed to be less than 0.9 Tesla, and the achievable gradient is $K = B_{\text{pole tip}}/a$. Show that the gradient field is

$$K = 2\mu_0 NI/a^2 \tag{4.2}$$

where NI is the number of ampere-turns per pole.



Solution:

Apply Ampere's law, but now recognize that

$$\int_A^B \vec{B}d\vec{s} = \Phi_B - \Phi_A \tag{4.3}$$

if the integral is in the gap **OR** in the iron. Thus

$$\oint \vec{H}d\vec{s} = \oint \frac{\vec{B}d\vec{s}}{\mu_0\mu_r} = -NI = (\Phi_B - \Phi_A)/\mu_0 \tag{4.4}$$

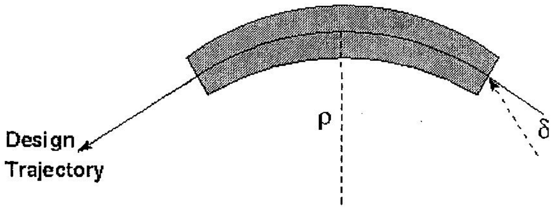
since I points down in the right side conductor in the figure above, $\mu_r \approx \infty$ in the iron and $\mu_r = 1$ in the gap. Using the result from the previous section then

$$\Phi_B = -\Phi_A = -K\,\frac{a^2}{2} \tag{4.5}$$

then it follows that

$$K = \frac{2\mu_0 NI}{a^2} \tag{4.6}$$

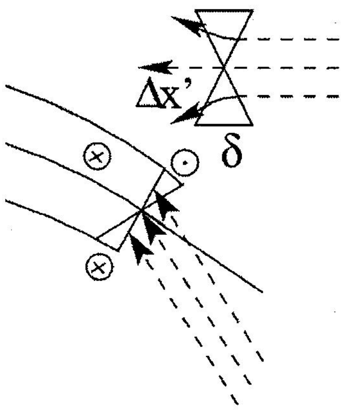
5 Dipole edge focusing (Lee 2.2.2, p. 73)



(10 points) Sector dipoles are bent so the end faces of the magnet are perpendicular to the design particle entry and exit angles. When a particle enters a sector dipole of bending radius ρ at an angle δ with respect to the design trajectory, it experiences some focusing. This phenomenon is usually referred to as edge focusing. Here we use the convention that $\delta > 0$ if the particle is closer to the center of the bending radius. Show that the transport matrices through the dipole for horizontal and vertical motion of the particle are

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{\tan\delta}{\rho} & 1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 1 & 0 \\ -\frac{\tan\delta}{\rho} & 1 \end{pmatrix} \tag{5.1}$$

The edge effect with $\delta > 0$ produces horizontal defocusing and vertical focusing.



Solution: The above figure shows the B field directions and the effect of the edge focusing in the horizontal plane. The difference between the standard reference trajectory and a reference trajectory with incident angle δ is that particles with position $-x$ relative to the reference orbit enter the magnet earlier, and so see more bending; particles with position $+x$ relative to the reference orbit enter the magnet later, and so see less bending. The overall effect is to bend particles with various offsets away from each other, or defocus.

To calculate $\Delta x'$, we can integrate the extra field B_y over the extra distance $x\tan\delta$ to find

$$\Delta x' = \frac{qB_y}{p}\tan\delta\, x = \frac{\tan\delta}{\rho}\, x \tag{5.2}$$

In the vertical plane, the naive conclusion is that there is no effect, but Maxwell's equations require that there is a fringe field that decays beyond the edge of the magnet, and this fringe field changes strength as one moves away from the magnet opening. Using a similar integral,

$$\Delta y' = \frac{q}{p}\int_{\text{fringe}} B_x\, ds \tag{5.3}$$

and assuming the fringe field decays linearly

$$B_y = B_0\,(1 - s/l) \tag{5.4}$$

for $0 < s < l$, Maxwell's equations give $B_x = -\frac{B_0 \sin\delta}{l}\, y$. This can be integrated over the path length for a particle with a vertically offset reference trajectory in the fringe field to find:

$$\Delta y' = \frac{q}{p}\int_0^{\frac{1}{\cos\delta}} \left(-\frac{b_o \sin\delta}{l}\, y\right)\, ds = -\left(\frac{\tan\delta}{\rho}\right)\, y \tag{5.5}$$

