Classical Mechanics and Electro-Magnetic Theory

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Literature

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Useful vector relations

$$\nabla(\mathbf{a}\varphi) = \varphi\nabla\mathbf{a} + \mathbf{a}\nabla\varphi$$

$$\nabla \times (\mathbf{a}\varphi) = \varphi(\nabla \times \mathbf{a}) - \mathbf{a} \times \nabla\varphi$$

$$\nabla(\mathbf{a} \times \mathbf{b}) = \mathbf{b}(\nabla \times \mathbf{a}) - \mathbf{a}(\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b}\nabla)\mathbf{a} - (\mathbf{a}\nabla)\mathbf{b} + \mathbf{a}(\nabla\mathbf{b}) - \mathbf{b}(\nabla\mathbf{a})$$

$$\nabla(\mathbf{ab}) = (\mathbf{b}\nabla)\mathbf{a} + (\mathbf{a}\nabla)\mathbf{b} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \times (\nabla\varphi) = 0$$

$$\nabla(\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla\mathbf{a}) - \Delta\mathbf{a}$$

$$\mathbf{a}(\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{c} \times \mathbf{a}) = \mathbf{c}(\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{ac}) - \mathbf{c}(\mathbf{ab})$$

$$(\mathbf{a} \times \mathbf{b})(\mathbf{c} \times \mathbf{d}) = (\mathbf{ac})(\mathbf{bd}) - (\mathbf{bc})(\mathbf{ad})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{c}[(\mathbf{a} \times \mathbf{b})\mathbf{d}] - \mathbf{d}[(\mathbf{a} \times \mathbf{b})\mathbf{c}]$$

Coordinate Transformations

cartesian coordinates	new coordinates
$x(u,v,w), y(u,v,w), z(u,v,w)$	(u,v,w)
$ds^2 = dx^2 + dy^2 + dz^2$	$ds^2 = \frac{du^2}{U^2} + \frac{dv^2}{V^2} + \frac{dw^2}{W^2}$
(dx, dy, dz)	$\left(\frac{du}{U}, \frac{dv}{V}, \frac{dw}{W}\right)$
$d\tau = dx \cdot dy \cdot dz$	$d\tau = \frac{du}{U} \cdot \frac{dv}{V} \cdot \frac{dw}{W}$
$\nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}\right)$	$\nabla\psi = \left(U\frac{\partial\psi}{\partial u}, V\frac{\partial\psi}{\partial v}, W\frac{\partial\psi}{\partial w}\right)$
$\nabla\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$	$\nabla\mathbf{a} = UVW\left[\frac{\partial}{\partial u}\left(\frac{a_u}{VW}\right) + \frac{\partial}{\partial v}\left(\frac{a_v}{UW}\right) + \frac{\partial}{\partial w}\left(\frac{a_w}{UV}\right)\right]$
$\nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}, \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$	$\nabla\times\mathbf{a} = \left(VW\left[\frac{\partial}{\partial v}\left(\frac{a_w}{W}\right) - \frac{\partial}{\partial w}\left(\frac{a_v}{V}\right)\right], \text{etc.}\right)$
$\Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$	$\Delta\psi = UVW\left[\frac{\partial}{\partial u}\left(\frac{U}{VW}\frac{\partial\psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{V}{UW}\frac{\partial\psi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{W}{UV}\frac{\partial\psi}{\partial w}\right)\right]$

where

$$U^{-1} = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}, \quad V^{-1} = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2}, \quad W^{-1} = \sqrt{\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2}$$

Integral Relations

$$\int_V \nabla\varphi \, \mathrm{d}\mathbf{r} = \oint_S \varphi \hat{\mathbf{u}} \, \mathrm{d}\sigma$$

$$\int_V \nabla \mathbf{a} \, \mathrm{d}\mathbf{r} = \oint_S \mathbf{a} \hat{\mathbf{u}} \, \mathrm{d}\sigma$$ Gauss's Law

$$\int_S (\nabla \times \mathbf{a}) \, \hat{\mathbf{u}} \, \mathrm{d}\sigma = \oint a \, \mathrm{d}\mathbf{s}$$ Stokes' Law

Primer in Electromagnetism

EM-fields, Maxwell's equations

Maxwell's Equations:

$$\nabla \mathbf{E} = \frac{4\pi}{[4\pi\epsilon_0]\epsilon_r}\rho$$ Coulomb's law

$$\nabla \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{[c]}{c}\frac{\partial \mathbf{B}}{\partial t}$$ Faraday's law

$$\nabla \times \mathbf{B} = \frac{4\pi\,\mu_r}{[4\pi c\epsilon_0]}\rho\boldsymbol{\beta} + \frac{\epsilon_r\,\mu_r}{[c]c}\frac{\partial \mathbf{E}}{\partial t}$$ Ampere's law

Lorentz Force:

$$\mathbf{F} = q\mathbf{E} + [c]\frac{q}{c}[\mathbf{v} \times \mathbf{B}]$$

(we will derive the Lorentz force equation from the Lagangian later)

Note! Use factors in [..] for MKS system and ignore [...] for cgs system

cgs-mks unit conversion

	cgs	mks
potential, electric field	$V, \mathbf{E}\vert_{\mathrm{cgs}}$	$\sqrt{4\pi\epsilon_0}\, V, \mathbf{E}\vert_{\mathrm{mks}}$
current, charge, densities	$I, j, q, \rho\vert_{\mathrm{cgs}}$	$\frac{1}{\sqrt{4\pi\epsilon_0}} I, j, q, \rho\vert_{\mathrm{mks}}$
magnetic inductance	$\mathbf{B}\vert_{\mathrm{cgs}}$	$\frac{\sqrt{4\pi}}{\sqrt{\mu_0}} \mathbf{B}\vert_{\mathrm{mks}}$
magnetic field	$H\vert_{\mathrm{cgs}}$	$\sqrt{4\pi\mu_0}\, H\vert_{\mathrm{mks}}$

$$\sqrt{\epsilon_0\mu_0} = \frac{1}{c}$$

constants

dielectric constant in vacuum

$$\epsilon_0 = \frac{10^7}{4\pi c^2}\frac{\mathrm{C}}{\mathrm{V\,m}} = 8.854187817\times 10^{-12}\frac{\mathrm{C}}{\mathrm{V\,m}}$$

vacuum permeability

$$\mu_0 = 4\pi\times 10^{-7}\frac{\mathrm{V\,s}}{\mathrm{A\,m}} = 1.256637061\times 10^{-6}\frac{\mathrm{V\,s}}{\mathrm{A\,m}}$$

in material environment $\epsilon_0\epsilon_\mathrm{r}\mu_0\mu_\mathrm{r}v^2 = 1$

velocity of light in matter $v = \frac{c}{\sqrt{\epsilon_\mathrm{r}\mu_\mathrm{r}}}$

in vacuum $\epsilon_0\mu_0 c^2 = 1$

vector/scalar potentials (cgs)

derive magnetic fields from
vector potential $\mathbf{A}$

$$\nabla\mathbf{B} = 0 \longrightarrow \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

because $\nabla(\nabla \times \mathbf{A}) = 0$

$$\nabla \times \mathbf{E} = -\tfrac{1}{c}\dot{\mathbf{B}} = -\tfrac{1}{c}\nabla \times \dot{\mathbf{A}} \longrightarrow \nabla \times (\mathbf{E} + \tfrac{1}{c}\dot{\mathbf{A}}) = 0$$

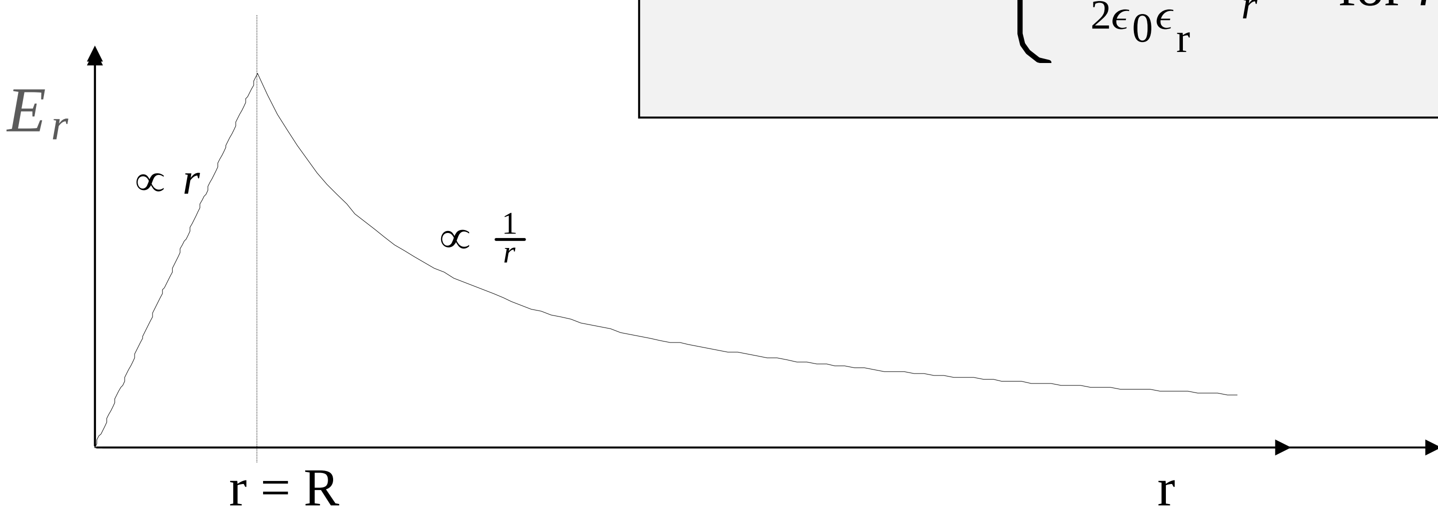
$$\boxed{\mathbf{E} = -\frac{[c]}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi}$$

φ : **scalar potential**

because $\nabla \times \nabla\varphi = 0$

electrical field within and outside a uniformly charged particle beam

$$E_r(r) = \begin{cases} \dfrac{\rho_0}{2\epsilon_0\epsilon_r} r & \text{for } r < R \\ \dfrac{\rho_0}{2\epsilon_0\epsilon_r} \dfrac{R^2}{r} & \text{for } r > R \end{cases}$$



magnetic field of a uniformly charged particle beam

start from $\nabla \times \mathbf{B} = \mu_0 \mu_\mathrm{r} \mathbf{j}$ (mks)

use Stokes' theorem and get

$$B_\varphi(r) = \begin{cases} \dfrac{j_0}{2\mu_0\mu_\mathrm{r}} r & \text{for } r < R \\ \dfrac{j_0}{2\mu_0\mu_\mathrm{r}} \dfrac{R^2}{r} & \text{for } r > R \end{cases}$$

Equations of Motion - 1

Lorentz force $\mathbf{F} = q\mathbf{E} + [c]\frac{q}{c}[\mathbf{v} \times \mathbf{B}]$

$$\left.\begin{aligned} \Delta\mathbf{p} &= \int \mathbf{F}_{\mathrm{L}}\mathrm{d}t \\ \Delta E_{\mathrm{kin}} &= \int \mathbf{F}_{\mathrm{L}}\mathrm{d}\mathbf{s} \end{aligned}\right\} \xrightarrow{\mathrm{d}\mathbf{s} = \mathbf{v}\mathrm{d}t} c\beta\Delta\mathbf{p} = \Delta E_{\mathrm{kin}}$$

$$\Delta E_{\mathrm{kin}} = \int \mathbf{F}_{\mathrm{L}}\mathrm{d}\mathbf{s} = \int \left(q\mathbf{E} + q\frac{[c]}{c}[\mathbf{v} \times \mathbf{B}] \right) \mathrm{d}\mathbf{s} = q\int \mathbf{E}\mathrm{d}\mathbf{s} + q\frac{[c]}{c}\int \underbrace{[\mathbf{v} \times \mathbf{B}]\mathbf{v}}_{=0}\mathrm{d}t$$

no work done by magnetic field!

equation of motion

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma A m \mathbf{v}) = eZ\mathbf{E} + eZ\frac{[c]}{c}[\mathbf{v} \times \mathbf{B}]}$$

A atomic number; Z charge multiplicity

Vector and scalar potential for a moving charge (cgs)

derive fields from vector- and scalar potential:

$$\mathbf{B} = \nabla \times \mathbf{A}$$
$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$$

insert into Maxwell's curl-equation (Ampere's law) $\frac{1}{\mu_r}\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{\epsilon_r}{c}\dot{\mathbf{E}}$

$$\frac{1}{\mu_r}\nabla \times (\nabla \times \mathbf{A}) = \frac{4\pi}{c}\mathbf{j} + \frac{\epsilon_r}{c}\left(-\frac{1}{c}\ddot{\mathbf{A}} - \nabla\dot{\varphi}\right)$$

with $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \mathbf{A}) - \nabla^2 \mathbf{A}$

Lorentz gauge

$$\nabla^2 \mathbf{A} - \frac{\epsilon_r \mu_r}{c^2}\ddot{\mathbf{A}} = -\frac{4\pi}{c}\mu_r \mathbf{j} - \nabla \underbrace{\left(\nabla \mathbf{A} + \frac{\epsilon_r \mu_r}{c}\dot{\varphi}\right)}_{=0}$$

