Primer in Special Relativity

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Relative Motion

Physical phenomena appear different for observers in different systems of reference. Yet, the laws of nature must be independent of the reference system. In classical mechanics we transform physical laws from one to another system of reference by way of the Galileo transformation $z^* = z - vt$ assuming that one system moves with velocity v along the z-axis of the other system.

As the velocities of bodies under study became faster, it became clear that this simple transformation must be modified. Maxwell's equations result in a velocity of electromagnetic waves equal to the velocity of light and do not contain any reference to a specific system of reference. Any attempt to find a variation of the "velocity of light" with respect to moving reference systems failed, most notably Michelson's experiment. The velocity of light is finite and it's value is

$$c = 299,792,458 \; m/s$$

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Lorentz Transformation

Any new transformation laws must include the observation that no element of energy can travel faster than the speed of light. The new transformation formulae combine space and time and are for a reference system moving with velocity v along the z-axis with respect to the inertial system.

$$\begin{aligned} x^* &= x \,, \\ y^* &= y \,, \\ z^* &= \frac{z - \beta_z\, ct}{\sqrt{1-\beta_z^2}} = \gamma\,(z - \beta_z\, ct) \;, \\ ct^* &= \frac{ct - \beta_z\, z}{\sqrt{1-\beta_z^2}} = \gamma\;(ct - \beta_z\, z) \;, \end{aligned}$$

where $\beta_z = v_z/c$, $\gamma = 1/\sqrt{1-\beta_z^2}$ and quantities designated with * are measured in the moving system L^*. These **Lorentz transformations** can be expressed in matrix formulation by

$$\begin{pmatrix} x^* \\ y^* \\ z^* \\ ct^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta\gamma \\ 0 & 0 & -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} .$$

Electromagnetic fields transform also in a special way between reference systems in relative motion:

$$\begin{pmatrix} E_x^* \\ E_y^* \\ E_z^* \\ cB_x^* \\ cB_y^* \\ cB_z^* \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & 0 & -\beta_z\gamma & 0 \\ 0 & \gamma & 0 & \beta_z\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \beta_z\gamma & 0 & \gamma & 0 & 0 \\ -\beta_z\gamma & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ cB_x \\ cB_y \\ cB_z \end{pmatrix} .$$

For example, a pure static magnetic field in the laboratory system $\mathcal{L}$ becomes an electromagnetic field in the electron system $\mathcal{L}^*$. An undulator field, therefore, looks to an electron the same as, for example, a laser field and both interactions can be described by Compton scattering.

Lorentz Contraction

Characteristic for relativistic mechanics is the Lorentz contraction and time dilatation, both of which become significant in the description of particle dynamics. To describe the Lorentz contraction, we consider a rod at rest in $\mathcal{L}$ along the z-coordinate with the length $\Delta z = z_2 - z_1$. In the system $\mathcal{L}^*$ this rod appears to have the length $\Delta z^* = z_2^* - z_1^*$ related to the length in the $\mathcal{L}-$system by

$$\Delta z \;=\; \gamma\,(z_2^* + v_z t^*) - \gamma\,(z_1^* + v_z t^*) \;=\; \gamma\,\Delta z^*$$

or

$$\Delta z \;=\; \gamma\,\Delta z^* .$$

The rod appears shorter in the moving particle system by the factor γ and is longest in it's rest system. For example, the periodicity of an undulator λ_p becomes Lorentz contracted to $\lambda_\mathrm{p}/\gamma$ for relativistic electrons.

Time Dilatation

Similarly, we may derive the time dilatation or the elapsed time between two events occurring at the same point in both coordinate systems. From the Lorentz transformations we get with $z_2^* = z_1^*$

$$\Delta t \;=\; t_2 - t_1 \;=\; \gamma\,\left(t_2^* + \frac{\beta_z z_2^*}{c}\right) - \gamma\,\left(t_1^* + \frac{\beta_z z_1^*}{c}\right)$$

or

$$\Delta t \;= \gamma\,\Delta t^* .$$

As a consequence, high energy, unstable particles, like pions and muons, live longer in the laboratory system as measured.

Space-Time

Imagine a light flash to originate at the origin of the coordinate system (x, y, z). At the time t, the edge of this expanding light flash has expanded with the velocity of light to

$$x^2 + y^2 + z^2 = c^2 t^2.$$

Applying a Lorentz transformation from the laboratory system $\mathcal{L}$ to a system $\mathcal{L}^*$ gives

$$x^{*2} + y^{*2} + z^{*2} = c^2 t^{*2}$$

demonstrating the invariance of the velocity of light.

4-Vector

Minkowski combined space and time to form a four-dimensional space-time coordinate system. The components of the **space-time 4-vector** are

$$\tilde{\mathbf{s}} = \left(x^0, x^1, x^2, x^3\right) = (\mathrm{i}ct, x, y, z)\,,$$

where the time component has been multiplied by c to give all components the same dimensions.From the **World time** defined as

$$\tau = \sqrt{-\tilde{\mathbf{s}}^2}$$

we get

$$\begin{aligned} c\,\mathrm{d}\tau &= \sqrt{c^2\,(\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2} \\ &= \sqrt{c^2 - \left(v_x^2 + v_y^2 + v_z^2\right)}\mathrm{d}t \\ &= \sqrt{c^2 - v^2}\,\mathrm{d}t = \sqrt{1-\beta^2}\,c\,\mathrm{d}t, \end{aligned}$$

a relation we known from the Lorentz transformation as time dilatation $\mathrm{d}\tau = \frac{1}{\gamma}\,\mathrm{d}t$. We may now form a velocity 4-vector or the **4-velocity**

$$\tilde{\mathbf{v}} = \frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}t} = \gamma\,(\mathrm{i}c, \dot{x}, \dot{y}, \dot{z})\ .$$

We may also derive the **4-acceleration**

$$\tilde{\mathbf{a}} = \frac{\mathrm{d}\tilde{\mathbf{v}}}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}t}\right)$$

or in component form $\tilde{\mathbf{a}} = (\mathrm{i}\,\tilde{a}_t, \tilde{a}_x, \tilde{a}_y, \tilde{a}_z)$ we get $\tilde{a}_x = \gamma^2 a_x + \gamma^4 \beta_x\,(\boldsymbol{\beta}\cdot\mathbf{a})$, where $\mathbf{a}$ is the ordinary acceleration. The other components can be obtained in a similar way.