Accelerator Physics: Homework 5

Due date: Tuesday, January 22, 2008

1. Coupling non-linear resonance

Consider an uncoupled linear motion in a storage ring, parameterized by

$$x = \sqrt{2I_x}\, w_x(s)\cos\left(\psi_x(s) + \varphi_x\right); \quad y = \sqrt{2I_y}\, w_y(s)\cos\left(\psi_y(s) + \varphi_y\right), \qquad (1)$$

in the presence of additional non-linear term in the Hamiltonian

$$H_{NL} = \alpha(s) x^n y^m . \qquad (2)$$

All coefficients above are periodical with the ring circumference, *C,* except the betatron phases

$$\psi_{x,y}(s + C) = \psi_{x,y}(s) + 2\pi Q_{x,y} .$$

Here: n, m, k, l are integer numbers.

(a) (5 points) write slow equation of motion for the action-angle variables;

(b) (5 points) consider resonant conditions $nQ_x + mQ_y = k + \delta Q; \quad |\delta Q| << 1$, i.e. a sum resonance, and find the expression for the resonant term in the Hamiltonian and the slow equations of motion;

(c) (5 points) consider resonant conditions $nQ_x - mQ_y = l + \delta Q; \quad |\delta Q| << 1$ i.e. a difference resonance, and find the expression for the resonant term in the Hamiltonian and the slow equations of motion;

(d) (10 points) show that in the resonance approximation $|\delta Q| << 1$, we have additional invariants of motion: $nI_y - mI_x = inv$ for the sum resonance and $nI_y + mI_x = inv$ for the difference resonance. Derive your conclusions on what resonance can be more dangerous from a perspective of continuous growth of the amplitudes (i.e. possibility to loose a beam at the walls of vacuum chamber)?

(e) (5 points) qualitatively answer the question if this term in the Hamiltonian can drive other $NQ_x \pm MQ_y = K + \delta Q; \quad N \neq n; M \neq m$ in the first order of perturbation theory?

Solution:

(a) after transition to the phase-action variables, we have only non-linear part of the Hamiltonian (2) is remaining, which we should express through new variables

$$H_{NL} = \alpha(s)x^n y^m = \alpha(s)\sqrt{2I_x}^n w_x{}^n(s)\cos^n\left(\psi_x(s)+\varphi_x\right)\sqrt{2I_y}^m w_y{}^m(s)\cos^m\left(\psi_y(s)+\varphi_y\right),$$

$$H_{NL} = \rho(s)I_x{}^{n/2}I_y{}^{m/2}\cos^n(\phi_x)\cos^m(\phi_y);\ \rho(s)=\alpha(s)2^{(n+m)/2}w_x{}^n(s)w_y{}^m(s);\ \rho(s+C)=\rho(s);$$

$$\phi_{x,y} \equiv \psi_{x,y}(s)+\varphi_{x,y};\quad \phi_{x,y}(s+C)=\phi_{x,y}(s)+2\pi Q_{x,y}$$

with trivial equation of motion:

$$\frac{d\varphi_x}{ds} = \frac{\partial H_{NL}}{\partial I_x} = \frac{n}{2}\rho(s)I_x^{\frac{n}{2}-1}I_y^{\frac{m}{2}}\cos^n(\phi_x)\cos^m(\phi_y);$$

$$\frac{d\varphi_y}{ds} = \frac{\partial H_{NL}}{\partial I_x} = \frac{m}{2}\rho(s)I_x^{\frac{n}{2}}I_y^{\frac{m}{2}-1}\cos^n(\phi_x)\cos^m(\phi_y);$$

$$\frac{dI_x}{ds} = -\frac{\partial H_{NL}}{\partial \varphi_x} = n\rho(s)I_x^{\frac{n}{2}}I_y^{\frac{m}{2}}\cos^{n-1}(\phi_x)\sin(\phi_x)\cos^m(\phi_y);$$

$$\frac{dI_y}{ds} = -\frac{\partial H_{NL}}{\partial \varphi_y} = n\rho(s)I_x^{\frac{n}{2}}I_y^{\frac{m}{2}}\cos^n(\phi_x)\sin(\phi_y)\cos^{m-1}(\phi_y);$$