Wave Equation

$$\nabla^2 \mathbf{A} - \frac{\epsilon_r \mu_r}{c^2}\ddot{\mathbf{A}} = -\frac{4\pi}{c}\mu_r \mathbf{j}$$

Wave Equation

in mks system $\nabla^2 \mathbf{A} - \epsilon_r \mu_r \epsilon_0 \mu_0 \ddot{\mathbf{A}} = -\mu_r \mu_0 \mathbf{j}$

similarly $\nabla^2 \varphi - \epsilon_r \mu_r \epsilon_0 \mu_0 \ddot{\varphi} = -\frac{\rho}{\epsilon_r \epsilon_0}$

what if $\ddot{\mathbf{A}} = \ddot{\varphi} = 0$?

electro-static fields $\nabla^2 \mathbf{A} = -\mu_r \mu_0 \mathbf{j}$ $\mathbf{B} = \nabla \times \mathbf{A}$

magneto-static fields $\nabla^2 \varphi = -\frac{\rho}{\epsilon_r \epsilon_0}$ $\mathbf{E} = -\nabla \varphi$

potentials for a moving charge

solutions:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi c^2 \epsilon_0} \int \frac{\mathbf{j}(x,y,z)}{R}\bigg|_{t_\mathrm{r}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\varphi(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(x,y,z)}{R}\bigg|_{t_\mathrm{r}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

R distance to observation point at retarded time !

and

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi c \epsilon_0} \frac{q}{R} \frac{\boldsymbol{\beta}}{1+\mathbf{n}\cdot\boldsymbol{\beta}}\bigg|_{t_\mathrm{r}}$$

$$\varphi(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \frac{q}{R} \frac{1}{1+\mathbf{n}\cdot\boldsymbol{\beta}}\bigg|_{t_\mathrm{r}}$$

Lienard – Wiechert potentials

Energy Conservation

Lorentz force $\mathbf{F}_{\mathrm{L}} = e\mathbf{E} + [c]\frac{e}{c}[\mathbf{v} \times \mathbf{B}] = e\mathbf{E} + [c]e\boldsymbol{\beta}\mathbf{B}$

rate of work: $\mathbf{F}_{\mathrm{L}}\mathbf{v} = (e\mathbf{E} + e[\mathbf{v} \times \mathbf{B}])\mathbf{v}$

$$\left.\begin{array}{l} [\mathbf{v} \times \mathbf{B}] \cdot \mathbf{v} = 0 \\ e\mathbf{E} \cdot \mathbf{v} = \mathbf{j} \cdot \mathbf{E} \end{array}\right\} \quad \int \mathbf{jE}\mathrm{d}V = \epsilon_0 \int (c^2\nabla \times \mathbf{B} - \dot{\mathbf{E}}) \cdot \mathbf{E}\,\mathrm{d}V$$

with $\nabla(\mathbf{a} \times \mathbf{b}) = \mathbf{b}(\nabla \times \mathbf{a}) - \mathbf{a}(\nabla \times \mathbf{b})$

$$\int \mathbf{j} \cdot \mathbf{E}\,\mathrm{d}V = \epsilon_0 \int \left[c^2\mathbf{B}\,\underbrace{\nabla \times \mathbf{E}}_{=-\dot{\mathbf{B}}} - c^2\nabla(\mathbf{E} \times \mathbf{B}) - \dot{\mathbf{E}}\mathbf{E} \right] \mathrm{d}V$$

$$= -\int \left[\frac{\mathrm{d}u}{\mathrm{d}t} + c^2\epsilon_0\nabla(\mathbf{E} \times \mathbf{B}) \right] \mathrm{d}V,$$

with field energy density:

$$u = \frac{\epsilon_0}{2}(E^2 + [c^2]B^2)$$

$$= \frac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2)$$

Poynting vector

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int u\,\mathrm{d}V}_{\text{change of field energy}} + \int \mathbf{j}\cdot\mathbf{E}\,\mathrm{d}V + \underbrace{\oint \mathbf{S}\cdot\mathbf{n}\,\mathrm{d}s}_{\text{radiation loss through closed surface } s} = 0$$

change of field energy

particle energy loss or gain

radiation loss through closed surface s

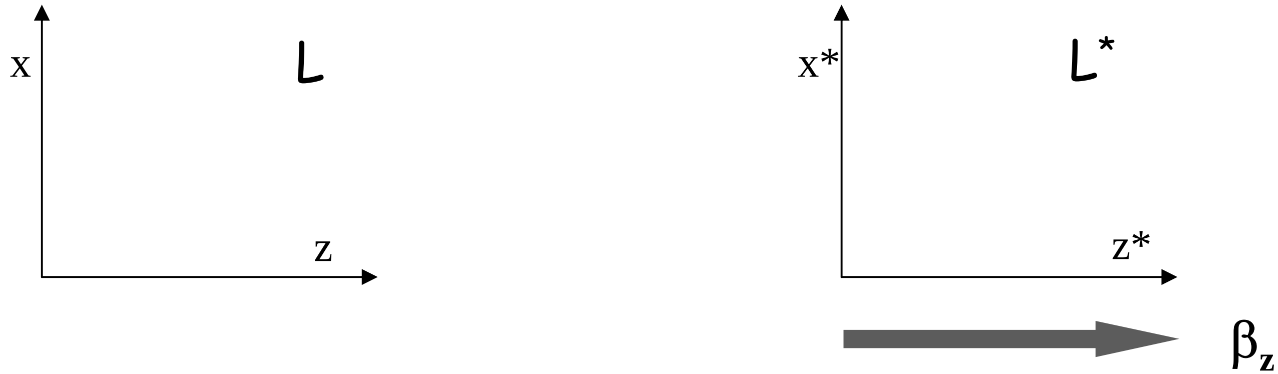
Poynting Vector: $\mathbf{S} = c^2\epsilon_0(\mathbf{E}\times\mathbf{B}) = c\epsilon_0\mathbf{E}^2\,\mathbf{n}$

since $(\mathbf{E}\perp\mathbf{n},\ \mathbf{B}\perp\mathbf{n})$ and $\mathbf{n}\times\mathbf{E} = c\mathbf{B}$

vectors **E,B,S** form a right handed orthogonal system

Primer in Special Relativity

Lorentz transformation:



laboratory system

moving system

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \beta\gamma \\ 0 & 0 & \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \\ ct^* \end{pmatrix}$$

$$\beta = \tfrac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

Contraction-dilatation

Lorentz contraction:

consider rod in lab system of length $\Delta z = z_2 - z_1$

$$\Delta z = \gamma\left(z_2^* + v_z t^*\right) - \gamma\left(z_1^* + v_z t^*\right) = \gamma\,\Delta z^*$$

Time dilatation:

consider two events happening at same place

$$\Delta t = t_2 - t_1 = \gamma\left(t_2^* + \frac{\beta z_2^*}{c}\right) - \gamma\left(t_1^* + \frac{\beta z_1^*}{c}\right) = \gamma\,\Delta t^*$$

Lorentz transformations of fields

$$\begin{pmatrix} E_x^* \\ E_y^* \\ E_z^* \\ cB_x^* \\ cB_y^* \\ cB_z^* \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & 0 & -\beta\gamma & 0 \\ 0 & \gamma & 0 & \beta\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \beta\gamma & 0 & \gamma & 0 & 0 \\ -\beta\gamma & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ cB_x \\ cB_y \\ cB_z \end{pmatrix}$$

magnetostatic field in lab system

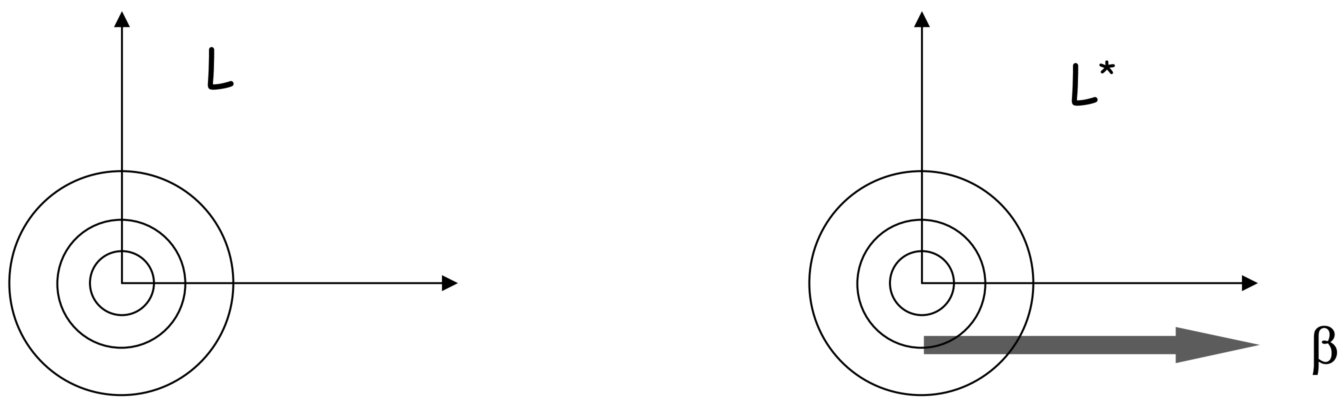
EM field in particle system

$$\begin{aligned} E_x^* &= -\beta\gamma cB_y \\ E_y^* &= 0 \\ E_z^* &= 0 \end{aligned}$$

we don't really know velocities ! can we express β,γ differently?

space – time

imagine a light flash to appear at time $t = 0$ from the origin of the lab coordinate system



at time t edge of light pulse has expanded to $\boxed{x^2 + y^2 + z^2 = c^2 t^2}$

observing from L* - system, we get from Lorentz transformations

$$\boxed{x^{*2} + y^{*2} + z^{*2} = c^2 t^{*2}} \; !$$

velocity of light is Lorentz invariant !

Minkowski combined space-time to
form a 4-dimensional coordinate system:

space – time 4-vector $\tilde{\mathbf{s}} = (x^0, x^1, x^2, x^3) = (\mathrm{i}ct, x, y, z)$ world point

all world points = world

variation of world point = world line

world time is defined by $c\tau = \sqrt{-\tilde{\mathbf{s}}^2}$ which is Lorentz invariant (homework?)

$$\left.\begin{aligned} c\,\mathrm{d}\tau &= \sqrt{c^2(\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2} \\ &= \sqrt{c^2 - (v_x^2 + v_y^2 + v_z^2)}\,\mathrm{d}t \\ &= \sqrt{c^2 - v^2}\,\mathrm{d}t = \sqrt{1-\beta^2}\,c\,\mathrm{d}t, \end{aligned}\right\} \quad \mathrm{d}\tau = \frac{1}{\gamma}\,\mathrm{d}t$$

γ: relativistic factor

length of 4 - vectors

length of 4-vectors is Lorentz invariant

examples $\tilde{\mathbf{s}}^2 = -c^2\tau^2$

actually

product of any two 4-vectors is Lorentz invariant

how do we know a vector is a 4-vector?

if the length of a vector is Lorentz invariant, it's a 4-vector

invariance of 4-vectors

$$\begin{aligned}\tilde{s}^{*2} &= x^{*2} + y^{*2} + z^{*2} - c^2 t^{*2} \\ &= x^2 + y^2 + (\gamma z - \beta\gamma\, ct)^2 - (-\beta\gamma\, z + \gamma\, ct)^2 \\ &= x^2 + y^2 + z^2 - c^2 t^2 \\ &= \tilde{s}^2\end{aligned}$$

any product of two 4-vectors is Lorentz invariant

$$\tilde{a}^* \tilde{b}^* = \tilde{a}\tilde{b}$$

homework?

