USPAS, Winter 2008

Accelerator Physics

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Problems in Home-works and Midterm Exam

Homework 1

Due date: Tuesday Jan 15, 2007

January 11, 2008

1 Basic relativity

(a) (3 points) In one dimension the work done by a force F acting through a distance dl is $dE = F\,dl$. Show directly that increasing the Lorentz factor of a particle of mass m by $\Delta\gamma$ changes the particle's energy by

$$\Delta E = \Delta\gamma\, mc^2 \tag{1.1}$$

where the rest energy of the particle is $E_0 = mc^2$. From this it follows that $E = \gamma E_0$. Use this to show that

$$E^2 = p^2c^2 + m^2c^4 \tag{1.2}$$

(b) (3 points) Show that an infinitesimal increase in energy dE is related to the infinitesimal increase in momentum dp by

$$\frac{dE}{E} = \beta^2 \frac{dp}{p} \tag{1.3}$$

where $\beta \equiv v/c$.

(c) (4 points) A unit charge particle of momentum p travels through a constant magnetic field B, and is bent in a circular arc of radius ρ. Show that

$$B\rho\,[\mathrm{T-m}] = 3.3357\, p\,[\mathrm{GeV/c}] \tag{1.4}$$

2 RHIC energy and current

Gold ions $^{197}Au^{+77}$ (A=197, Z=79) are injected into the Brookhaven Alternating Gradient Synchrotron (AGS) with a kinetic energy of 100 GeV/u (i. e. GeV per nucleon). The AGS accelerates protons up to a kinetic energy of 22.9 GeV for injection into Relativistic Heavy Ion Collider (RHIC). The circumference of the AGS is 807.1 m, and the rest mass of a gold ($^{197}Au^{+77}$) ion is 183.434 GeV/c^2.

(a) (4 points) What is the velocity of the injected gold ions?

(b) (3 points) What is the corresponding kinetic energy for $^{197}Au^{+77}$ ions extracted from the AGS for RHIC?

(c) (3 points) Why does the beam current increase although the circulating charge stays constant during acceleration?

3 Basic collision kinematics

(a) (3 points) Show that the total energy for a head-on collision of two particles, each with center of mass energy $\gamma_{\mathrm{cm}} mc^2$, is equal to the total energy of a fixed-target collision, where one particle is at rest and the other has energy $\gamma_{\mathrm{fixed}} mc^2$, where

$$\gamma_{\mathrm{fixed}} = 2\gamma_{\mathrm{cm}}^2 - 1 \tag{3.1}$$

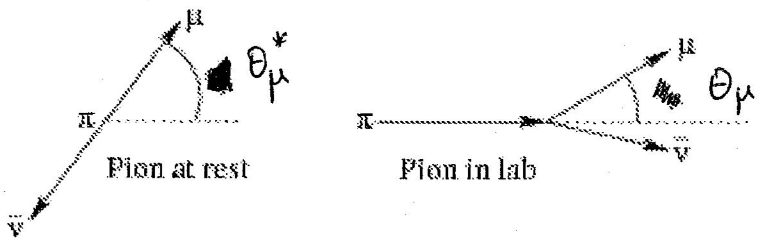
Consider a charged pion decaying into a muon plus an antineutrino:

$$\pi^- \rightarrow \mu^- + \bar{\nu}_\mu \tag{3.2}$$

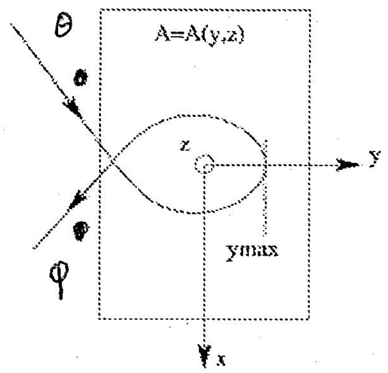
Use $M_{\pi^\pm} = 140\mathrm{MeV/c^2}$, $m_\mu = 106\mathrm{MeV/c^2}$, and $m_{\bar{\nu}} = 0$.

(b) (3 points) In the rest system of the pion, what are the energies and momenta of the muon and antineutrino?