(b,d) $nQ_x + mQ_y = k + \delta Q;\quad |\delta Q| << 1$, we need to expand cos() via exponents to see the oscillating terms:

$$\cos^n(\phi_x) = 2^{-n}\left(e^{i\phi_x}+e^{-i\phi_x}\right)^n = 2^{-n}\sum_{k=0}^{n} C_n^k e^{i(n-2k)\phi_x};$$

$$\cos^m(\phi_y) = 2^{-m}\left(e^{i\phi_y}+e^{-i\phi_y}\right)^m = 2^{-m}\sum_{j=0}^{m} C_n^j e^{i(m-2j)\phi_y};$$

$$H_{NL} = 2^{-(n+m)}\rho(s)I_x{}^{n/2}I_y{}^{m/2}\sum_{k=0}^{n}\sum_{j=0}^{m} C_n^j C_n^k \exp\left\{i\left((n-2k)\phi_x+(m-2j)\phi_y\right)\right\}$$

Each term in the double sum oscillates with its own frequency: the $(n-2k)\phi_x+(m-2j)\phi_y$ advances by $2\pi\left\{(n-2k)Q_x+(m-2j)Q_y\right\}$. Thus, the phase in the resonant term corresponding to our conditions (k=0, j=0), advances by only $2\pi\delta Q << 1$, i.e. it is a resonant terms. The term opposite to in (k=n, j=n) does the same thing, it advances by $-2\pi\delta Q$. One thing is easy $C_n^0=1$

$$H_{sum} = 2^{1-(n+m)}\rho(s)I_x{}^{n/2}I_y{}^{m/2}\cos\left(n\phi_x+m\phi_y\right)$$

$$\frac{d\varphi_x}{ds} = \frac{\partial H_{NL}}{\partial I_x} = \frac{n}{2} 2^{1-(n+m)} \rho(s) I_x^{\frac{n}{2}-1} I_y^{\frac{m}{2}} \cos(n\phi_x + m\phi_y)$$

$$\frac{d\varphi_y}{ds} = \frac{\partial H_{NL}}{\partial I_x} = \frac{m}{2} 2^{1-(n+m)} \rho(s) I_x^{\frac{n}{2}} I_y^{\frac{m}{2}-1} \cos(n\phi_x + m\phi_y);$$

$$\frac{dI_x}{ds} = -\frac{\partial H_{NL}}{\partial \varphi_x} = -n \cdot 2^{1-(n+m)} \rho(s) I_x^{\frac{n}{2}} I_y^{\frac{m}{2}} \sin(n\phi_x + m\phi_y);;$$

$$\frac{dI_y}{ds} = -\frac{\partial H_{NL}}{\partial \varphi_y} = -m \cdot 2^{1-(n+m)} \rho(s) I_x^{\frac{n}{2}} I_y^{\frac{m}{2}} \sin(n\phi_x + m\phi_y).$$

Hence,

$$\frac{d(mI_x - nI_y)}{ds} = 0 \ \Rightarrow mI_x - nI_y = inv\,,$$

(c,d) – the only difference in the change of the sign in front of y-phase:

$$H_{dif} = 2^{1-(n+m)} \rho(s) I_x^{n/2} I_y^{m/2} \cos(n\phi_x - n\phi_y)$$

$$\frac{d\varphi_x}{ds} = \frac{\partial H_{NL}}{\partial I_x} = \frac{n}{2} 2^{1-(n+m)} \rho(s) I_x^{\frac{n}{2}-1} I_y^{\frac{m}{2}} \cos(n\phi_x - m\phi_y)$$

$$\frac{d\varphi_y}{ds} = \frac{\partial H_{NL}}{\partial I_x} = \frac{m}{2} 2^{1-(n+m)} \rho(s) I_x^{\frac{n}{2}} I_y^{\frac{m}{2}-1} \cos(n\phi_x - m\phi_y);$$

$$\frac{dI_x}{ds} = -\frac{\partial H_{NL}}{\partial \varphi_x} = -n \cdot 2^{1-(n+m)} \rho(s) I_x^{\frac{n}{2}} I_y^{\frac{m}{2}} \sin(n\phi_x - m\phi_y);;$$

$$\frac{dI_y}{ds} = -\frac{\partial H_{NL}}{\partial \varphi_y} = +m \cdot 2^{1-(n+m)} \rho(s) I_x^{\frac{n}{2}} I_y^{\frac{m}{2}} \sin(n\phi_x - m\phi_y).$$

$$\frac{d(mI_x + nI_y)}{ds} = 0 \ \Rightarrow mI_x + nI_y = inv$$

(e) the answer on the question is easy – at difference resonance both amplitudes can not grow above some limit imposed by initial conditiona;

$$I_x \le \frac{inv}{m};\ I_y \le \frac{inv}{n};,$$

while at the they both can (in principle) grow infinitely:

$$I_x = \frac{n}{m}(I_y + inv).$$

2. Twisted quadrupole

(a) (20 points) Find 4x4 matrix of twisted quadrupole, i.e. a quadrupole whose poles have torsion. The transverse Hamiltonian of this magnet is:

$$h = \frac{\pi_x^2 + \pi_y^2}{2} + K_1 \frac{x^2 - y^2}{2} + \kappa\left(y\pi_x - x\pi_y\right)$$