4-velocity

4-velocity: $\tilde{\mathbf{v}} = \frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}t} = \gamma(\mathrm{i}c, \dot{x}, \dot{y}, \dot{z})$

in moving system $(\gamma = 1; \dot{x} = \dot{y} = \dot{z} = 0)$: $\tilde{\mathbf{v}}^2 = -c^2 = \mathrm{const}$

velocity of light is Lorentz invariant !

$$c = 299{,}792{,}458\ m/s$$

4-acceleration

$$\tilde{a} = \frac{d\tilde{v}}{d\tau} = \gamma \frac{d}{dt}\left(\gamma \frac{d\tilde{s}}{dt}\right)$$

$$\tilde{a}^2 = \gamma^6 \left\{\mathbf{a}^2 - [\boldsymbol{\beta} \times \mathbf{a}]^2\right\} = \tilde{a}^{*2}$$

4-momentum

energy – momentum 4-vector: $c\tilde{\mathbf{p}} = (\mathrm{i}E, cp_x, cp_y, cp_z)$

with $E_0 = Amc^2$

$$c^2\tilde{\mathbf{p}}^2 = -E^2 + c^2p_x^2 + c^2p_y^2 + c^2p_z^2$$

total energy

$$\boxed{E^2 = c^2p^2 + A^2m^2c^4}$$

Relativistic factor depends on particle velocity, but generally we don't know velocities.

look for different expression

relativistic factor

$$\left.\begin{array}{r}(\mathrm{i}E, c\mathbf{p}) \\ \gamma(\mathrm{i}c, \dot{\mathbf{r}})\end{array}\right\} \quad \begin{array}{l}\longrightarrow \quad E^2 = c^2p^2 + A^2m^2c^4 \qquad (1) \\ \longrightarrow \quad -c\gamma E + c\gamma\dot{\mathbf{r}}\mathbf{p} = -cAmc^2\end{array}$$

$$-\gamma E + c\gamma\boldsymbol{\beta}\mathbf{p} = -Amc^2 \longrightarrow cp = \frac{\gamma E - Amc^2}{\gamma\beta}$$ since $\mathbf{p} \parallel \boldsymbol{\beta}$

insert into (1) $\quad E^2 = \left(\frac{\gamma E - Amc^2}{\gamma\beta}\right)^2 + (Amc^2)^2$

with $\beta^2\gamma^2 = \gamma^2 - 1$ we get $E - \gamma Amc^2 = 0$ or

relativistic factor

$$\boxed{\gamma = \frac{E}{Amc^2}}$$

Examples of 4-Vectors

space-time 4-vector: $\tilde{\mathbf{s}} = (\mathbf{r}, \mathrm{i}ct)$

energy-momentum 4-vector $(c\mathbf{p}, \mathrm{i}E)$

EM field 4-vector $(\mathbf{A}, \mathrm{i}\phi)$

derived 4-vectors

velocity 4-vector: $\tilde{\mathbf{v}} = \frac{\mathrm{d}\tilde{\mathbf{r}}}{\mathrm{d}\tau} = \gamma(\dot{\mathbf{r}}, \mathrm{i}c)$

acceleration 4-vector $\tilde{a} = \frac{\mathrm{d}\tilde{v}}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma \frac{\mathrm{d}\tilde{s}}{\mathrm{d}t}\right)$

$$\tilde{a}^2 = \gamma^6 \left\{\mathbf{a}^2 - [\boldsymbol{\beta} \times \mathbf{a}]^2\right\} = \tilde{a}^{*2}$$

conventions

particles: electrons, protons, ions

energy: eV, keV, MeV, GeV

1 eV = kin.energy gained while traveling through potential difference of 1 Volt

momentum: eV/c, keV/c, MeV/c, GeV/c

mostly we use $c\mathbf{p}$ for the momentum

proton rest mass: $m_p c^2 = 938.272$ MeV

electron rest mass: $m_e c^2 = 0.510999$ MeV

$$cp \approx \sqrt{2Amc^2E_{\text{kin}}} = Amc^2\beta \approx cAmv$$ nonrelativistic case

examples

$$20\ \text{keV A}^+ : A = 40$$
$$Amc^2 = 37531\,\text{MeV} \gg 0.020\,\text{MeV}$$
$$v = c\sqrt{\frac{2\cdot 0.02}{37531}} = 0.00103c$$

non relativistic

$$400\ \text{keV He}^+ : A = 2$$
$$Amc^2 = 1876.56\,\text{MeV} \gg 0.4\,\text{MeV}$$
$$v = c\sqrt{\frac{2\cdot 0.4}{1876.56}} = 0.02065c$$

starting to become relativistic

$$20\ \text{MeV electrons}: A = 1$$
$$mc^2 = 0.511\ \text{MeV} \ll 20\ \text{MeV}$$
$$v = c\sqrt{1 - \frac{0.511^2}{20^2}} = 0.99967c$$

highly relativistic

summary of formulas

relativistic factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ or $\gamma = \frac{E}{Amc^2} = 1 + \frac{E_{\mathrm{kin}}}{Amc^2}$

total energy $E^2 = c^2p^2 + A^2m^2c^4$

momentum $cp = \beta E$

velocity $\beta = \sqrt{1-\gamma^{-2}} = \sqrt{1 - \frac{1}{\left(1+\frac{E_{\mathrm{kin}}}{Amc^2}\right)}} = \frac{\sqrt{E_{\mathrm{kin}}^2 + 2Amc^2E_{\mathrm{kin}}}}{E_{\mathrm{kin}} + Amc^2}$

consider EM wave in particle system $E^* = E_0^* e^{i\Phi^*}$

phase of the wave is: $\Phi^* = \omega^*[t^* - \frac{1}{c}(n_x^* x^* + n_y^* y^* + n_z^* z^*)]$

Phase is product of two 4-vectors! $(i\frac{1}{c}E, \mathbf{p})(ict, \mathbf{r}) = -tE + \mathbf{rp}$

with $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k} = \hbar k\mathbf{n}$

$$(i\frac{1}{c}E, \mathbf{p})(ict, \mathbf{r}) = -tE + \mathbf{rp} = -t\hbar\omega + \hbar k\mathbf{nr} = -t\hbar\omega + \hbar\frac{\omega}{c}\mathbf{nr}$$

Radiation phase is Lorentz invariant:

$$\omega^*[ct^* - n_x^* x^* - n_y^* y^* - n_z^* z^*] = \omega[ct - n_x x - n_y y - n_z z]$$

Now apply Lorentz transformation and collect coefficients of (t,x,y,z)

Doppler effect

$$\omega^*[(-\beta\gamma z + \gamma ct) - n_x^* x - n_y^* y - n_z^*(\gamma z - \beta\gamma ct)] = \omega[ct - n_x x - n_y y - n_z z]$$

coefficients must be zero !

example ct-term: $\omega^*\gamma(1 + n_z^*\beta) = \omega$

$$\boxed{\omega = \gamma(1 + n_z^*\beta)\omega^*}$$

<u>relativistic Doppler effect</u>

example: Undulator Radiation

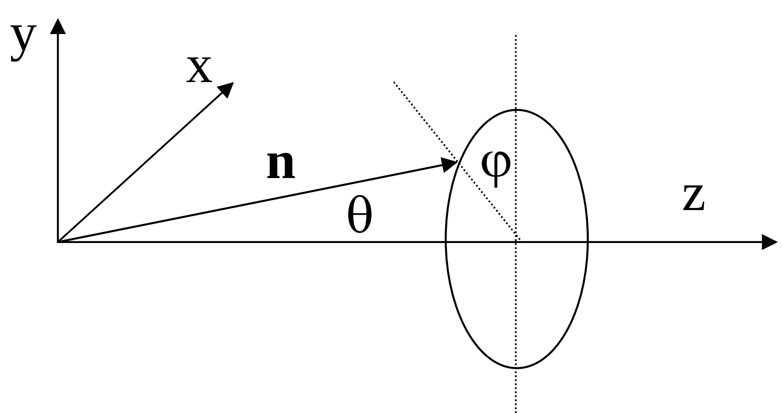
$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{1}{2}\gamma^2\theta^2\right)$$

Collimation

from coefficients of spatial terms, we get:

$$n_{x,y} = \frac{n^*_{x,y}}{\gamma(1+n^*_z\beta)} \qquad \text{and} \qquad n_z = \frac{\beta+n^*_z}{(1+n^*_z\beta)}$$

n is a unit vector and therefore:

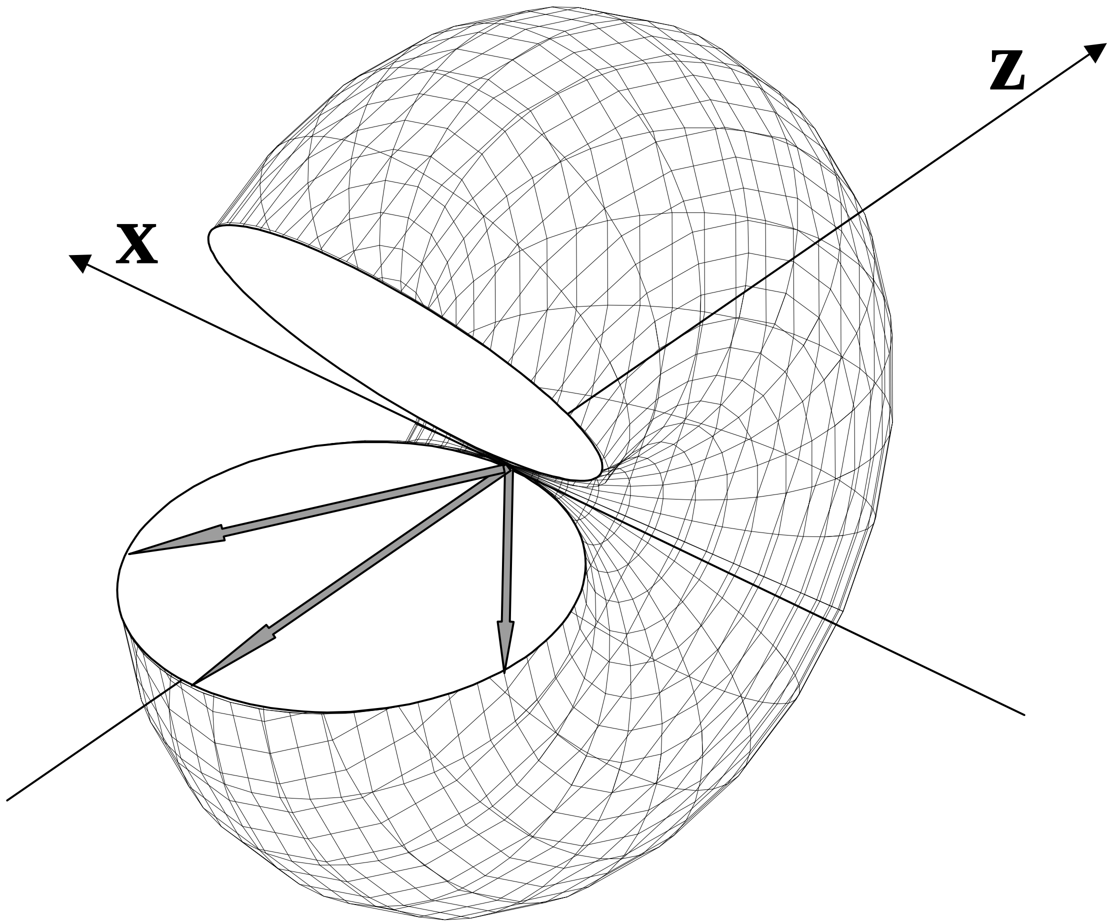


$$\boxed{\begin{aligned} n_x &= \sin\theta\sin\varphi \\ n_y &= \sin\theta\cos\varphi \\ n_z &= \cos\theta \end{aligned}}$$

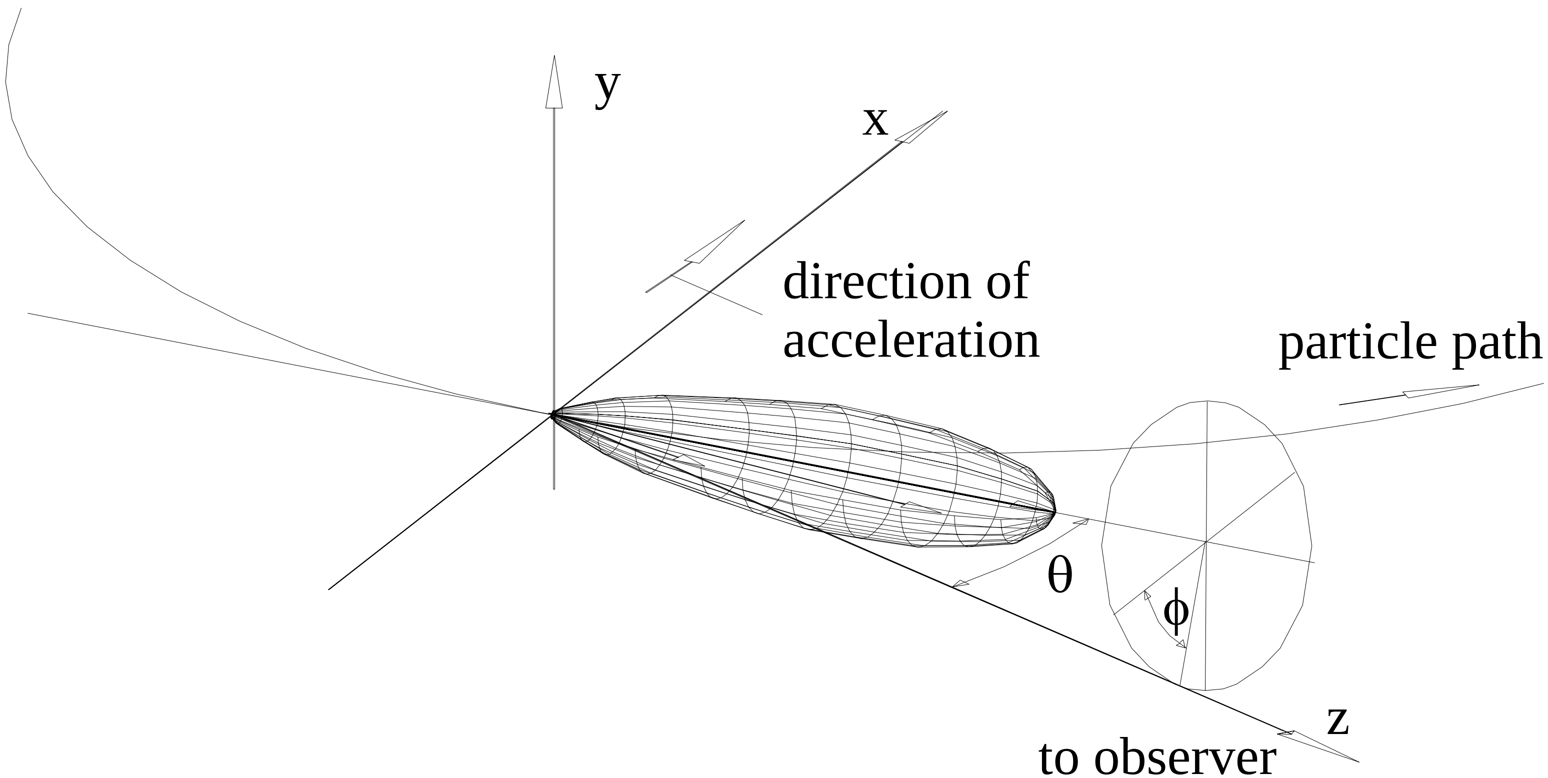
$\sin\theta \approx \theta \approx \dfrac{\sin\theta^*}{\gamma(1+\beta\cos\theta^*)}$ or for $-\pi/2 < \theta^* < \pi/2$

$$\boxed{|\theta| \le \pm\frac{1}{\gamma}}$$

radiation emission in particle system



radiation emission in laboratory system



Lagrange function

Lagrange Function or Lagrangian

for any mechanical system, a function $L = L(q_i, \dot{q}_i, t)$ exists with the

property $\delta \int_{t_0}^{t_1} L \, \mathrm{d}t = 0$

The **action** $\int L \, \mathrm{d}t$ assumes a minimum for any real path

formulating the Lagrangian L is a creative act to describe *a* physical phenomenon