(c) (4 points) For a moving pion with total energy $U_\pi = \gamma M_\pi c^2$ find an expression for the direction, θ_μ of the muon relative to the pion in the lab in terms of the angle θ_μ^* in the in the pion's rest system.



4 Magnetic Mirror



(a) (2 points) An electron moves through a magnetic field with vector potential $\vec{A} = \vec{A}(y, z)$. Find an additional invariant of motion from the independence of $\vec{A}$ from x. Write an explicit expression for p_x using this invariant.

(b) (2 points) Consider a magnet with mid-plane symmetry, $\vec{H} = \hat{e}_z H(y)$ at $z = 0$, shown above, with $\vec{A} = \vec{A}(y, z)$ inside the magnet and $\vec{A} = 0$ outside the magnet. Consider an electron entering the magnet in the midplane $z = 0$ with mechanical momentum

$$\vec{p} = \hat{e}_x p_x + \hat{e}_y p_y = p(\hat{e}_x \cos\theta + \hat{e}_y \sin\theta) \tag{4.1}$$

which enters the magnet, turns around in the magnet, and comes back out.

(c) (2 points) Show that the trajectory of the electron remains in the $z = 0$ plane.

(d) (2 points) Find the equation of the angle φ of the electron's exit trajectory relative to $\hat{e}_x$ direction.

(e) (2 points) Find the equation defining the penetration depth y_{max} of the electron in the magnet in terms of $A(y, z = 0)$. You don't have to solve this equation generally, but do write down an equation that you could solve numerically for y_{max}.

Hint : Use the Lorentz force to find (b), and use the canonical momentum to connect the mechanical momentum with $\vec{A} = \vec{A}(y, z = 0)$ for (c,d).

5 The Lorentz Group

(a) (4 points) For the Lorentz boost

$$\vec{K} \equiv \hat{e}_x \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \hat{e}_y \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \hat{e}_z \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{5.1}$$

and rotation matrix

$$\vec{S} \equiv \hat{e}_x \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \hat{e}_y \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + \hat{e}_z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.2}$$

given in class, show that

$$(\vec{\epsilon}\,\vec{S})^3 = -\vec{\epsilon}\,\vec{S} \quad (\vec{\epsilon}\,\vec{K})^3 = \vec{\epsilon}\,\vec{K} \quad \text{for } \forall\vec{\epsilon} = \vec{\epsilon}^\star \quad \text{where } |\vec{\epsilon}| = 1 \tag{5.3}$$

and, more generally,

$$(\vec{a}\,\vec{S})^3 = -\vec{a}\,\vec{S}|\vec{a}|^2 \quad (\vec{a}\,\vec{K})^3 = \vec{a}\,\vec{K}|\vec{a}|^2 \quad \text{for } \forall\vec{a} = \vec{a}^\star \tag{5.4}$$

(b) (4 points) Use these results to show that

$$e^{\vec{\omega}\vec{S}} = I + \frac{\vec{\omega}\vec{S}}{|\vec{\omega}|}\sin|\vec{\omega}| - \frac{\left(\vec{\omega}\vec{S}\right)^2}{|\vec{\omega}|^2}(\cos|\vec{\omega}| - 1) \tag{5.5}$$

and

$$e^{\vec{\beta}\vec{K}} = I + \frac{\vec{\beta}\vec{K}}{|\vec{\beta}|}\sinh|\vec{\beta}| - \frac{\left(\vec{\beta}\vec{K}\right)^2}{|\vec{\beta}|^2}\left(\cosh|\vec{\beta}| - 1\right) \tag{5.6}$$

(c) (2 points) Are $\vec{S}$ and/or $\vec{K}$ symplectic?

Accelerator Physics: Homework 2

Due date: Wednesday January 16, 2007

1 EM similarity to Lorentz group

Consider an invariant equation of motion of a charged particle in a constant electromagnetic field:

$$mc\frac{du^i}{ds} = \frac{e}{c}F^i_k \cdot u^k \qquad \frac{d}{ds}[u] = D[u] \qquad [D] = \frac{e}{mc^2}F^i_k \tag{1.1}$$

where $[u]$ is a 4-vector, and which has the general solution

$$[u] = e^{Ds}[u_0] \tag{1.2}$$

(a) (4 points) Write matrix $[D]$. Identify the similarity of $[D]$ with the generator of the Lorentz group, and find the analogy between boost, rotations, and components of the electromagnetic field.