(b) (5 points) Identify when motion in both x and y direction is stable, i.e. there are no growing solutions?

Solution: We should use classification all magnetic element done in the conclusion to Lecture 4:

$$\lambda^2 = a \pm b;\; a = -\frac{f + g + 2L^2}{2};\; b^2 = \frac{(f-g)^2}{4} + 2L^2(f+g) + n^2 \qquad \text{(L4-10)}$$

In our case: $f - g = K_1; n = 0; L = \kappa; \quad \lambda^2 = a \pm b;\; a = -\kappa^2;\; b^2 = K_1^{\,2};$
$\lambda_1 = \sqrt{-(K_1 + \kappa^2)}; \lambda_2 = \sqrt{K_1 - \kappa^2};$

It means that there are two cases (iv and v in out classification:

I) when both eigen values are imaginary, i.e. the conditions we needed to find in the (b) part of the problem

$$K_1 > -\kappa^2;\; \lambda_1 = i\sqrt{\kappa^2 + K_1} = i\omega_1$$
$$K_1 < \kappa^2;\; \lambda_2 = i\sqrt{\kappa^2 - K_1} = -i\omega_2$$

which means $-\kappa^2 < K_1 < \kappa^2;\; |K_1| < \kappa^2$, focusing should not be too strong for rotation to give strong focusing effect. In this case one can use very long quadrupole of this type without worrying about beam stability. I hope one of you will use such devise in your accelerator.

The matrix is from the same lecture eq (IV-1):

$$\mathbf{M}_{4x4} = \frac{1}{\omega_1^{\,2} - \omega_2^{\,2}}\left\{\left(\mathbf{I}\cos\omega_1 s + \mathbf{D}\frac{\sin\omega_1 s}{\omega_1}\right)\left(\mathbf{D}^2 + \omega_2^{\,2}\mathbf{I}\right) - \left(\mathbf{I}\cos\omega_2 s + \mathbf{D}\frac{\sin\omega_2 s}{\omega_2}\right)\left(\mathbf{D}^2 + \omega_1^{\,2}\mathbf{I}\right)\right\}$$

We did not asked you to open the brackets – do it using one of tools which guarantee correctness of multiplications.

II) if $|K_1| > \kappa^2$ you have both cos, sin and cosh and sinh term, that later of which are growing exponentially at large distances:

$$\mathbf{M}_{4x4} = \frac{1}{\omega_1^{\,2} + \omega_2^{\,2}}\left\{\left(\mathbf{I}\cos\omega_1 s + \mathbf{D}\frac{\sin\omega_1 s}{\omega_1}\right)\left(\mathbf{D}^2 - \omega_2^{\,2}\mathbf{I}\right) - \left(\mathbf{I}\cosh\omega_2 s + \mathbf{D}\frac{\sinh\omega_2 s}{\omega_2}\right)\left(\mathbf{D}^2 + \omega_1^{\,2}\mathbf{I}\right)\right\} \qquad \text{(V-1)}$$

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Accelerator Physics: Homework 6

Due date: Wednesday, January 21, 2008

Problem 1. Sextupole terms
Consider a linear oscillator

$$x = \sqrt{\frac{2I}{\omega}}\cos(\omega s + \varphi); \quad \pi_x = x' = -\sqrt{2\omega I_y}\sin(\omega s + \varphi),$$
$$x = A\cos(\omega s + \varphi); \quad \pi_x = x' = -\omega A\sin(\omega s + \varphi),$$

in the presence of quadratic non-linear term (sextupole term) in the Hamiltonian:

$$h = \frac{\pi_x^2}{2} + \omega^2\frac{x^2}{2} + K_2\frac{x^3}{3}$$

(a) (20 points) Find first perturbation order terms in I, φ.