Lagrange Equations - 1

the Lagrange principle is valid in all reference systems and we write:

$$\delta \int_{\tau_0}^{\tau_1} L^* \mathrm{d}\tau = 0$$

where L^* is the Lagrange function in the system $\mathcal{L}^*$ and τ the world time

For simplicity we use t as the general time for any reference system

$$\int \delta L \, \mathrm{d}t = \int \sum_i \frac{\partial L}{\partial q_i} \delta q_i \mathrm{d}t + \int \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \mathrm{d}t$$

second term:

$$\int \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \mathrm{d}t = \int \frac{\partial L}{\partial \dot{q}_i} \frac{\mathrm{d}}{\mathrm{d}t} \delta q_i \mathrm{d}t = \underbrace{\frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_0}^{t_1}}_{=0} - \int \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \mathrm{d}t$$

Lagrange Equations - 2

$$\delta \int L \, \mathrm{d}t = \int \sum_i \left(\frac{\partial L}{\partial q_\mathrm{i}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_\mathrm{i}} \right) \delta q_i \, \mathrm{d}t = 0$$

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = 0}$$

Lagrange equations

generalized momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i}$

Lagrange function has dimension of energy

how do we get a Lagrangian ?

example: **particle at rest**

use a quantity which is Lorentz invariant !

??????????

4-Vector ?

how do we formulate a Lagrange function?

example: **particle at rest**

use a quantity which is Lorentz invariant !

energy-momentum 4-vector for particle at rest:

$$\frac{1}{c}(cp_x^*, cp_y^*, cp_z^*, \mathrm{i}E^*) = (0, 0, 0, \mathrm{i}mc)$$

here, particle system is most convenient

space-time 4-vector $(\mathrm{d}x^*, \mathrm{d}y^*, \mathrm{d}z^*, \mathrm{i}c\mathrm{d}t^*)$

product of both

$$(0, 0, 0, \mathrm{i}mc)(\mathrm{d}x^*, \mathrm{d}y^*, \mathrm{d}z^*, \mathrm{i}c\mathrm{d}t^*) = -mc^2\mathrm{d}t^* = -mc^2\sqrt{1-\beta^2}\,\mathrm{d}t$$

and

$$\boxed{L = -mc^2\sqrt{1-\beta^2}}$$

how do we formulate a Lagrange function?

example: **charged particle plus EM field**

interaction depends only on field, charge and relative velocity

product of the EM-field and velocity 4-vectors:

$$(A_x, A_y, A_z, \mathrm{i}\phi)(\gamma\beta_x, \gamma\beta_y, \gamma\beta_z, \mathrm{i}\gamma) = (\mathbf{A\beta} - \phi)\gamma$$

with $\mathrm{d}\tau = \frac{1}{\gamma}\mathrm{d}t$

the Lagrangian

$$\boxed{L = -mc^2\sqrt{1-\beta^2} + e\mathbf{A\beta} - e\phi}$$

canonical momentum ***P*** of a charged particle in an EM-field:

$$\boxed{\mathbf{P} = \frac{m\dot{\mathbf{q}}}{\sqrt{1-\beta^2}} + \frac{e}{c}\mathbf{A} = \gamma m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A} = \mathbf{p} + \frac{e}{c}\mathbf{A}}$$

Charged particle in an EM Field

Lagrange Equations: $$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0$$

from $\frac{\partial L}{\partial \mathbf{r}} = \nabla L = \frac{e}{c}\nabla(\mathbf{Av}) - e\nabla\phi$ and with $\nabla(\mathbf{ab}) = (\mathbf{b}\nabla)\mathbf{a} + (\mathbf{a}\nabla)\mathbf{b} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$

$$\frac{\partial L}{\partial \mathbf{r}} = \frac{e}{c}(\mathbf{v}\nabla)\mathbf{A} + \frac{e}{c}\mathbf{v}\times(\nabla\times\mathbf{A}) - e\nabla\phi$$

insert into $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \mathbf{v}} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right) = \frac{e}{c}(\mathbf{v}\nabla)\mathbf{A} + \frac{e}{c}\mathbf{v}\times(\nabla\times\mathbf{A}) - e\nabla\phi$$

and get with $\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\nabla)\mathbf{A}$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{e}{c}\mathbf{v}\times(\nabla\times\mathbf{A}) - e\nabla\phi$$

finally with:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$\boxed{F_{\mathrm{L}} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = e\mathbf{E} + \frac{e}{c}\mathbf{v}\times\mathbf{B}}$$

Lorentz force

Acceleration

energy change due to Lorentz force:

$$\Delta e = \int \mathbf{F}_L \, d\mathbf{s} = e \int (\mathbf{E} + [\boldsymbol{\beta} \times \mathbf{B}]) \, d\mathbf{s}$$

$$\begin{aligned} \Delta e &= e \int \mathbf{E} d\mathbf{s} + e \int [\boldsymbol{\beta} \times \mathbf{B}] \, d\mathbf{s} \\ &= e \int \mathbf{E} d\mathbf{s} + e \int \underbrace{[\boldsymbol{\beta} \times \mathbf{B}] \, \mathbf{v}}_{=0} \, dt \end{aligned}$$

use electrical fields for particle acceleration

no acceleration due to magnetic fields

Hamiltonian

Hamiltonian

use canonical variables: q_i, p_i

introduce coordinate transformation: $(q_i, \dot{q}_i, t) \Rightarrow (q_i, p_i, t)$

Hamiltonian function:

$$\boxed{H(q_i, p_i) = \sum \dot{q}_i p_i - L(q_i, \dot{q}_i)}$$

Hamiltonian equations:

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i$$

$$\frac{\partial H}{\partial p_i} = +\dot{q}_i$$

Hamiltonian from Lagrangian

use canonical coordinates in Lagrangian:

with $L = -mc^2\sqrt{1-\beta^2} + e\mathbf{A}\boldsymbol{\beta} - e\phi$

Hamiltonian becomes: $H(q_i, p_i) = \sum \dot{q}_i p_i - L(q_i, \dot{q}_i)$

$$= \sum \dot{q}_i p_i + mc^2\sqrt{1-\beta^2} - e\mathbf{A}\dot{\boldsymbol{\beta}} + e\phi$$

with canonical momentum

$$\mathbf{p} = \frac{m\dot{\mathbf{q}}}{\sqrt{1-\beta^2}} + \frac{e}{c}\mathbf{A} = \gamma m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A}$$

$$\boxed{H(q_i, p_i) = \sum \gamma m\dot{q}_i^2 + mc^2\sqrt{1-\beta^2} + e\phi}$$

Hamiltonian for charged particle in EM-field

$$H(q_i, p_i) = \sum_i \frac{m\dot{q}_i^2}{\sqrt{1-\beta^2}} + mc^2\sqrt{1-\beta^2} + e\phi$$

or

$$(H - e\phi)^2 = \frac{m^2c^4}{1-\beta^2}$$

and after some manipulation

$$\boxed{(c\mathbf{p} - e\mathbf{A})^2 - (H - e\phi)^2 = -m^2c^4}$$

equal to length of 4-vector: $[c\tilde{\mathbf{p}}, \mathrm{i}(E - e\phi)]$, where $H = E$

and $\tilde{\mathbf{p}} = \gamma m\dot{\mathbf{q}}$ the ordinary momentum

Particle mass m is Lorentz invariant

Hamiltonian for particle in EM field

from $$(c\mathbf{p} - e\mathbf{A})^2 - (H - e\phi)^2 = -m^2c^4$$

the more familiar form

$$\boxed{H = e\phi + \sqrt{(c\mathbf{p} - e\mathbf{A})^2 + m^2c^4}}$$

where we use the cartesian coordinate system: $(\bar{x}, \bar{y}, \bar{z})$

and $\mathbf{p} = \gamma m\dot{\mathbf{q}} + \frac{e}{c}\mathbf{A}$ are the conjugate momenta $[\mathbf{q} = (\bar{x}, \bar{y}, \bar{z})]$.

Cyclic Variables - 1

Assume the Hamiltonian does not depend on the coordinate q_i

$$H = H(q_1, \dots q_{i-1}, q_{i+1} \dots, p_1, p_2 \dots, p_i, \dots)$$

then: $\dfrac{\partial H}{\partial q_i} = -\dot{p}_i = 0$ or $\boxed{p_i = \text{const}}$

<u>constant of motion !</u>

and from $\dfrac{\partial H}{\partial p_i} = \dot{q}_i = \text{const}$

$$\boxed{q_i(t) = \omega_i t + c_i}$$

Example:

assume that Hamiltonian does not depend explicitly on the time t

like $(H - e\phi)^2 = \frac{m^2 c^4}{1-\beta^2}$ or $H = \gamma mc^2 + e\phi$

then $\frac{\partial H}{\partial t} = 0$ and the momentum conjugate to the time is a constant of motion

from the second Hamiltonian equation

$$\frac{\partial H}{\partial p_t} = \frac{\mathrm{d}}{\mathrm{d}t} t = 1$$

and the momentum conjugate to the time is $\boxed{p_t = H = \text{const}}$

which is the total energy of the system.

Canonical Transformations – 1

finding cyclical variables is highly desirable and we use canonical coordinate transformations to find them

transformation of coordinates: $(q_i, p_i, t) \Rightarrow (\bar{q}_i, \bar{p}_i, t)$

$$\bar{q}_k = f_k(q_i, p_i, t)$$
$$\bar{p}_k = g_k(q_i, p_i, t)$$

$$\delta \int \left(\sum_k \dot{q}_k p_k - H \right) \mathrm{d}t = 0$$
$$\delta \int \left(\sum_k \dot{\bar{q}}_k \bar{p}_k - \bar{H} \right) \mathrm{d}t = 0$$

integrants can differ only by total time derivative of arbitrary function

$$\sum_k \dot{q}_k p_k - H = \sum_k \dot{\bar{q}}_k \bar{p}_k - \bar{H} + \frac{\mathrm{d}G}{\mathrm{d}t}$$

G is called the *generating function*

Generating Function - 1

$$G = G(q_k, \bar{q}_k, p_k, \bar{p}_k, t) \qquad 0 \le k \le N$$

only $2N$ variables are independent, others define transformation

possible generating functions

$$\boxed{\begin{aligned} G_1 &= G_1(q, \bar{q}, t), \qquad & G_3 &= G_3(p, \bar{q}, t), \\ G_2 &= G_2(q, \bar{p}, t), \qquad & G_4 &= G_4(p, \bar{p}, t). \end{aligned}}$$

use, for example, G_1: $\quad G = G_1(q_k, \bar{q}_k, t)$

$$\sum_k \dot{q}_k p_k - H = \sum_k \overline{\dot{q}_k}\, \bar{p}_k - \overline{H} + \frac{\mathrm{d}G}{\mathrm{d}t}$$

$$\frac{\mathrm{d}G_1}{\mathrm{d}t} = \sum_k \frac{\partial G_1}{\partial q_k}\frac{\partial q_k}{\partial t} + \sum_k \frac{\partial G_1}{\partial \bar{q}_k}\frac{\partial \bar{q}_k}{\partial t} + \frac{\partial G_1}{\partial t}$$

Generating Function - 2

$$\sum_k \dot{q}_k \left(p_k - \frac{\partial G}{\partial q_k}\right) - \sum_k \overline{\dot{q}_k} \left(\bar{p}_k + \frac{\partial G}{\partial \bar{q}_k}\right) - \left(H - \overline{H} + \frac{\partial G}{\partial t}\right) = 0$$

$$p_k = \frac{\partial G_1}{\partial q_k}$$

$$\bar{p}_k = -\frac{\partial G_1}{\partial \bar{q}_k}$$

$$\overline{H} = H + \frac{\partial G_1}{\partial t}$$

Generating Function - 3

general equations:

$$y_k = \pm \frac{\partial}{\partial x_k} G(x, \bar{x}, t),$$

$$\bar{y}_k = \mp \frac{\partial}{\partial \bar{x}_k} G(x, \bar{x}, t),$$

$$\overline{H} = H + \frac{\partial}{\partial t} G(x, \bar{x}, t).$$

x and y are mutually conjugate variables

use upper signs if derivation is with respect to coordinates and lower signs if derivative with respect to momenta

Generating Function - 4

How do we know that the new coordinates are canonical?