(b) (6 points) Write the explicit expression for $M = e^{Ds}$ in the case of a pure constant electric field ($B = 0$) and pure constant magnetic field ($E = 0$).

2 Cos-theta magnet

(10 points) Show that current distributed in a thin cylindrical shell with a strength

$$I(\theta) = \frac{I_0}{n\pi}\cos(n\theta) \tag{2.1}$$

will produce a pure $2n$-multipole distribution inside the cylinder.

3 Quadrupole gradient, inductance (Lee 1.12, p. 29)

From Maxwell, $\nabla \times \vec{B} = 0$ in a current-free region, and the magnetic field can be derived from a magnetic potential Φ_m with $\vec{B} = -\nabla\Phi_m$.

(a) (2 points) For a quadrupole field with $B_z = Kx$, $B_x = Kz$, show that the magnetic potential is $\Phi_m = -Kxz + c$ where c is a constant. We can choose a gauge where $c = 0$.

(b) (5 points) Equipotential curves are therefore xz =constant. The pole shape of a quadrupole is therefore a hyperbolic curve described by $xz = a^2/2$ where a is the half-aperture of the quadrupole. The magnitude of the field at the surface of the pole is $B_{\text{pole tip}} = Ka$. To avoid magnetic field saturation in the (typically iron) pole tip, the pole tip field in a quadrupole is normally designed to be less than 0.9 Tesla, and the achievable gradient is $B_1 = B_{\text{pole tip}}/a$. Show that the gradient field is

$$B_1 = 2\mu_0 NI/a^2 \tag{3.1}$$

where NI is the number of ampere-turns per pole.

(c) (3 points) Show that the inductance of a quadrupole of length l is

$$L = \frac{8\mu_0 N^2 l}{a^2}\left(x_c^2 - \frac{a^4}{12x_c^2}\right) \tag{3.2}$$

where x_c is the distance of the conductor from the center of the quadrupole.

4 Dipole edge focusing (Lee 2.2.2, p. 73)

(10 points) Sector dipoles are bent so the end faces of the magnet are perpendicular to the design particle entry and exit angles. When a particle enters a sector dipole at an angle δ with respect to the design trajectory, it experiences some focusing. This phenomenon is usually referred to as edge focusing. We use the convention that $\delta > 0$ if the particle is closer to the center of the bending radius. Show that the transport matrices through the dipole for horizontal and vertical motion of the particle are

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{\tan\delta}{\rho} & 1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 1 & 0 \\ -\frac{\tan\delta}{\rho} & 1 \end{pmatrix} \tag{4.1}$$

The edge effect with $\delta > 0$ produces horizontal defocusing and vertical focusing.

5 Gaussian luminosity (Lee 1.7(b), p. 27)

The total counting rate of a physical interaction at a single collision point is given by $R = \mathcal{L}\sigma$, where σ is the cross-section of the interaction and the luminosity $\mathcal{L}$ (in units of cm^{-2} s^{-1}) is a measure of the interaction probability per unit area and time. When two accelerator bunches with relativistic velocities β collide head-on,

$$\mathcal{L} = 2fN_1N_2 \int \rho_1(x,y,s_1)\,\rho_2(x,y,s_2)\,dx\,dy\,d(\beta ct) \tag{5.1}$$

where $s_1 = s + \beta ct$, $s_2 = s - \beta ct$, f is the collision frequency, N_1 and N_2 are the number of particles in each bunch, and ρ_1 and ρ_2 are the normalized distribution functions for both bunches.

(a) (5 points) Using a Gaussian bunch distribution,

$$\rho(x,y,s) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_s}\exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{s^2}{2\sigma_s^2}\right] \tag{5.2}$$

where σ_x, σ_y, and σ_s are the rms horizontal, vertical, and longitudinal beam sizes, show that the luminosity for two bunches with identical distributions is

$$\mathcal{L} = \frac{fN_1N_2}{4\pi\sigma_x\sigma_y} \tag{5.3}$$

(b) (5 points) Show that if two bunches collide with a vertical offset of Δy, the luminosity is reduced by a factor of $\exp(-\Delta y^2/4\sigma_y^2)$.