(b) (5 points) Show that far from resonances, there is no average growth in the actions and there is no tune dependence on the actions.

(c) (25 points) Write second order perturbation term for φ and calculate the tune shift proportional to the action and to the second order of K_2.

Suggestions: (a) note that $\omega = const$, (b) you may use Canonical pair (I,φ) or use reduced equation of motion derived by Dr.Pozdeyev in his lecture for (A,φ). Both methods will give you the same result.

Solution: Transformation to action-angle variable give shortened Hamiltonian:

$$h = \frac{K_2}{3}\sqrt{\frac{2I}{\omega}}^3\cos^3(\omega s + \varphi) = \alpha I^{3/2}(\cos 3\phi + 3\cos\phi); \quad \phi = \omega s + \varphi;$$
$$\alpha = \frac{K_2}{12}\sqrt{\frac{2}{\omega}}^3;$$
$$\cos^3(\phi) = \frac{1}{8}\left(e^{i\phi} + e^{i\phi}\right)^3 = \frac{1}{8}\left(e^{3i\phi} + 3e^{i\phi} + 3e^{-i\phi} + e^{-3i\phi}\right)$$

Equations of motion are simple:

$$h_1 = \frac{K_2}{3}\sqrt{\frac{2I}{\omega}}^3\cos^3(\omega s + \varphi) = \alpha I^{3/2}(\cos 3\phi + 3\cos\phi); \quad \phi = \omega s + \varphi;$$
$$\varphi' = \frac{\partial h_1}{\partial I} = \frac{3}{2}\alpha I^{1/2}(\cos 3\phi + 3\cos\phi); \quad I' = -\frac{\partial h_1}{\partial \varphi} = 3\alpha I^{3/2}(\sin 3\phi + \sin\phi); \qquad (1)$$

where all terms do oscillate and average to zero. Integrating first order oscillating term:

$$\varphi_1 = \varphi_0 + \tilde{\varphi}; \quad I_1 = I_0 + \tilde{I};$$

$$\tilde{\varphi} = \frac{3}{2}\alpha I^{1/2}\int^t (\cos 3\phi + 3\cos\phi)dt \;= \frac{1}{2\omega}\alpha I^{1/2}(\sin 3\phi + 9\sin\phi) \qquad (2)$$

$$\tilde{I} = 3\alpha I^{3/2}\int^t (\sin 3\phi + \sin\phi)dt = -\frac{\alpha I^{3/2}}{\omega}(\cos 3\phi + 3\cos\phi);$$

Now we need to put the above expression into the r.h.s. of (1) assuming that perturbation is weak $\tilde{\varphi}; \; \tilde{I}$ and we can expand the terms $\varphi_1 = \varphi_0 + \tilde{\varphi}; \;\; I_1 = I_0 + \tilde{I}$ to write in second order perturbation:

$$\begin{aligned}
\varphi' &= \frac{3}{2}\alpha\sqrt{I_o + \tilde{I}}\left(\cos 3(\phi_o + \tilde{\varphi}) + 3\cos(\phi_o + \tilde{\varphi})\right) = \frac{3}{2}\alpha I_o^{\;1/2}(\cos 3\phi_o + 3\cos\phi_o) + \\
&\quad \frac{3}{4}\alpha I_o^{\;-1/2}(\cos 3\phi_o + 3\cos\phi_o)\tilde{I} - \frac{9}{2}\alpha I_o^{\;1/2}(\sin 3\phi_o + \sin\phi_o)\tilde{\varphi} \\
I' &= 3\alpha\left(I_o + \tilde{I}\right)^{3/2}\left(\sin 3(\phi_o + \tilde{\varphi}) + \sin(\phi_o + \tilde{\varphi})\right) = 3\alpha(I_o)^{3/2}(\sin 3\phi_o + \sin\phi_o) + \\
&\quad \frac{9}{8}\alpha I_o^{\;1/2}(\sin 3\phi_o + 3\sin\phi_o)\tilde{I} + 3\alpha(I_o)^{3/2}(3\cos 3\phi_o + \cos\phi_o)\tilde{\varphi}
\end{aligned} \qquad (3)$$

Comparing the terms in (2) and (3) one can see that terms in I' have products of sin and cos terms, which give zero average value. Hence, as expected away from the resonances, particle's amplitude does not grow.