Poisson brackets:

$$[f(q,p,t),g(q,p,t)] = \sum_{k=0}^{n}\left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\right)$$

transformations are canonical if and only if

$$[Q_i,Q_j]_{qp} = 0 \qquad [P_i,P_j]_{qp} = 0$$

and $[Q_i,P_j]_{qp} = \lambda\,\delta_{ij}$

for $i,j = 0,1,2\ldots,n$

λ is scale factor. We consider only scale preserving transformations $\lambda = 1$

Harmonic Oscillator

Find coordinates that makes phase space motion circular

$$H = \frac{1}{2m}p^2 + \frac{k}{2}q^2 = \text{const.}$$

use transformation: $Q = \sqrt{\frac{k}{2}}\,q; \quad P = \sqrt{\frac{1}{2m}}\,p \qquad H = Q^2 + P^2$

$$[Q,P]_{qp} = \sqrt{\frac{k}{2}}\sqrt{\frac{1}{2m}} - 0 \bullet 0 = \frac{1}{2}\sqrt{\frac{k}{m}} \neq 1$$ not scale preserving !

scale preserving transformation:

we try $Q = \sqrt{\frac{k}{2}}\,\alpha q; \quad P = \frac{\alpha}{\sqrt{2m}}p$ and $[Q,P]_{qp} = \sqrt{\frac{k}{2}}\,\alpha \bullet \frac{\alpha}{\sqrt{2m}} = 1 \qquad \alpha^2 = 2\sqrt{\frac{m}{k}}$

or $Q = \sqrt[4]{mk}\,q; \quad P = \frac{1}{\sqrt[4]{mk}}p \qquad H = \frac{1}{2}\sqrt{\frac{k}{m}}\left(Q^2 + P^2\right)$

Curvilinear Coordinates - 1

example: curvilinear coordinate system of beam dynamics $(\bar{x}, \bar{y}, \bar{z}) \longrightarrow (x, y, z)$

$$\mathbf{r}(\bar{x}, \bar{y}, \bar{z}) = \mathbf{r}_0(\bar{z}) + x\mathbf{u}_x(\bar{z}) + y\mathbf{u}_y(\bar{z}) \quad \text{with} \begin{cases} \mathbf{u}_z(\bar{z}) = \dfrac{\mathrm{d}}{\mathrm{d}\bar{z}}\mathbf{r}_0(\bar{z}) \\ \dfrac{\mathrm{d}}{\mathrm{d}\bar{z}}\mathbf{u}_x(\bar{z}) = \kappa_x \mathbf{u}_z(\bar{z}) \\ \mathbf{u}_y(\bar{z}) = \mathbf{u}_z(\bar{z}) \times \mathbf{u}_x(\bar{z}) \end{cases}$$

find canonical momenta from contact transformation defined by generating function: $G(z, x, y, \overline{p}_z, \overline{p}_x, \overline{p}_y) = -(c\overline{\mathbf{p}} - e\overline{\mathbf{A}})\left[\mathbf{r}_0(z) + x\mathbf{u}_x(z) + y\mathbf{u}_y(z)\right]$

new canonical momenta:

$$\boxed{\begin{aligned} cp_z - eA_z &= -\frac{\partial G}{\partial z} = (c\overline{\mathbf{p}} - e\overline{\mathbf{A}})(1 + \kappa_x x + \kappa_y y)\mathbf{u}_z, \\ cp_x - eA_x &= -\frac{\partial G}{\partial x} = (c\overline{\mathbf{p}} - e\overline{\mathbf{A}})\mathbf{u}_x(z) \\ cp_y - eA_y &= -\frac{\partial G}{\partial y} = (c\overline{\mathbf{p}} - e\overline{\mathbf{A}})\mathbf{u}_y(z) \end{aligned}}$$

with curvatures $\kappa_{x,y} = \dfrac{1}{\rho_{x,y}}$

Curvilinear Coordinates - 2

Hamiltonian in cartesian coordinates
$$H = e\phi + \sqrt{(c\bar{\mathbf{p}} - e\bar{\mathbf{A}})^2 + m^2c^4}$$

Hamiltonian in beam dynamics coordinates:

$$H = e\phi + \sqrt{\left(\frac{cp_z - eA_z}{1+\kappa_x x}\right)^2 + (cp_x - eA_x)^2 + (cp_y - eA_y)^2 + m^2c^4}$$

for a flat beam transport system with only horizontally bending fields:

$$\mathbf{A} = (0, 0, A_z) = (0, 0, -B_y x)$$

and

$$\boxed{H(x,y,z,t) = e\phi + \sqrt{\left(\frac{cp_z - eA_z}{1+\kappa_x x}\right)^2 + (cp_x)^2 + (cp_y)^2 + m^2c^4}}$$

Extended Hamiltonian – 1

start with Hamiltonian $H = H\left(q_1, q_2, \ldots q_\mathrm{f}, p_1, p_2, \ldots p_\mathrm{f}, t\right)$

introduce independent variables as coordinates:

$$q_0 = t \qquad \text{and} \qquad p_0 = -H$$

and formulate new Hamiltonian:

$$\mathcal{H}(q_0, q_1, q_2 \ldots q_f, p_0, p_1, p_2 \ldots p_f) = H + p_0 = 0$$

or in cartesian coordinates:

$$\mathcal{H}(t, x, y, z, \ldots q_f, -H, p_x, p_y, p_z, \ldots p_f) = 0$$

Hamilton's equations:

$$\frac{\mathrm{d}q_l}{\mathrm{d}\tau} = \frac{\partial \mathcal{H}}{\partial p_l}; \qquad \frac{\mathrm{d}p_l}{\mathrm{d}\tau} = -\frac{\partial \mathcal{H}}{\partial q_l}; \qquad \text{for} \quad l = 0, 1, 2, \ldots$$

Extended Hamiltonian – 2

for $l = 0$

$$\frac{\mathrm{d}q_0}{\mathrm{d}\tau} = 1 \quad \Rightarrow \quad q_0 = \tau + C_1 = t$$

and $$\frac{\mathrm{d}p_0}{\mathrm{d}\tau} = -\frac{\partial \mathcal{H}}{\partial q_0} = -\frac{\partial H}{\partial q_0} = -\frac{\partial H}{\partial t} = -\frac{\mathrm{d}H}{\mathrm{d}t} \quad \Rightarrow \quad p_0 = -H + C_2$$

$$H \neq H(\tau) \ \Rightarrow \ \frac{\mathrm{d}p_0}{\mathrm{d}\tau} = 0 \ \Rightarrow \ H = \mathrm{const.}$$

Change of Independent Variable

we want to change the independent variable from t to, say z or, generally, from q_i to q_j

to define, say q_3, as new independent variable, go backward by solving

$$\boxed{\mathcal{H}(q_0, q_1, q_2 \ldots q_f, p_0, p_1, p_2 \ldots p_f) = 0}$$

for $p_3 = -K(q_0, q_1, q_2 \ldots q_f, p_0, p_1, p_2, p_4 \ldots p_f)$

and define new extended Hamiltonian $\mathcal{K} = p_3 + K = 0$

Equations:

$$\frac{\mathrm{d}q_3}{\mathrm{d}q_3} = \frac{\partial \mathcal{K}}{\partial p_3} = 1 \qquad \frac{\mathrm{d}p_3}{\mathrm{d}q_3} = -\frac{\partial \mathcal{K}}{\partial q_3} = -\frac{\partial K}{\partial q_3}$$

$$\frac{\mathrm{d}q_{i\neq 3}}{\mathrm{d}q_3} = \frac{\partial \mathcal{K}}{\partial p_{i\neq 3}} = \frac{\partial K}{\partial p_{i\neq 3}} \qquad \frac{\mathrm{d}p_{i\neq 3}}{\mathrm{d}p_3} = -\frac{\partial \mathcal{K}}{\partial q_{i\neq 3}} = -\frac{\partial K}{\partial q_{i\neq 3}}$$

are in Hamiltonian form with the new Hamiltonian $K = -p_3$

Independent Variable z

switch to z as independent variable

from $$H(x,y,z,t) = e\phi + \sqrt{\frac{(cp_z - eA_z)^2}{(1+\kappa x)^2} + c^2 p_\perp^2 + m^2 c^4}$$

we solve for $$cp_z = eA_z + (1+\kappa x)\sqrt{(H - e\phi)^2 - c^2 p_\perp^2 - m^2 c^4}$$

$$= eA_z + (1+\kappa x)\sqrt{T^2 - m^2 c^4 - c^2 p_\perp^2}$$

$$= eA_z + (1+\kappa x)\sqrt{c^2 p^2 - c^2 p_\perp^2}$$

we also divide by the momentum and use slopes rather than momenta

and the new Hamiltonian is with $\mathcal{H}(x,x',y,y',z) = -\frac{cp_z}{cp}$

or $$\boxed{\mathcal{H}(x,x',y,y',z) = -\frac{eA_z}{cp} - (1+\kappa x)\sqrt{1 - x'^2 - y'^2}}$$

Hamiltonian for Beam Dynamics

from $\mathcal{H}(x,x',y,y',z) = -\frac{eA_z}{cp} - (1+\kappa x)\sqrt{1-x'^2-y'^2}$

we get with $\frac{1}{cp} = \frac{1}{cp_0(1+\delta)} \approx \frac{1}{cp_0}(1-\delta)$, where $\delta = \frac{\mathrm{d}p}{p_0}$ and $\kappa = \frac{1}{\rho_0}$

the Hamiltonian for beam dynamics

$$\boxed{\mathcal{H}(x,x',y,y',z) = -\frac{e\overline{A}_z}{cp_0}(1+\kappa x)(1-\delta) - (1+\kappa x)\sqrt{1-x'^2-y'^2}}$$

note, we used here! $A_z(x,y,z) = (1+\kappa x)\overline{A}_z(\bar{x},\bar{y},\bar{z})$

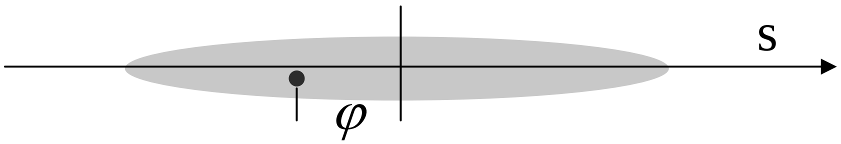
where $\overline{A}_z(\bar{x},\bar{y},\bar{z})$ is the vector potential in cartesian coordinates (straight magnets)

for many specific examples consult:

Derivation of Hamiltonians for Accelerators, K.R. Symon

available on: http://www.aps.anl.gov/APS/frame_search.html

Longitudinal Motion



$\psi = \psi_s + \varphi$ phase

ψ_s : synchronous phase

$\delta = \frac{\mathrm{d}cp}{cp}$ relative momentum deviation

η_c momentum compaction

ω_0 revolution frequency

per turn:

$$\boxed{\begin{aligned} \dot{\varphi} &= \omega_{rf}\,\eta_c\,\delta \\ \dot{\delta} &= \frac{e\hat{V}}{T_0 E_s \beta^2}\left[\sin\left(\psi_s + \varphi\right) - \sin\psi_s\right] \end{aligned}}$$

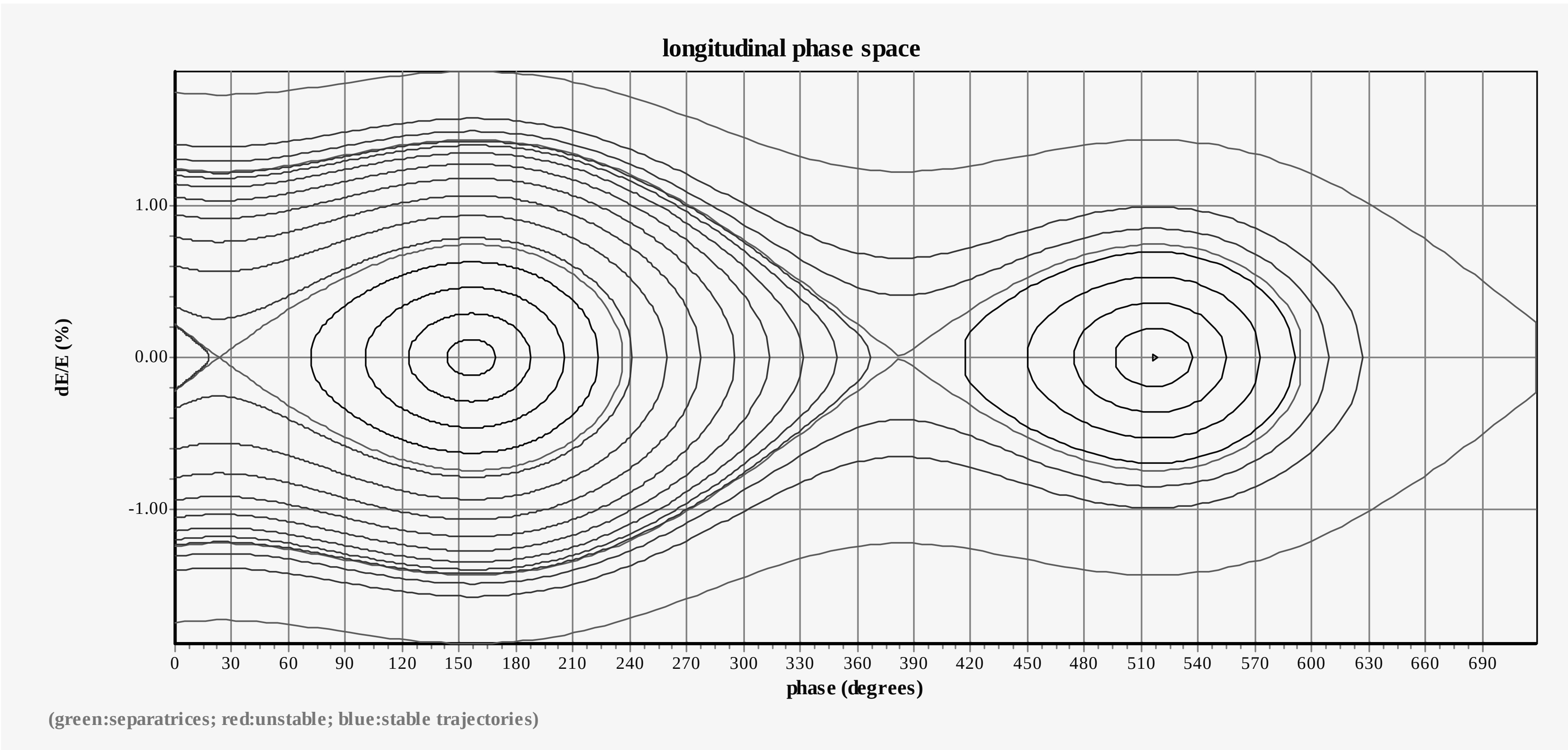
these equations of motion can be derived from Hamiltonian:

