Use of Null Vectors (Corrector Ironing)

When there are more correctors than BPMS a multiplicity of solutions is possible. The response matrix R now has dimensionality m < n (more columns than rows). For simplicity, assume R has rank m. Since (rank of the column space) = (rank of the row space), the additional n - m column vectors do not extend the dimensionality. The singular value decomposition looks like:

$$\begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & \boldsymbol{R} & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \\ \\ \boldsymbol{\theta} \\ \\ \\ \end{bmatrix}$$

$$\boldsymbol{R} = \boldsymbol{U} \cdot \boldsymbol{W} \cdot \boldsymbol{V}^T = \begin{bmatrix} | & | & | \\ \boldsymbol{u}_p & | & \boldsymbol{u}_n \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{w} & | & 0 \\ - & + & - \\ 0 & | & 0 \end{bmatrix} \cdot \begin{bmatrix} - & \boldsymbol{v}_p & - \\ - & - & -- \\ - & \boldsymbol{v}_n & - \end{bmatrix}$$

The corresponding sets of orbit- and corrector-eigenvectors have been partitioned into *particular* eigenvectors and *null* eigenvectors. Application of a linear combination of particular corrector-eigenvectors is observable at the BPMS. Application of a null corrector-eigenvector is not observable at the BPMS (hence the zero singular value). In terms of linear algebra, the particular corrector-eigenvectors span the row space of R while the remaining vectors span the complementary null space of R.

Typically, the total corrector pattern in a storage ring may contain both particular eigenvector and null eigenvector components:

$$\boldsymbol{\theta}_t = \sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p + \sum_i \boldsymbol{b}_j \boldsymbol{v}_j^n$$

The null vector components are sometimes called 'fighting' correctors and can be reduced or eliminated. The orbit distortion caused by the correctors is

$$\boldsymbol{x} = \boldsymbol{R} \cdot \boldsymbol{\theta}_t = (\boldsymbol{U} \cdot \boldsymbol{W} \cdot \boldsymbol{V}^T)(\sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p + \sum_i \boldsymbol{b}_j \boldsymbol{v}_j^n)$$

Since the null vectors do not contribute to observable orbit distortion

$$\boldsymbol{x} = (\boldsymbol{U} \cdot \boldsymbol{W} \cdot \boldsymbol{V}^T) \sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p$$

or, by applying the orthonormal properties of vectors $\mathbf{v}_i$,

$$\boldsymbol{x} = \sum_i (\boldsymbol{w}_i \boldsymbol{a}_i) \boldsymbol{u}_i \; .$$

In practice, we can replace the complete corrector set $\boldsymbol{\theta}_t = \sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p + \sum_i \boldsymbol{b}_j \boldsymbol{v}_j^n$ with a reduced set

$$\boldsymbol{\theta}_p = \sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p$$

This procedure is often referred to as 'corrector ironing' [Ziemann].

To isolate to scalar coefficients a_i, use SVD matrix mechanics as before:

$$\boldsymbol{x} = \boldsymbol{R} \cdot \boldsymbol{\theta}_p = (\boldsymbol{U} \cdot \boldsymbol{W} \cdot \boldsymbol{V}^T)(\sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p)$$

$$(\boldsymbol{V} \cdot \boldsymbol{W}^{-1} \cdot \boldsymbol{U}^T) \cdot \boldsymbol{x} = \sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p \qquad \text{(matrix inverse)}$$

$$\boldsymbol{V} \cdot \boldsymbol{W}^{-1} \cdot \begin{bmatrix} - & \boldsymbol{u}_1 & - \\ & \dots & \\ - & \boldsymbol{u}_n & - \end{bmatrix} \cdot \boldsymbol{x} = \sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p \qquad \text{(expand } \mathbf{U}\text{)}$$

$$\boldsymbol{V} \cdot \begin{bmatrix} \boldsymbol{w}_1^{-1} & & 0 \\ & \dots & \\ 0 & & \boldsymbol{w}_n^{-1} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u}_1 \cdot \boldsymbol{x} \\ | \\ \boldsymbol{u}_n \cdot \boldsymbol{x} \end{bmatrix} = \sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p \qquad \text{(multiply x, expand } \mathbf{W}\text{)}$$

$$\begin{bmatrix} | & & | \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_m \\ | & & | \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{w}_1^{-1} \boldsymbol{u}_1 \cdot \boldsymbol{x} \\ | \\ \boldsymbol{w}_n^{-1} \boldsymbol{u}_n \cdot \boldsymbol{x} \end{bmatrix} = \sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p \qquad \text{(expand } \mathbf{V}\text{)}$$

$$\boldsymbol{a}_i = \frac{\boldsymbol{u}_i \cdot \boldsymbol{x}}{\boldsymbol{w}_i} \qquad \text{(equate coeffecients of } \mathbf{V}\text{)}$$

In physical terms, the measured orbit is projected into the non-null eigenvectors (row-space) and scaled by the corresponding singular value to find the coefficients a_i. These coefficients are used to generate the corrector vector $\boldsymbol{\theta}_p = \sum_i \boldsymbol{a}_i \boldsymbol{v}_i^p$. The new corrector set contains no null-vector components.

An example of where this technique was used on SPEAR is the case where the position of 10 photon beams positions were held constant while the strength of approximately 30 corrector magnets were reduced (10 constraints, 30 variables).

Note - to see that the solution θ_p is the minimum-square corrector strength set, compute the modulus of the corrector vector:

$$(\boldsymbol{\theta}_p + \boldsymbol{\theta}_n) \cdot (\boldsymbol{\theta}_p + \boldsymbol{\theta}_n) = \boldsymbol{\theta}^2{}_p + 2\boldsymbol{\theta}_p \cdot \boldsymbol{\theta}_n + \boldsymbol{\theta}^2{}_n$$

Since the θ_n component is rejected, the expression is minimized.