Accelerator Physics: Homework 3

Due date: Thurssday January 17, 2007

1 Dual-Lens Focusing

(10 points) Consider a lens system made up of a thin focusing quadrupole of focusing length f and a thin defocusing quadrupole of focusing length $-f$ separated by a drift space L. Show that in either order of the quadrupoles, the system is net focusing if $|f| > L$.

2 Quadrupole Transport Matrix Using Sylvester's Formula

(10 points) Find the transport matrix for a quadrupole with

$$H(x, p_x, y, p_y) = \left(\frac{p_x^2}{2} + kx^2\right) + \left(\frac{p_y^2}{2} - ky^2\right) \tag{2.1}$$

for both positive and negative k, using the matrix exponent and Sylvester's formula as discussed in class. Compare your solution to the equation in Lee between (2.38) and (2.39). Hint: Use the fact that the Hamiltonian is decoupled.

3 Hamiltonian Symplecticity

Recall that the symplecticity condition for a transport matrix M is

$$\tilde{M}SM = S \tag{3.1}$$

where $\tilde{M}$ is the transpose of M and S is the block-diagonal symplectic form

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_n \tag{3.2}$$

where $S^2 = -I$. (S is rather like a matrix equivalent of $i = \sqrt{-1}$.) A Hamiltonian system has a Hamiltonian H that produces the equations of motion, Hamilton's equations:

$$X' = S\frac{\partial H}{\partial X} \tag{3.3}$$

where $X = (q_1, p_1, \ldots, q_n, p_n)$ are canonical coordinates and the prime indicates differentiation with respect to s. The transport matrix or Jacobian M_{ij} is defined as how particles change with respect to their initial coordinates:

$$M_{ij} = \frac{\partial X_i}{\partial (X_0)_j} \tag{3.4}$$

where $(X_0)_j$ is the j^{th} component of the initial coordinates of a particle at $s = 0$, including both positions and momenta. This is all we need to prove an important result: all Hamiltonian systems (even nonlinear ones) are symplectic.

(a) (2 points) What is $M(s=0)$? (This should be self-evident from Eqn. (2.4).)

(b) (3 points) Demonstrate that $M' = S\,Y\,M$ using Hamilton's eqations, where Y is the symmetric matrix that has components

$$Y_{ij} \equiv \frac{\partial H}{\partial X_i \partial X_j} \tag{3.5}$$

(c) (3 points) Show that $(\tilde{M}\,S\,M)' = 0$ by distributing the differentiation among $\tilde{M}$ and M, and using the result of part (b).

(d) (2 points) $(\tilde{M}\,S\,M)' = 0$ implies that $(\tilde{M}\,S\,M)$ is independent of s and therefore a constant. Evaluate this constant at $s = 0$ using part (a) to show that M is symplectic – that is, obeys Eqn. (2.1) for all time s.

4 Perturbed Harmonic Oscillator and Canonical Coordinates

For the perturbed harmonic oscillator Hamiltonian

$$H(p,q) = \frac{p^2}{2} + \omega^2\frac{q^2}{2} + \epsilon \cdot H_p(p,q) \tag{4.1}$$

where $\epsilon \cdot H_p(p,q)$ is a weak perturbation with $\epsilon \ll 1$, the system can be parameterized with an amplitude a and phase φ as:

$$q = \frac{a}{\sqrt{\omega}}\cos(\omega t + \varphi) \quad p = -a\sqrt{\omega}\sin(\omega t + \varphi) \tag{4.2}$$

(a) (2 points) Show that the transformation $(q,p) \to (Q = \varphi, P = I = a^2/2)$ is canonical.

(b) (3 points) Show that the area of the ellipse inside the the trajectory in the phase space (q,p) is πI.