$$\mathcal{H} = \tfrac{1}{2}\omega_{rf}\,\eta_c\,\delta^2 - \frac{e\hat{V}}{T_0 E_s \beta^2}\left[\cos\left(\psi_s + \varphi\right) + \varphi\sin\psi_s - \cos\psi_s\right]$$

$$= \tfrac{1}{2}\dot{\varphi} + \Omega^2\left[1 - \frac{\cos\left(\psi_s + \varphi\right)}{\cos\psi_s} - \varphi\tan\psi_s\right]$$

where the synchrotron oscillation frequency $\Omega^2 = \omega_0^2 \frac{\eta_c h e\hat{V}\cos\psi_s}{2\pi\beta\, cp}$

Longitudinal Phase Space



Magnetic Fields

Vector Potential

general vector potential

$$A_z = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + \ldots$$

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad A_x = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + \ldots$$

$$A_y = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + \ldots$$

Maxwell's Equations:

$\nabla \mathbf{B} = 0$ imposes no restriction on A

$$\nabla \times \mathbf{B} = 0 \Longrightarrow \nabla \times (\nabla \times \mathbf{A}) = 0$$

$$\frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z} + \frac{\partial^2 A_z}{\partial x \partial z} = 0$$

etc.

too complicated for most situations (see K. Symon)

Magnet Fields

in vacuum $\nabla \times \mathbf{B} = 0$ field can be derived from a scalar potential ψ

with $\mathbf{B} = -\nabla\psi$

$$\left.\begin{array}{l} \mathbf{B} = \nabla \times \mathbf{A} \\ \nabla\mathbf{B} = 0 \end{array}\right\} \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

for any vector potential

$\nabla\mathbf{B} = 0$

$$\boxed{\Delta\psi = 0 \quad \text{Laplace Equation}}$$

in cylindrical coordinates

$$\Delta\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} \equiv 0$$

with solution:

$$\boxed{\psi(r,\varphi,z) = -\frac{cp}{e}\sum_{n>0}\frac{1}{n!}A_n(z)r^n \mathrm{e}^{in\varphi}}$$

Multipole Field Expansion

use cartesian coordinates:

$$\psi(x,y,z) = -\frac{cp}{e}\sum_{n>0}\frac{1}{n!}A_n(z)(x+\mathrm{i}y)^n = -\frac{cp}{e}\sum_{n>0}\sum_{j=0}^{n}\mathrm{i}^j A_{n-j,j}(z)\frac{x^{n-j}}{(n-j)!}\frac{y^j}{j!}$$

real and imaginary terms represent two basic field orientations:

in beam dynamics horizontal focusing should be the same for y or $-y$

this is called the "mid-plane symmetry

electric field: $E_x(x,y) = E_x(x,-y)$

potential must be **symmetric** in y
use only real terms

magnetic field: $B_y(x,y) = B_y(x,-y)$

potential must be **anti-symmetric** in y
use only imaginary terms

Magnetic Multipole Fields

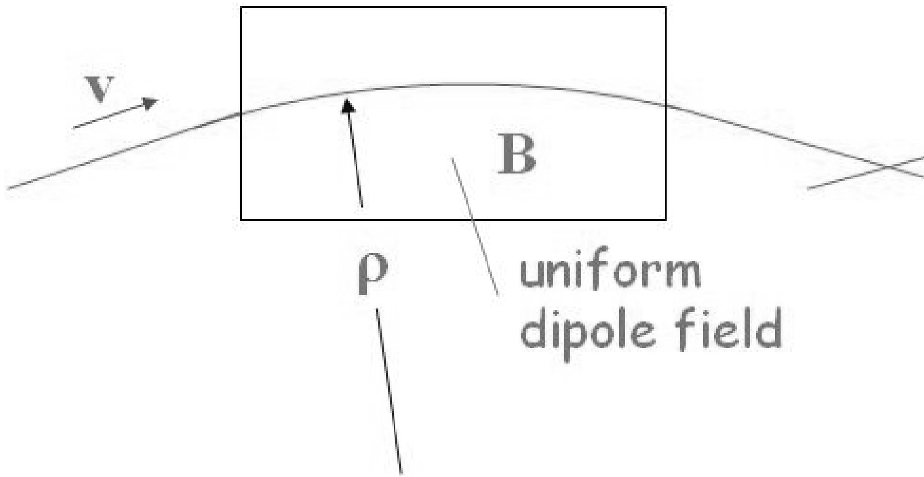
			rotated magnets	ordinary beam dynamics magnets
dipole	$-\frac{e}{cp}\psi_1$	$=$	$-\kappa_y x$	$+\kappa_x y$
quadrupole	$-\frac{e}{cp}\psi_2$	$=$	$-\frac{1}{2}\underline{k}(x^2-y^2)$	$+kxy$
sextupole	$-\frac{e}{cp}\psi_3$	$=$	$-\frac{1}{6}\underline{m}(x^3-3xy^2)$	$+\frac{1}{6}m(3x^2y-y^3)$
octupole	$-\frac{e}{cp}\psi_4$	$=$	$-\frac{1}{24}\underline{r}(x^4-6x^2y^2+y^4)$	$+\frac{1}{6}r(x^3y-xy^3)$

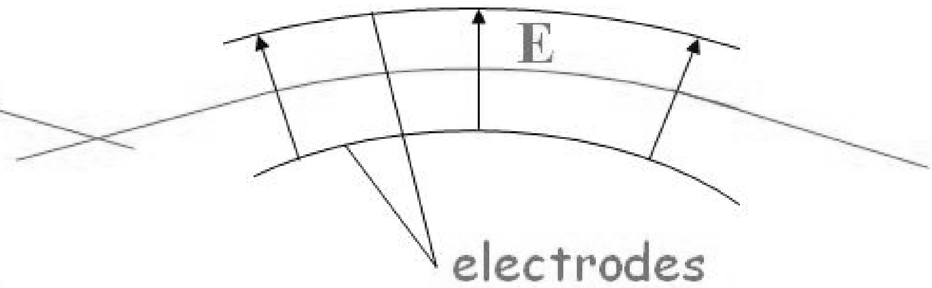
<u>vector potentials</u>

			ordinary beam dynamics magnets	rotated magnets
dipole	$\frac{e}{cp}\overline{A}_{z1}$	$= -\frac{1}{2}$	$-\frac{1}{2}\kappa_x x$	$-\frac{1}{2}\kappa_y y$
quadrupole	$\frac{e}{cp}\overline{A}_{z2}$	$=$	$-\frac{1}{2}k(x^2-y^2)$	$-\underline{k}\,xy$
sextupole	$\frac{e}{cp}\overline{A}_{z3}$	$=$	$-\frac{1}{6}m(x^3-3xy^2)$	
octupole	$\frac{e}{cp}\overline{A}_{z4}$	$=$	$-\frac{1}{24}r(x^4-6x^2y^2+y^4)$	

Beam deflection

bending - 1





Lorentz force = centrifugal force

$$\mathbf{F}_{\mathrm{L}} = e\mathbf{E} + [c]e\boldsymbol{\beta}\mathbf{B} = \mathbf{F}_{\mathrm{cf}} = \frac{\gamma A m c^2 \beta^2}{\rho}$$

bending - 2

$$\mathbf{F}_{\mathrm{L}} = e\mathbf{E} + [c]e\boldsymbol{\beta}\mathbf{B} = \mathbf{F}_{\mathrm{cf}} = \frac{\gamma A m c^2 \beta^2}{\rho}$$

curvature of trajectory

$$\boxed{\frac{1}{\rho} = \frac{e|\mathbf{E}|}{\gamma A m c^2 \beta^2} + \frac{[c]eB}{\gamma A m c^2 \beta}}$$

for constant electric and/or magnetic field
trajectory is a segment of a circle, an arc

radius of circle = bending radius: ρ

deflection angle: $\varphi = \frac{\ell_{\mathrm{b}}}{\rho}$

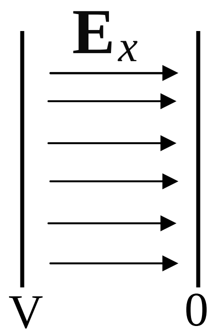
ℓ_{b} arc length of bending magnet general curvature: $\frac{1}{\rho} = \frac{e|\mathbf{E}|}{\gamma A m c^2 \beta^2} + \frac{[c]eB}{\gamma A m c^2 \beta}$

electro-static dipole (real terms) n=1

$-\frac{e}{cp}\psi_1(x,y,z) = A_{10}\, x$

equipotential (2G aperture):

$x_{\mathrm{eq}} = \pm G = \mathrm{const}$



$$A_{10} = \kappa = \frac{1}{\rho_x}$$

$$\boxed{\frac{1}{\rho} = \frac{eV}{2GAmc^2\gamma\beta^2} = \frac{eV}{E_{\mathrm{kin}}}\,\frac{\gamma}{\gamma+1}\,\frac{1}{2G}}$$

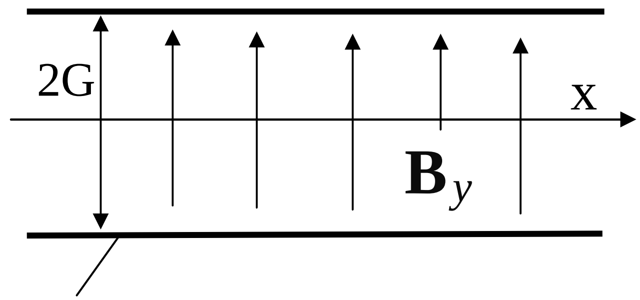
magneto-static field

magneto-static dipole (imaginary terms)

$$-\tfrac{e}{cp}\psi_1(x,y,z) = A_{01}\, y$$

equipotential (2G aperture):

$$y_{\text{eq}} = \pm G_{\text{aperture}} = \text{const}$$



ferromagnetic surface =
equipotential surface

$$A_{01} = \kappa = \tfrac{1}{\rho_x}$$

$$\boxed{\frac{1}{\rho} = \frac{[c]eB_y}{Amc^2\beta\gamma} = \frac{[c]eB_y}{\sqrt{Amc^2 E_{\text{kin}}}\,\sqrt{\gamma+1}}}$$

numerical expressions for dipoles

electrical dipole:

$$\frac{1}{\rho} = \frac{e\mathbf{V}}{2G\gamma Amc^2\beta^2} = \frac{eV}{E_{\text{kin}}}\frac{\gamma}{\gamma+1}\frac{1}{2G} = \frac{50}{G(\text{cm})}\frac{\gamma}{\gamma+1}\frac{eV}{E_{\text{kin}}}$$

magnetic dipole:

$$\frac{1}{\rho} = \frac{[c]eB_y}{Amc^2\beta\gamma} = 0.299792458\frac{B_y(\text{T})}{cp(\text{GeV})}$$

$$= \frac{[c]eB_y}{\sqrt{Amc^2E_{\text{kin}}}\sqrt{\gamma+1}} = 310.6209\frac{B_y(\text{T})}{\sqrt{AE_{\text{kin}}(\text{keV})}\sqrt{\gamma+1}}$$

note: $A_{\text{electron}} = \frac{1}{1822.9}$

bending –non relativistic beam

non relativistic beams $\gamma \approx 1$ and $E_{\text{kin}} \ll Amc^2$

electric field only: $\frac{1}{\rho} = \frac{e|\mathbf{E}|}{Amc^2\beta^2} = \frac{e|\mathbf{E}|}{2E_{\text{kin}}}$ with $E_{\text{kin}} = \frac{1}{2}Amc^2\beta^2$

$$\frac{1}{\rho}\left(\mathrm{m}^{-1}\right) = 0.5\frac{\left|\mathbf{E}\left(\mathrm{V/m}\right)\right|}{E_{\text{kin}}\left(\mathrm{V}\right)}$$

magnetic field only: $\frac{1}{\rho} = \frac{[c]eB}{Amc^2\beta} = \frac{[c]eB}{cp}$ with $cp = cAmv$