$$\langle I' \rangle = 0; \quad i.e.\ \langle I \rangle = const$$

In contrast, averaging the first equation in (3) yields non-zero result:

$$\langle \varphi' \rangle = \; -\frac{3}{4}\frac{\alpha^2 I_o}{\omega}\;\left(\left\langle (\cos 3\phi_o + 3\cos\phi_o)^2 \right\rangle + \left\langle (\sin 3\phi_o + 9\sin\phi_o)(\sin 3\phi_o + \sin\phi_o) \right\rangle\right) = \; -15\frac{\alpha^2 I_o}{\omega}$$

$$\alpha = \frac{K_2}{12}\sqrt{\frac{2}{\omega}}^{\,3} \Rightarrow \langle \varphi' \rangle = -\frac{120}{12^2}\frac{K_2^{\;2}}{\omega^2}I_o = -\frac{5}{6}\frac{K_2^{\;2}}{\omega^2}I_o$$

$$\frac{\partial\omega}{\partial I_o} = -\frac{5}{6}\frac{K_2^{\;2}}{\omega^2}$$

predicting that there will be negative nonlinear frequency (tune) shift with the square of the amplitude:

$$\omega(a) - \omega(0) = -\frac{5}{12}\frac{K_2^{\;2}}{\omega^2}a^2$$

Accelerator Physics: Midterm Exam Solutions

Friday, January 18 2007

1 Zero Trace Matrix

(10 points) Show that if M is a 2x2 matrix with unit determinant and Tr(M)=0, then $M^2 = -I$.
Solution: We can write M generally as

$$M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad M^2 = \begin{pmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{pmatrix} \tag{1.1}$$

where $\det(M) = -a^2 - bc = 1$. The result immediately follows.

2 Multiple FODO Cell Concatenation

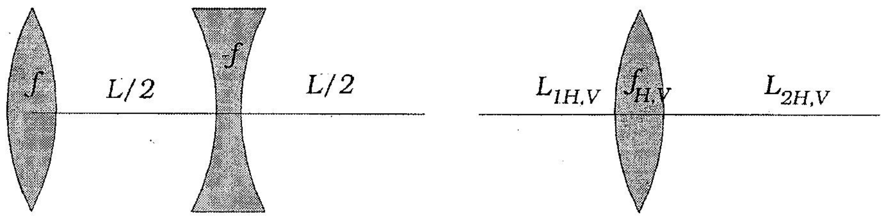
(10 points) Consider a FODO cell with phase advance of $2\pi/n$ in each plane. Show that the transport matrix of the concatenation of n of these FODO cells is I.
Solution: The FODO cell is periodic, so its transport matrix can be written in the form

$$M = I\cos\mu + J\sin\mu = I\cos(2\pi/n) + J\sin(2\pi/n) = \mathrm{e}^{2\pi J/n} \tag{2.1}$$

Concatenating n matrices together gives

$$M^n = \mathrm{e}^{2\pi J} = I\cos(2\pi n) + J\sin(2\pi n) = I \tag{2.2}$$

3 FODO Cell Equivalence



(15 points) Consider a FODO cell with drift lengths $L/2$ and quadrupole focal lengths $\pm f$ as shown on the left. The transport matrix of this FODO lattice, starting with the focusing quadrupole, was given in class as

$$\begin{aligned} M_{\mathrm{FODO}}(\text{horizontal}) &= \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{L}{2f} - \frac{L^2}{4f^2} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 + \frac{L}{2f} \end{pmatrix} \end{aligned} \tag{3.1}$$

in the horizontal plane.

(a) Show that M_{FODO}(horizontal) can be written as M_{OFO}(horizontal) – that is, as the horizontal transport matrix of a single quadrupole of focal length f_H between two straight sections of (possibly different) lengths L_{1H} and L_{2H}, as shown on the right. How do (f_H, L_{1H}, L_{2H}) relate to (f, L) of the FODO cell?
Solution: The OQO lattice transport matrix can be written as

$$M_{\text{OQO}} = \begin{pmatrix} 1 & L_{2H} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f_H & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{1H} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{L_{2H}}{f_H} & L_{1H} + L_{2H} + \frac{L_{1H}L_{2H}}{f_H} \\ \frac{1}{f_H} & 1 + \frac{L_{1H}}{f_H} \end{pmatrix} \tag{3.2}$$