(c) (2 points) Show (directly!) that in the new canonical coordinates, the new Hamiltonian can be written as

$$\bar{H}(p,q) = \epsilon \cdot H_p\left(-\sqrt{2I\omega}\sin(\omega t + \varphi), \sqrt{2I/\omega}\cos(\omega t + \varphi)\right) \tag{4.3}$$

(d) (3 points) Using Hamilton's equations and the solution for $\bar{H}$, write the equations of motion for (φ, I). These should look like harmonic oscillator equations, relating, for example, $\ddot{\varphi}$ to φ.

Accelerator Physics: Homework 4

Due date: Monday January 21, 2008

1 Propagating Courant-Snyder Parameters

Suppose you are given the Courant-Snyder parameters, or equivalently the J-matrix,

$$J(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \tag{1.1}$$

at one point in a ring where the one-turn transport matrix is

$$M(s) = I\cos\mu + J\sin\mu = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{1.2}$$

The Courant-Snyder parameters can be found at other points with the use of the appropriate transfer matrices. Suppose $J(s_1)$ is the matrix representing a known set of parameters at point $s = s_1$, and we want to find $J(s_2)$. Let $M(s_1, s_2)$ be the matrix propogating the motion from point $s = s_1$ to point $s = s_2$. $M(s_1, s_2)$ cannot be written in the form of (1.2) because this transport is not periodic, but we can find it anyway.

(a) (3 points) Show that the J-matrices at the two points are related by

$$J(s_2) = M(s_1, s_2)\ J(s_1)\ M^{-1}(s_1, s_2) \tag{1.3}$$

(b) (7 points) Show that the parameter relations are

$$\begin{aligned} \beta(s_2) &= m_{11}^2\beta(s_1) - 2m_{11}m_{12}\alpha(s_1) + m_{12}^2\gamma(s_1) \\ \alpha(s_2) &= -m_{11}m_{21}\beta(s_1) + (m_{11}m_{22} + m_{12}m_{21})\alpha(s_1) - m_{12}m_{22}\gamma(s_1) \\ \gamma(s_2) &= m_{21}^2\beta(s_1) - 2m_{21}m_{22}\alpha(s_1) + m_{22}^2\gamma(s_1) \end{aligned} \tag{1.4}$$

where the m_{ij} are the matrix elements of $M(s_1, s_2)$.

2 Phase Advance From Transport Matrix

(a) (7 points) Show that the phase advance from point $s = s_1$ to point $s = s_2$ through a section described by the transport matrix $M(s_1, s_2)$ is given by:

$$\Delta\psi = \tan^{-1}\left(\frac{m_{12}}{\beta(s_1)\ m_{11} - \alpha(s_1)\ m_{12}}\right) \tag{2.1}$$

where the m_{ij} are the matrix elements of $M(s_1, s_2)$ and $\beta(s_1)$ and $\alpha(s_1)$ are the Courant-Snyder parameters at point $s = s_1$.

(b) (3 points) Check this expression by demonstrating that $\Delta\psi = \mu$ for $M(s_1, s_1 + C)$, where C is the accelerator circumference.

An observation that you don't have to do for homework: (1.4) can be solved to find the elements m_{ij} of $M(s_1, s_2)$ in terms of the Courant-Snyder parameters at s_1 and s_2 and the phase advance $\Delta\psi$ given by (2.1). We will cover this next week in class.

Accelerator Physics: Homework 5

Due date: Tuesday, January 22, 2008

1. Coupling non-linear resonance

Consider an uncoupled linear motion in a storage ring, parameterized by

$$x = \sqrt{2I_x}\, w_x(s)\cos\left(\psi_x(s) + \varphi_x\right); \quad y = \sqrt{2I_y}\, w_y(s)\cos\left(\psi_y(s) + \varphi_y\right),$$

in the presence of additional non-linear term in the Hamiltonian

$$H_{NL} = \alpha(s) x^m y^n.$$

All coefficients above are periodical with the ring circumference, *C,* except the betatron phases

$$\psi_{x,y}(s + C) = \psi_{x,y}(s) + 2\pi Q_{x,y}.$$

Here: n, m, k, l are integer numbers.