$$\frac{1}{\rho} \approx \frac{[c]eB}{\sqrt{2Amc^2 E_{\text{kin}}}}$$

$$\frac{1}{\rho} \approx 219.64\frac{B(\mathrm{T})}{\sqrt{AE_{\text{kin}}(\mathrm{keV})}}$$

bending magnet-coils

$$\nabla \times \frac{\mathbf{B}}{\mu_r} = \frac{4\pi}{c}\mathbf{j},$$

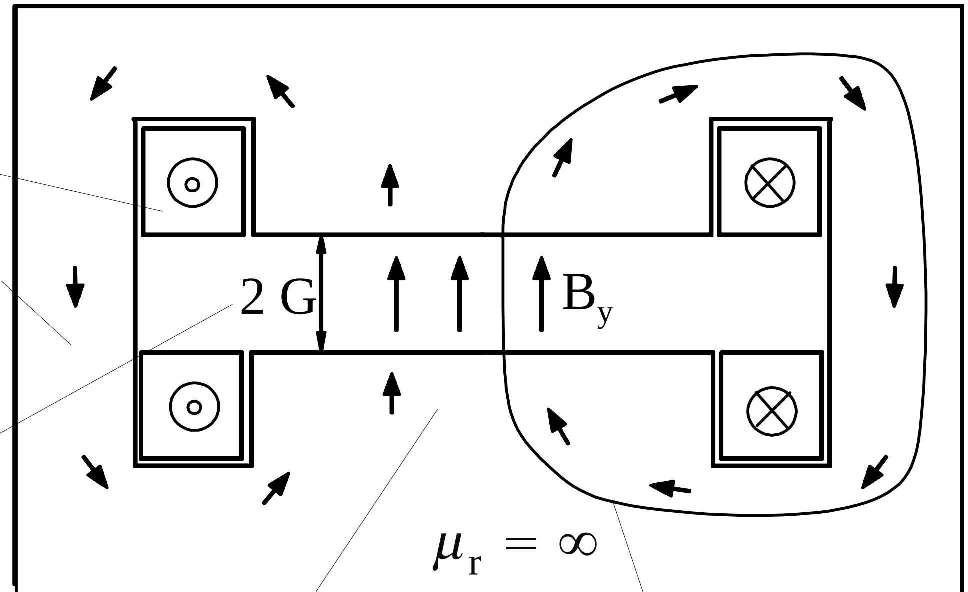
mks

$$\nabla \times \frac{\mathbf{B}}{\mu_r} = \mu_0 \mathbf{j}$$

excitation coil

return yoke

pole gap



magnet pole

integration path

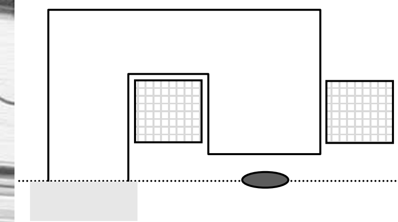
$$\int \left(\nabla \times \frac{\mathbf{B}}{\mu_r}\right) \mathrm{d}A = \oint \frac{\mathbf{B}}{\mu_r} \mathrm{d}s = \mu_0 \int \mathbf{j}\, \mathrm{d}A$$

$$\cancel{2}GB_y = \mu_0 \cancel{2}I_{\mathrm{coil}}$$

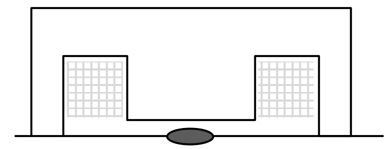
$$\boxed{I_{\mathrm{coil}}[A] = \frac{1}{\mu_0} B_y[\mathrm{T}]G[\mathrm{m}]}$$

for n_t windings per coil, the power supply current is: $I_{ps}[A] = I_{\mathrm{coil}}[\mathrm{A}]/n_t$

bending magnet-photo



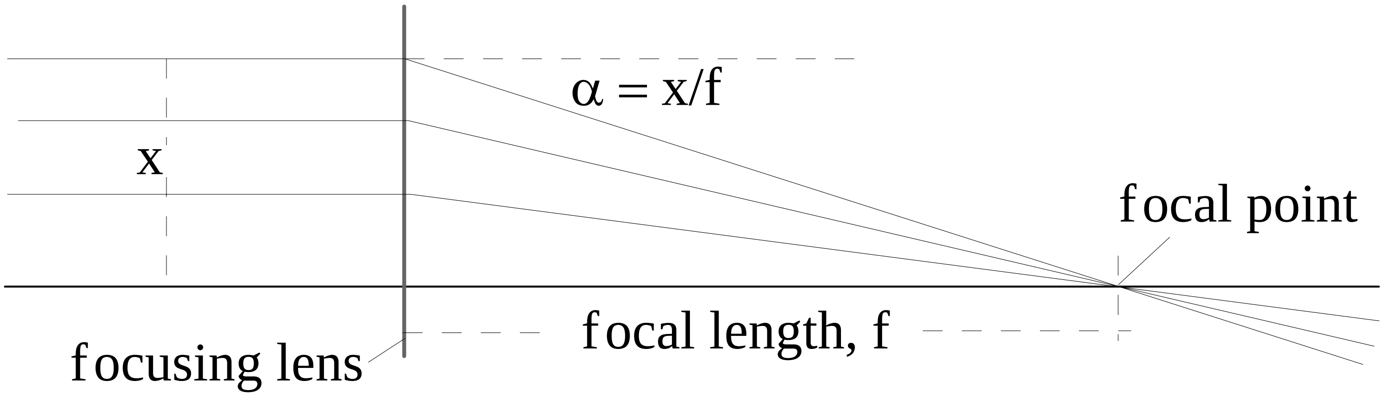
C-magnet



H-magnet

Beam focusing

principle of focusing



deflection of trajectory $$\alpha = \frac{\ell_b}{\rho} = \frac{e|\mathbf{E}|\ell_b}{\gamma A m c^2 \beta^2} + \frac{[c]eB\ell_b}{\gamma A m c^2 \beta}$$

need field like $E_x = g_e x$ or $B_y = g_m x$

which gives desired deflection

$$\alpha = \frac{eg\ell_b}{cp} x = k\ell_b x \quad \begin{cases} k_e = \dfrac{eg_e}{\gamma A m c^2 \beta^2} \\ k_m = \dfrac{[c]eg_m}{\gamma A m c^2 \beta} \end{cases}$$

real n=2 terms:

$$-\frac{e}{cp}\psi_2(x,y,z) = A_{20}\,\tfrac{1}{2}x^2 - A_{02}\,\tfrac{1}{2}y^2 = A_{20}\,\tfrac{1}{2}(x^2-y^2) = k\,\tfrac{1}{2}(x^2-y^2)$$

equipotential: $x_{\mathrm{eq}}^2 - y_{\mathrm{eq}}^2 = \pm R_{\mathrm{aperture}} = \mathrm{const}$

$$E_x = \frac{cp}{e}kx = gx$$
$$E_y = -\frac{cp}{e}ky = -gy$$

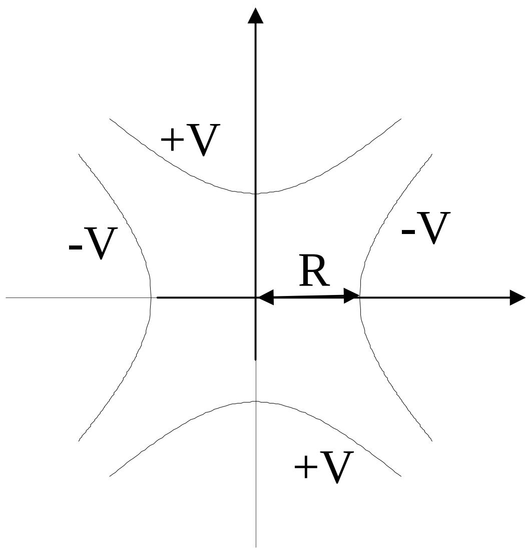
electro-static quadrupole

electrode potential

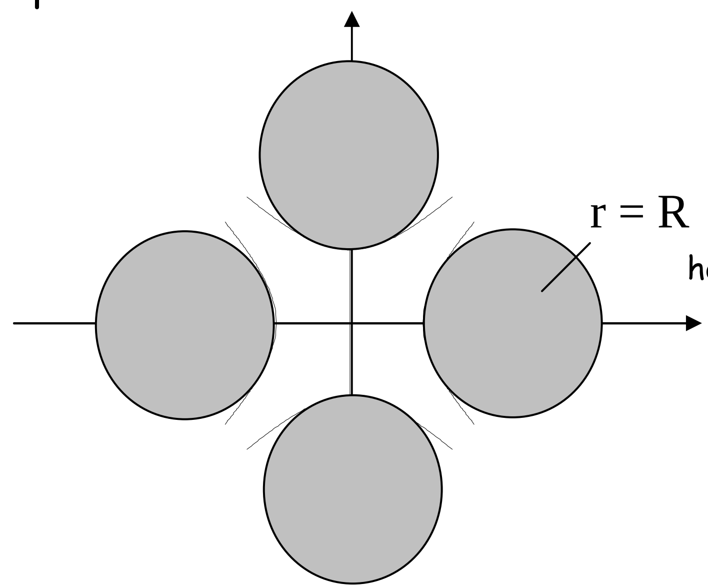
$$V = \pm\tfrac{1}{2}R^2 g$$

field gradient

$$g = \frac{\partial E_x}{\partial x}$$



more practical rendition: 4 round rods

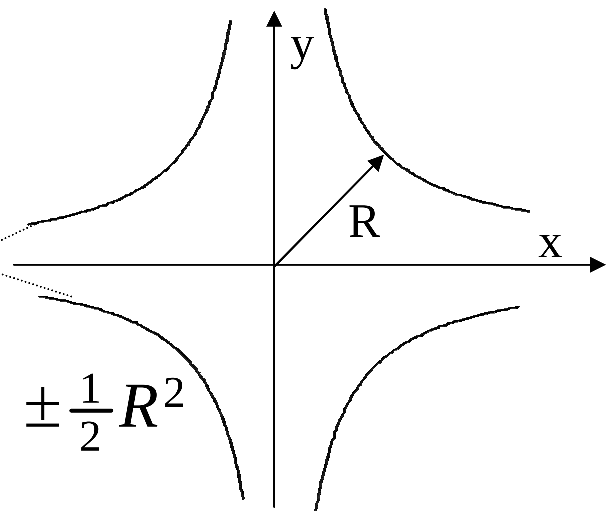


magnetic quadrupole

use imaginary term: $-\frac{e}{cp}\psi_2(x,y,z) = A_{11}\,xy = k\,xy$

magnetic quadrupole:

ferromagnetic surfaces



pole profile: $xy = \pm\frac{1}{2}R^2$

definition of k

$$\frac{cp}{e}k = -\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} = \frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = g$$

g field gradient

quadrupole magnet

focal length: $\frac{1}{f} = k\ell_q$ ℓ_q : length of quadrupole

electrostatic quadrupole

$$\left.\begin{aligned} k_e &= \frac{eg_e}{\gamma A m c^2 \beta^2} \\ g_e &= \frac{2V}{R^2} \end{aligned}\right\}$$

$$\boxed{k_e = \frac{e\gamma}{E_{kin}(\gamma+1)} \frac{2V}{R^2} = 2 \cdot 10^4 \frac{eV}{E_{kin}} \frac{\gamma}{\gamma+1} \frac{1}{R(\mathrm{cm})^2}}$$

magnetic quadrupole strength: $k = \frac{[c]eg}{cp} = \frac{[c]eg}{\sqrt{Amc^2 E_{kin}}\sqrt{\gamma+1}}$