An important 4-vector is the 4-momentum or energy-momentum 4-vector defined by

$$c\tilde{\mathbf{p}} = (\mathrm{i}E, cp_x, cp_y, cp_z).$$

Invariance to Lorentz Equations

Remarkably, the length of a 4-vector, or the product of any two 4-vectors is Lorentz invariant. The proof is straight forward calculating the length of a 4-vector in one system and applying a Lorentz transformation. Particularly, calculating the length of the momentum we get

$$-E^2 + c^2 p_x^2 + c^2 p_y^2 + c^2 p_z^2 = -mc^2,$$

where we have set $E_0 = mc^2$ for a particle at rest. The rest mass of a particle is Lorentz invariant. From this we get

$$E^2 = c^2 p^2 + m^2 c^4.$$

From the velocity 4-vector we get $-c^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = -c^2$ or the invariance of the velocity of light. Light travels at the velocity c independent from the system of reference. That is the reason why the flash of light discussed earlier expands like a sphere seen from any system of reference.

Energy - Momentum

Einstein tells us that the total rest energy of a particle is

$$E_0 = mc^2,$$

and the relativistic factor

$$\gamma_0 = \frac{E_0}{mc^2} = 1.$$

Momentum and total energy of a particle are connected by

$$E^2 = (cp)^2 + \left(mc^2\right)^2,$$

and the velocity is

$$\beta = \sqrt{1 - \gamma^{-2}}.$$

Combining the last two equations we get

$$cp = \beta\, E.$$

Spatial and Spectral Distribution of Radiation

Although the acceleration and the creation of radiation fields is not periodic, we may Fourier-decompose the radiation pulse and obtain a spectrum of plane waves

$$E^* = E_0^* \mathrm{e}^{\mathrm{i}\,\Phi^*} ,$$

where the phase is defined by

$$\Phi^* = \omega^* \left[t^* - \tfrac{1}{c}\left(n_x^* x^* + n_y^* y^* + n_z^* z^*\right)\right] .$$

The phase of an electromagnetic wave is proportional to the product of the momentum-energy and space-time 4-vectors and its length is therefore invariant under Lorentz transformations. We have

$$\omega^* \left[ct^* - n_x^* x^* - n_y^* y^* - n_z^* z^*\right] = \omega \left[ct - n_x x - n_y y - n_z z\right]$$

between the phases as measured in both the laboratory $\mathcal{L}$ and the particle frame of reference $\mathcal{L}^*$. To derive the relationships between similar quantities in both systems, we use the Lorentz transformations noting that the particle reference frame is the frame, where the particle or radiation source is at rest. We use the Lorentz transformations to replace the coordinates (x^*, y^*, z^*, ct^*) by those in the laboratory system. Since the space-time coordinates are independent from each other, we may equate their coefficients on either side of the equation separately.

Spectral distribution

In so doing, we get from the ct-coefficients for the oscillation frequency

$$\omega^* \gamma \left(1 + \beta_z\, n_z^*\right) = \omega ,$$

which expresses the **relativistic Doppler effect**. Looking parallel to the direction of particle motion $n_z^* = 1$ the observed oscillation frequency is increased by the factor $(1 + \beta_z)\,\gamma \approx 2$ for highly relativistic particles. The Doppler effect is reduced if the radiation is viewed at some finite angle Θ with respect to the direction of motion of the source. In these cases $n_z^* = \cos\Theta^*$.

Spatial Distribution

Similarly, we obtain also the transformation of spatial directions

$$\begin{aligned} n_x &= \frac{n_x^*}{\gamma\,(1 + \beta_z\, n_z^*)} , \\ n_y &= \frac{n_y^*}{\gamma\,(1 + \beta_z\, n_z^*)} , \\ n_z &= \frac{\beta_z + n_z^*}{(1 + \beta_z\, n_z^*)} . \end{aligned}$$

These transformations define the spatial distribution of radiation in the laboratory system. In case of transverse acceleration the radiation in the particle rest frame is distributed like $\cos^2 \Theta^*$ about the direction of motion. This distribution becomes greatly collimated into the forward direction in the laboratory system. With $n_x^{*2} + n_y^{*2} = \sin^2 \Theta^*$ and $n_x^2 + n_y^2 = \sin^2 \Theta \approx \Theta^2$ and $n_z^* = \cos \Theta^*$ we find

$$\Theta \approx \frac{\sin \Theta^*}{\gamma(1 + \beta \cos \Theta^*)} .$$

In other words, radiation from relativistic particles, emitted in the particle system into an angle $-\pi/2 < \Theta^* < \pi/2$ appears in the laboratory system highly collimated in the forward direction within an angle of

$$\Theta \approx \frac{1}{\gamma} .$$

This angle is very small for highly relativistic electrons like those in a storage ring, where γ is of the order of $10^3 - 10^4$.