(a) (5 points) write slow equation of motion for the action-angle variables;

(b) (5 points) consider resonant conditions $nQ_x + mQ_y = k + \delta Q; \quad |\delta Q| << 1$, i.e. a sum resonance, and find the expression for the resonant term in the Hamiltonian and the slow equations of motion;

(c) (5 points) consider resonant conditions $nQ_x - mQ_y = l + \delta Q; \quad |\delta Q| << 1$ i.e. a difference resonance, and find the expression for the resonant term in the Hamiltonian and the slow equations of motion;

(d) (10 points) show that in the resonance approximation $|\delta Q| << 1$, we have additional invariants of motion: $nI_y - mI_x = inv$ for the sum resonance and $nI_y + mI_x = inv$ for the difference resonance. Derive your conclusions on what resonance can be more dangerous from a perspective of continuous growth of the amplitudes (i.e. possibility to loose a beam at the walls of vacuum chamber)?

(e) (5 points) qualitatively answer the question if this term in the Hamiltonian can drive other $NQ_x \pm MQ_y = K + \delta Q; \quad N \neq n; M \neq m$ in the first order of perturbation theory?

2. Twisted quadrupole

(a) (20 points) Find 4x4 matrix of twisted quadrupole, i.e. a quadrupole whose poles have torsion. The transverse Hamiltonian of this magnet is:

$$h = \frac{\pi_x^2 + \pi_y^2}{2} + K_1 \frac{x^2 - y^2}{2} + \kappa\left(y\pi_x - x\pi_y\right)$$

(b) (5 points) Identify when motion in both x and y direction is stable, i.e. there are no growing solutions?

3. Sextupole terms

Consider an uncoupled linear motion in a storage ring, parameterized by

$$x = \sqrt{2I_x}\, w_x(s)\cos(\psi_x(s) + \varphi_x); \quad y = \sqrt{2I_y}\, w_y(s)\cos(\psi_y(s) + \varphi_y),$$

in the presence of sextupole fields:

$$h = \frac{\pi_x^2 + \pi_y^2}{2} + K_1(s)\frac{x^2 - y^2}{2} + K_2(s)\frac{x^3 - 3y^2 x}{2}$$

(a) (20 points) Find (in a form of integrals) first perturbation order terms in $I_x,\ \varphi_x,\ I_y,\ \varphi_y$.

(b) (5 points) Show that far from resonances, there is no average growth in the actions and there is no tune dependence on the actions.

(c) (25 points) Write second order perturbation term for φ_x and demonstrate that there will tune shift proportional to the action and to the second order of sextupole strength.

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Accelerator Physics: Homework 6

Due date: Wednesday, January 21, 2008

Problem 1. Sextupole terms
Consider a linear oscillator

$$x = \sqrt{\frac{2I}{\omega}}\cos(\omega s + \varphi); \quad \pi_x = x' = -\sqrt{2\omega I_y}\sin(\omega s + \varphi),$$

$$x = A\cos(\omega s + \varphi); \quad \pi_x = x' = -\omega A\sin(\omega s + \varphi),$$

in the presence of quadratic non-linear term (sextupole term) in the Hamiltonian:

$$h = \frac{\pi_x^2}{2} + \omega^2\frac{x^2}{2} + K_2\frac{x^3}{3}$$

(a) (20 points) Find first perturbation order terms in $I,\ \varphi$.

(b) (5 points) Show that far from resonances, there is no average growth in the actions and there is no tune dependence on the actions.

(c) (25 points) Write second order perturbation term for φ and calculate the tune shift proportional to the action and to the second order of K_2.

Suggestions: (a) note that $\omega = const$, (b) you may use Canonical pair (I,φ) or use reduced equation of motion derived by Dr. Pozdeyev in his lecture for (A,φ). Both methods will give you the same result.

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Accelerator Physics: Homework 6

Due date: Wednesday January 23, 2008

1 Quadrupole Lie Map

(a) (15 points) Demonstrate that the 1D thin-lens quadrupole Lie operator

$$\exp\left(:-\frac{1}{2f}x^2:\right) \tag{1.1}$$

gives the correct thin-lens quadrupole kick when applied to particle initial coordinates $(x_0, p_{x,0})$. (Hint: Review sections (2.2) and (2.4) of the notes, in particular the first few equations of (2.4).)