Setting the lower left components of (3.1) and (3.2) equal gives $\boxed{f_H = -2f^2/L}$ which is negative definite. Setting the lower right components equal gives $L_{1H} = \frac{f_H L}{2f}$ or $\boxed{L_{1H} = -f}$. (Yes, this is negative since we assumed $f > 0$ for the horizontal transport case.) Finding L_{2H} is probably easiest done setting the upper left components equal:

$$\begin{aligned} 1 - \frac{LL_{2H}}{2f^2} &= 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \\ \Rightarrow L_{2H} &= f + \frac{L}{2} \end{aligned} \tag{3.3}$$

(b) Show that the vertical transport matrix of the same FODO cell, M_{FODO}(vertical) can be written as M_{OQO}(vertical) with quadrupole focal length f_V and lengths L_{1V} and L_{2V}. How do these relate to the horizontal OQO cell parameters found in part (a)?
Solution: The transport matrix for the vertical motion replaces f with $-f$ in (3.1). This gives $\boxed{f_V = f_H = -2f^2/L,\ L_{1V} = f}$, and $\boxed{L_{2V} = -f + L/2}$. We can observe that $\boxed{L_{1H} + L_{2H} = L_{1V} + L_{2V} = L/2}$. The two transports are different between horizontal and vertical , so we cannot replace a pair of quadrupoles in a FODO cell with a single quadrupole.

Note that in general, any 2D single-plane transport can be modeled by this drift-quad-drift model, since we have three parameters to fit and three unknowns. The fourth degree of freedom in the general 2x2 matrix is constrained by symplecticity.

4 Beta Function At A Waist

(20 points) The transport of the envelope function $w(s)$ from a local minimum of value w_0 through a drift space of length s is given by:

$$\begin{pmatrix} w(s) \\ w'(s) + \frac{i}{w(s)} \end{pmatrix} e^{i\Delta\psi} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ \frac{i}{w_0} \end{pmatrix} \tag{4.1}$$

(a) Show that

$$w^2(s) = w_0^2 + \frac{s^2}{w_0^2} \quad \text{or} \quad \beta(s) = \beta_0 + \frac{s^2}{\beta_0} \tag{4.2}$$

Thus the beta function near a waist (or local minimum) in a drift region is quadratic.
Solution: From (4.1), we have

$$w(s)e^{i\Delta\psi} = w_0 + \frac{is}{w_0} \tag{4.3}$$

$w(s)$ is by definition real, so we can multiply by the complex conjugate to give

$$w^2(s) = \left(w_0 + \frac{is}{w_0}\right)\left(w_0 - \frac{is}{w_0}\right) = w_0^2 + \frac{s^2}{w_0^2} \quad (4.4)$$

We have defined $\beta(s) = w^2(s)$ in the notes, so (4.2) follows.

(b) From (4.1), calculate the phase advance $\Delta\psi$ as $s \to \infty$. This is half of the maximum phase advance of a field-free region.
Solution: There are multiple ways to do this. One way is to use the definition of the phase advance as an integral of the inverse of the beta function:

$$\Delta\psi = \int_0^\infty \frac{ds}{\beta(s)} = \beta_0 \int_0^\infty \frac{ds}{\beta_0^2 + s^2} = \tan^{-1}\left(\frac{s}{\beta_0}\right)\Big|_0^\infty = \frac{\pi}{2} \quad (4.5)$$

You can also evaluate (4.4) in the limit of $s \to \infty$:

$$e^{i\Delta\psi} = \frac{w_0 + is/w_0}{\sqrt{w_0^2 + s^2/w_0^2}} \quad (4.6)$$

$$= \frac{is/w_0}{s/w_0} = i \quad \text{as } s \to \infty \quad (4.7)$$

which gives $\Delta\psi = \frac{\pi}{2}$.

5 Solenoid Transport Matrix

(25 points) Consider a solenoid of length L with only longitudinal field B_s. The torsion to decouple the Hamiltonian is

$$\kappa = -\frac{eB_s}{2pc} \quad (5.1)$$

and the resulting Hamiltonian in rotating frame of reference canonical coordinates (x, π_x) and (y, π_y) is

$$H(x, \pi_x, y, \pi_y) = \frac{\pi_x^2 + \pi_y^2}{2} + \kappa^2 \frac{x^2 + y^2}{2} \quad (5.2)$$