(b) (20 points) Using the Cayley-Hamilton theorem, demonstrate that the 1D thick-lens focusing quadrupole Lie operator

$$exp\left(:-\frac{L}{2}\left(kx^2+p^2\right):\right) \tag{1.2}$$

gives the correct thick-lens focusing quadrupole kick when applied to particle initial coordinates $(x_0, p_{x,0})$. Here you should write the Lie operator in a bilinear form and use the Cayley-Hamilton theorem to reduce the problem to the sum of two matrices.

Note how similar this process is to the process already performed in class using the D matrix and Sylvester's formula!

(c) (5 points) Repeat part (b) for the thin-lens defocusing quadrupole Lie operator and corresponding matrix kick. What happens if instead of changing the sign of p^2, we change the sign of x^2, or x^2 and p^2 together?

Space charge effects in accelerators.

Home Work

January 23, 2008

1 Problem (10 Points)

KV distribution is given by

$$f(x, x', y, p') = f_0 \delta\left(\frac{A_x}{\epsilon_x} + \frac{A_y}{\epsilon_y} - 1\right) \tag{1}$$

where

$$A_x = \gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 \tag{2}$$
$$A_y = \gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2 \tag{3}$$

Show that projection of this distribution on the x,y plane is a uniform ellipse

$$f(x, y) = \frac{e\lambda}{\pi ab} H\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \tag{4}$$

where $H(u) = 1$ if $u > 0$ and 0 if $u < 0$. This is a uniform distribution within an ellipse with semi-axis a and b. The parameters $\epsilon_{x,y}$ are the horizontal and vertical beam emittances at the edge of the envelopes.

2 Problem (10 Points)

Show that for a Gaussian beam the envelope equations are

$$\sigma_x'' + K_x \sigma_x = \frac{\epsilon_x^2}{\sigma_x^3} + \frac{\xi}{2(\sigma_x + \sigma_y)} \tag{5}$$

$$\sigma_y'' + K_y \sigma_y = \frac{\epsilon_y^2}{\sigma_y^3} + \frac{\xi}{2(\sigma_x + \sigma_y)} \tag{6}$$

4 Play Homework

4.1 Numerical Chirikov Overlap of Beam-Beam Resonances

`http://www.rhichome.bnl.gov/AP/Java/beamtune.html`

Open the beam-beam map Java example on the website in the URL above and play with it a bit. This is a simplified model of the beam-beam interaction of two beams as was discussed in class. Keeping the tune at $Q = 0.331$, set the beam-beam tune shift ξ to zero, and click within the black area to launch particles and produce Poincaré plots. With no nonlinearity, these are all horizontal lines, consistent with constant action.

(a) Produce phase space plots by gradually increasing ξ by 0.001. With What ξ do you start to see resonance islands?

(b) Vary ξ up to about 0.03. How do the resonance island locations and widths seem to scale with ξ? What other harmonics of phase space distortion appear at small, medium, and large amplitudes?

(c) As ξ gets even larger, you will see more and more resonance islands appear at small amplitudes. These islands remain small, but at some point ξ is large enough that resonances start to overlap, and stochastic motion occurs consistent with the Chirikov overlap criterion. Experimentally find the lowest value of ξ where stochasticity occurs to two significant figures. (Hint: It's between 0 and 1.)

(d) **Bonus:** Using the equations on this web page, determine the amplitudes of stable and unstable fixed points for $Q = 0.331$ and $\xi = 0.01$. Is this consistent with the claim that the displayed amplitude ranges from 0-6?

Accelerator Physics: Midterm Exam

Friday, January 18 2007

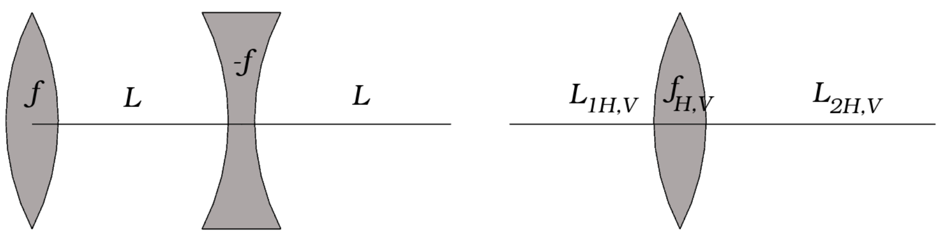
1 Zero Trace Matrix

(10 points) Show that if M is a 2x2 matrix with unit determinant and Tr(M)=0, then $M^2 = -I$.