Find the transport matrix of this solenoid.
Solution: This problem is almost exactly the same as problem 2 of homework set 3, with the difference of a sign, and a different interpretation of the focusing strength coefficient k or κ. The problem is separable, so we can solve each plane independently, and they will have the same form. The one-dimensional single-plane Hamiltonian can be written as

$$H(s) = \begin{pmatrix} \kappa & 0 \\ 0 & 1 \end{pmatrix} \quad (5.3)$$

(this corrects a typo in the homework 3 solution), so we have

$$D = S \cdot H(s) = \begin{pmatrix} 0 & 1 \\ -\kappa & 0 \end{pmatrix} \qquad M_{\text{solenoid}} = \exp[D \cdot L] \quad (5.4)$$

Comparing to the quadrupole problem, $k = \kappa^2$ which is positive definite, so the transport matrix in each plane of this solenoid in the rotated coordinate system is simply that of a focusing quadrupole with effective strength $k = \kappa^2 = (e^2 B_s^2)(4p^2c^2)$, or

$$M_{\text{solenoid}} = \begin{pmatrix} \cos\kappa L & \frac{\sin\kappa L}{\kappa} \\ -\kappa\sin\kappa L & \cos\kappa L \end{pmatrix} \quad (5.5)$$

6 Azimuthally Symmetric Optics

(30 points) Consider an azimuthally symmetric ring with orbit radius $\rho = \frac{1}{K_0}$ and a field gradient

$$\frac{e}{p_0 c}\frac{\partial B_y}{\partial x} = -nK_0^2 \tag{6.1}$$

where n is known as the field index. The Hamiltonian for transverse motion does not depend on s, and is given by

$$H(x, \pi_x, y, \pi_y) = \frac{\pi_x^2 + \pi_y^2}{2} + K_0^2(1-n)\frac{x^2}{2} + nK_0^2\frac{y^2}{2} \tag{6.2}$$

(a) Find the one-turn matrices for horizontal and vertical motion.
Solution:

Here the problem is akin to that of the quadrupole again, where the total length of the quadrupole is the circumference of the accelerator $C = 2\pi\rho$, and the effective quadrupole strengths are $K_x = K_0^2(1-n)$ and $K_y = K_0^2 n$. This gives the one-turn matrices as

$$\mathbf{T}_\mathrm{x} = I\cos|\lambda_x|C + D\frac{\sin|\lambda_x|C}{|\lambda_x|} = \begin{pmatrix} \cos|\lambda_x|C & \frac{\sin|\lambda_x|C}{|\lambda_x|} \\ -|\lambda_x|\sin|\lambda_x|C & \cos|\lambda_x|C \end{pmatrix} \tag{6.3}$$

where $\lambda_x = K_0\sqrt{1-n}$, and

$$\mathbf{T}_\mathrm{y} = I\cos|\lambda_y|C + D\frac{\sin|\lambda_y|C}{|\lambda_y|} = \begin{pmatrix} \cos|\lambda_y|C & \frac{\sin|\lambda_y|C}{|\lambda_y|} \\ -|\lambda_y|\sin|\lambda_y|C & \cos|\lambda_y|C \end{pmatrix} \tag{6.4}$$

where $\lambda_y = K_0\sqrt{n}$. Note that we reqire that $0 < n < 1$ for stability in both planes.

(b) Find the horizontal and vertical tunes, $\nu_{x,y}$, and show that $\nu_x^2 + \nu_y^2 = 1$.
Solution: The tune is related to the trace of the one-turn matrix:

$$Tr(T) = 2\cos\mu = 2\cos(2\pi\nu) \tag{6.5}$$

which immediately gives $\nu_x = K_0\sqrt{1-n}\rho$ and $\mu_y = K_0\sqrt{n}\rho$. But $K_0 = 1/\rho$, so we have $\boxed{\nu_x = \sqrt{1-n} \text{ and } \mu_y = \sqrt{n}}$. From this result it is obvious that $\nu_x^2 + \nu_y^2 = 1$.

(c) Find the beta functions of this storage ring.
Solution: Since there is azimuthal symmetry, the beta functions are constant around the ring. We then have

$$\nu_x = \oint \frac{ds}{\beta_x} = \frac{2\pi C}{\beta_x} = K_0\sqrt{1-n} \quad \Rightarrow \quad \beta_x = \frac{2\pi C}{K_0\sqrt{1-n}} \tag{6.6}$$

$$\nu_y = \oint \frac{ds}{\beta_y} = \frac{2\pi C}{\beta_y} = K_0\sqrt{n} \quad \Rightarrow \quad \beta_y = \frac{2\pi C}{K_0\sqrt{n}} \tag{6.7}$$