2 Multiple FODO Cell Concatenation

(10 points) Consider a FODO cell with phase advance of $2\pi/n$ in each plane. Show that the matrix of the concatenation of n of these FODO cells is I.

3 FODO Cell Equivalence



(15 points) Consider a FODO cell with drift lengths $L/2$ and quadrupole focal lengths $\pm f$ as shown on the left. The transport matrix of this FODO lattice, starting with the focusing quadrupole, was given in class as

$$M_{\mathrm{FODO}}(\text{horizontal}) = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \tag{3.1}$$

$$= \begin{pmatrix} 1 - \frac{L}{2f} - \frac{L^2}{4f^2} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 + \frac{L}{2f} \end{pmatrix} \tag{3.2}$$

in the horizontal plane.

(a) Show that M_{FODO}(horizontal) can be written as M_{OFO}(horizontal) – that is, as the horizontal transport matrix of a single quadrupole of focal length f_H between two straight sections of (possibly different) lengths L_{1H} and L_{2H}, as shown on the right. How do (f_H, L_{1H}, L_{2H}) relate to (f, L) of the FODO cell?

(b) Show that the vertical transport matrix of the same FODO cell, M_{FODO}(vertical) can be written as M_{OQO}(vertical) with quadrupole focal length f_V and lengths L_{1V} and L_{2V}. How do these relate to the horizontal OFO cell parameters found in part (a)?

4 Beta Function At A Waist

(20 points) The transport of the envelope function $w(s)$ from a local minimum of value w_0 through a drift space of length s is given by:

$$\begin{pmatrix} w(s) \\ w'(s) + \frac{i}{w(s)} \end{pmatrix} e^{i\Delta\psi} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ \frac{i}{w_0} \end{pmatrix} \tag{4.1}$$

(a) Show that

$$w^2(s) = w_0^2 + \frac{s^2}{w_0^2} \tag{4.2}$$

or, equivalently,

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0} \tag{4.3}$$

Thus the beta function near a waist (or local minimum) in a drift region is quadratic.

(b) From (4.1), calculate the phase advance $\Delta\psi$ as $s \to \infty$. This is half of the maximum phase advance of a field-free region.

5 Solenoid Transport Matrix

(25 points) Consider a solenoid of length L with only longitudinal field B_s. The torsion to decouple the Hamiltonian is

$$\kappa = -\frac{eB_s}{2pc} \tag{5.1}$$

and the resulting Hamiltonian in rotating frame of reference canonical coordinates (x, π_x) and (y, π_y) is

$$H(x, \pi_x, y, \pi_y) = \frac{\pi_x^2 + \pi_y^2}{2} + \kappa^2 \frac{x^2 + y^2}{2} \tag{5.2}$$

Find the transport matrix of this solenoid.

6 Azimuthally Symmetric Optics

(30 points) Consider an azimuthally symmetric ring with orbit radius $\rho = \frac{1}{K_0}$ and a field gradient

$$\frac{e}{p_0 c} \frac{\partial B_y}{\partial x} = -nK_0^2 \tag{6.1}$$

where n is known as the field index. The Hamiltonian for transverse motion does not depend on s, and is given by

$$H(x, \pi_x, y, \pi_y) = \frac{\pi_x^2 + \pi_y^2}{2} + K_0^2(1-n)\frac{x^2}{2} + nK_0^2\frac{y^2}{2} \tag{6.2}$$

(a) Find the one-turn matrices for horizontal and vertical motion.

(b) Find the horizontal and vertical tunes, $\nu_{x,y}$, and show that $\nu_x^2 + \nu_y^2 = 1$.

(c) Find the beta function of this storage ring.