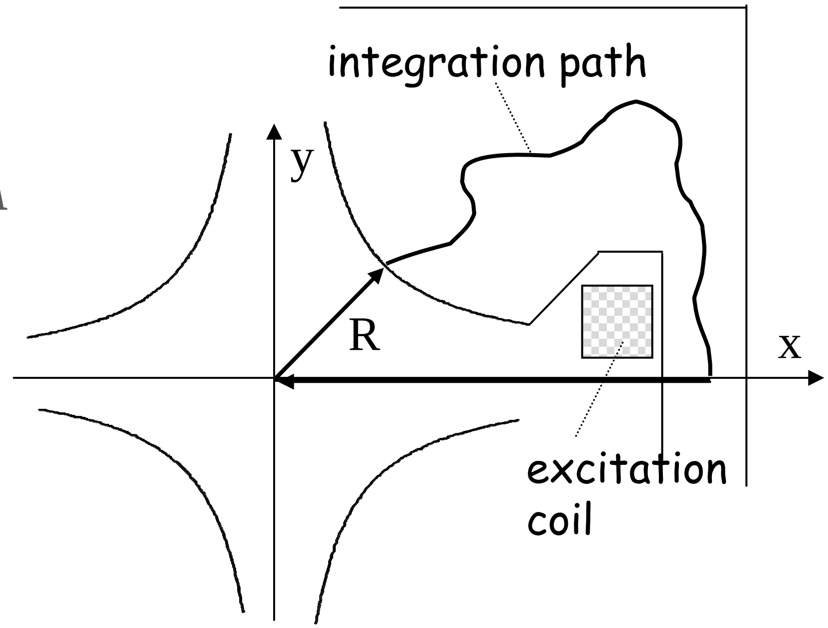
$$\boxed{k = 0.299792458 \frac{g(\mathrm{T/m})}{cp(\mathrm{GeV})} = 310.6209 \frac{g(\mathrm{T/m})}{\sqrt{AE_{kin}(\mathrm{keV})}\sqrt{\gamma+1}}}$$

note: $A_{\mathrm{electron}} = \frac{1}{1822.9}$

similar to bending magnet, integrate

$$\int \left(\nabla \times \frac{\mathbf{B}}{\mu_r} \right) dA = \oint \frac{\mathbf{B}}{\mu_r} ds = \mu_0 \int \mathbf{j}\, dA$$

$$\oint \frac{\mathbf{B}}{\mu_r} ds = \underbrace{\int_0^R \frac{\mathbf{B}_r}{\mu_r} d\mathbf{r}}_{=\frac{1}{2} g R^2} + \underbrace{\int_{\text{iron}} \frac{\mathbf{B}}{\mu_r} d\mathbf{r}}_{\approx 0\ (\mu_r = \infty)} + \underbrace{\int_{\text{iron}}^{0} \frac{\mathbf{B}_y}{\mu_r} d\mathbf{x}}_{=0\ (\mathbf{B}_y \perp d\mathbf{x})} = \frac{1}{2} g R^2$$



with r.h.s. $\frac{1}{2} g R^2 = \mu_0 I_{\text{coil}}$

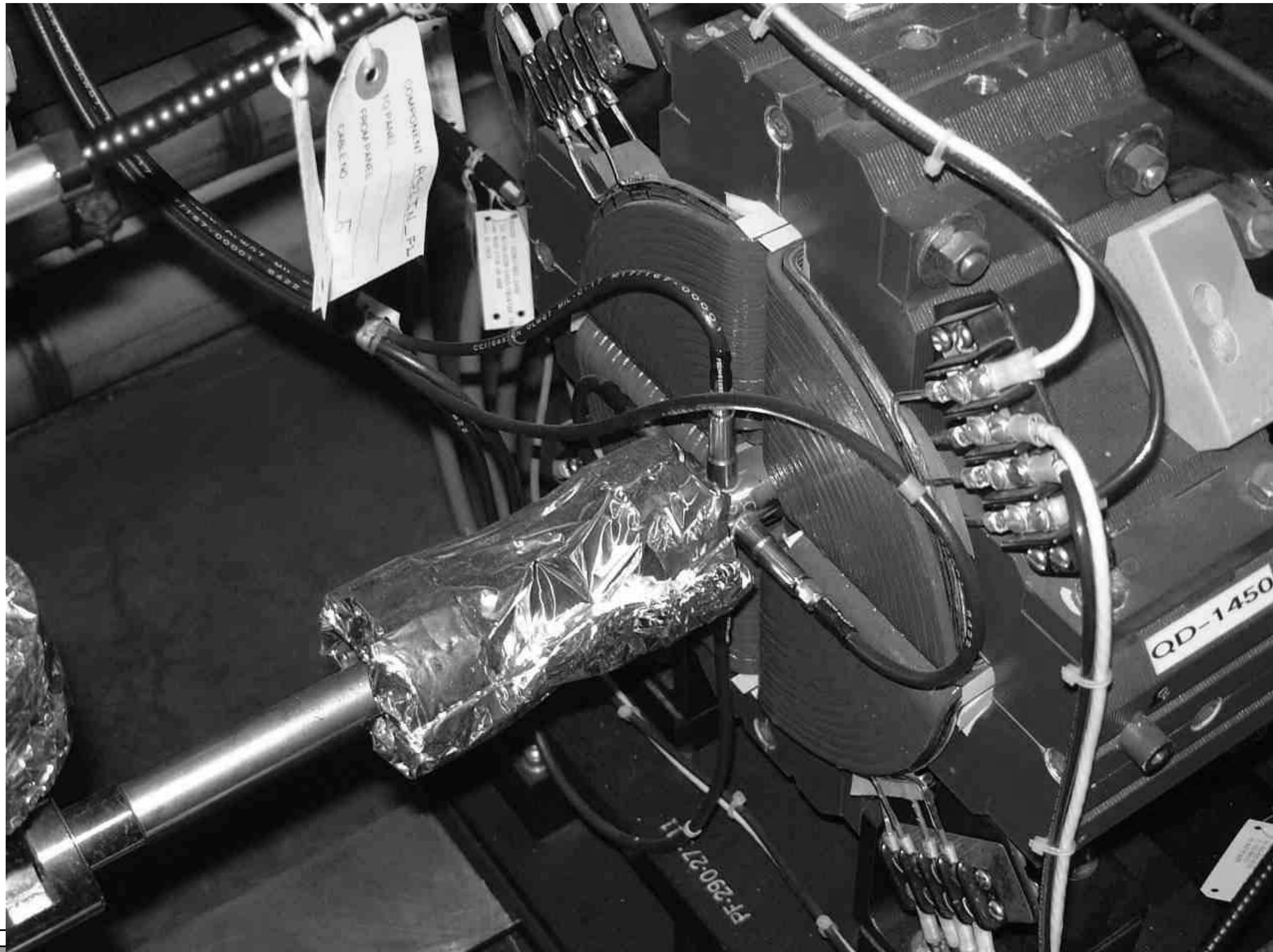
total current per coil

$$I_{\text{coil}}[\text{A}] = \frac{1}{2\mu_0} g[\text{T/m}] R[\text{m}]^2 = 39.789\, g[\text{T/m}] R[\text{cm}]^2$$

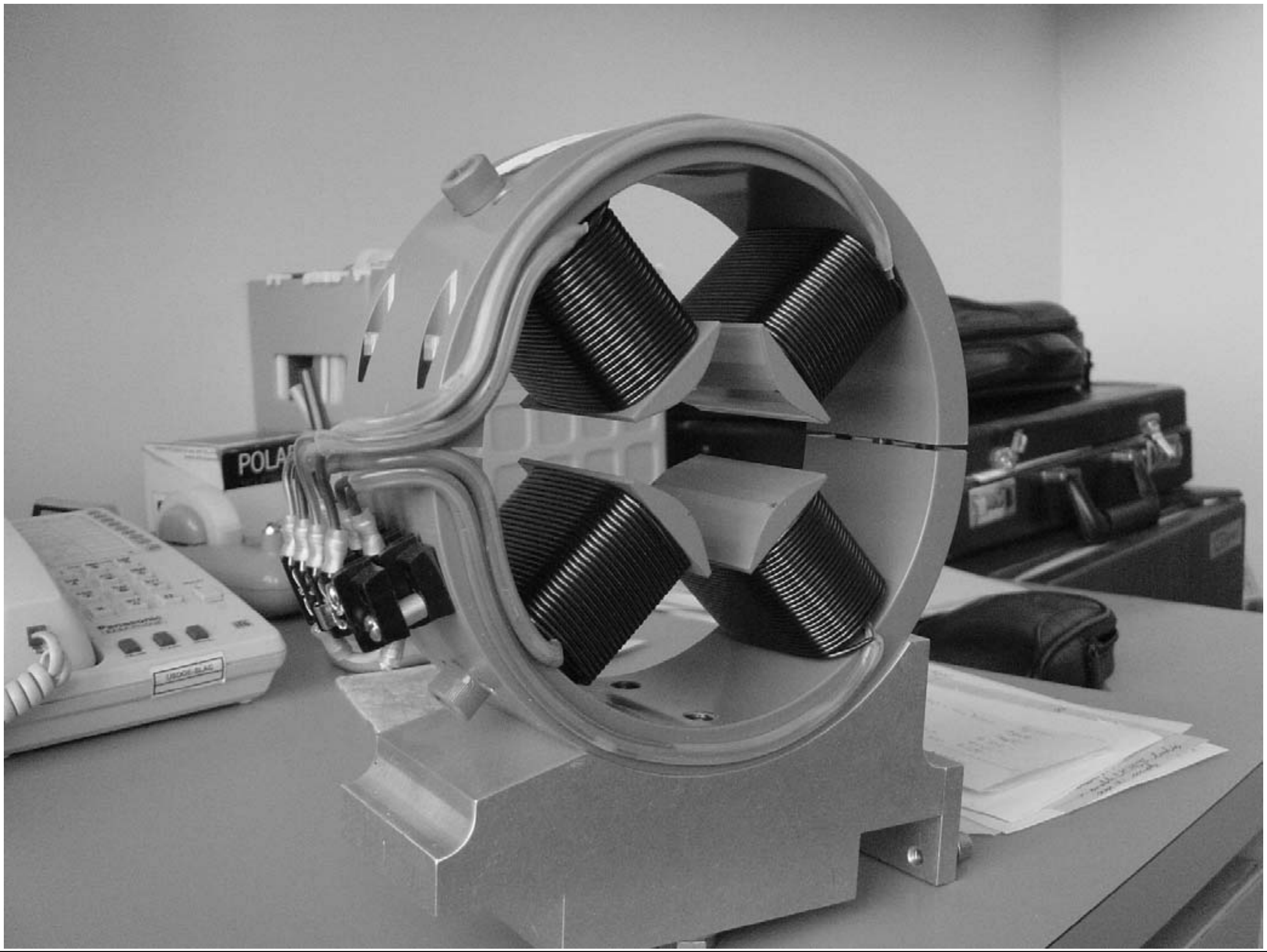
for n_t windings per coil, the power supply current is:

$$I_{\text{ps}}[A] = I_{\text{coil}}[\text{A}]/n_t$$

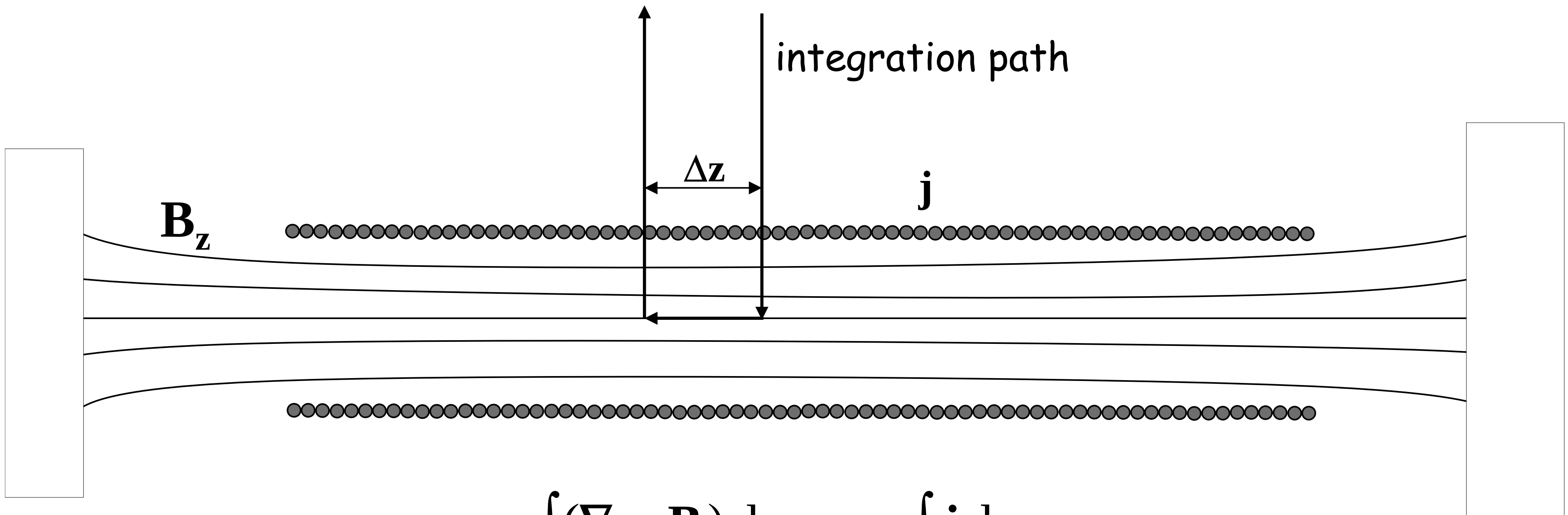
photo of quadrupole



real quadrupole - 1



Solenoid



$$\int (\nabla \times \mathbf{B})\, d\mathbf{a} = \mu_0 \int \mathbf{j}\, d\mathbf{a}$$

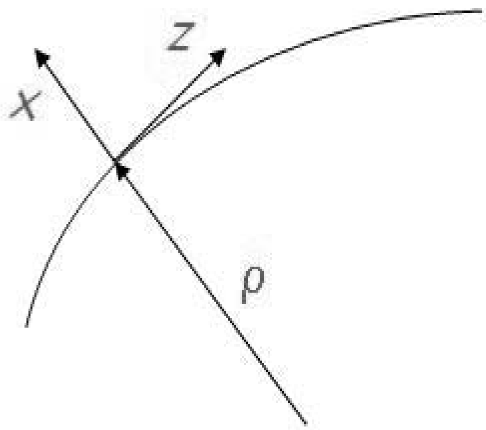
$$\int (\nabla \times \mathbf{B})\, d\mathbf{a} = \oint \mathbf{B}\, d\mathbf{s} = B_z \Delta z \qquad \mu_0 \int \mathbf{j}\, d\mathbf{a} = \mu_0\, j\, \Delta z$$

$$\boxed{B_z = \mu_0\, j}$$

$$B_z(\mathrm{T}) = 4\pi 10^{-7}\, j\,(\mathrm{A/m})$$

Equations of Motion

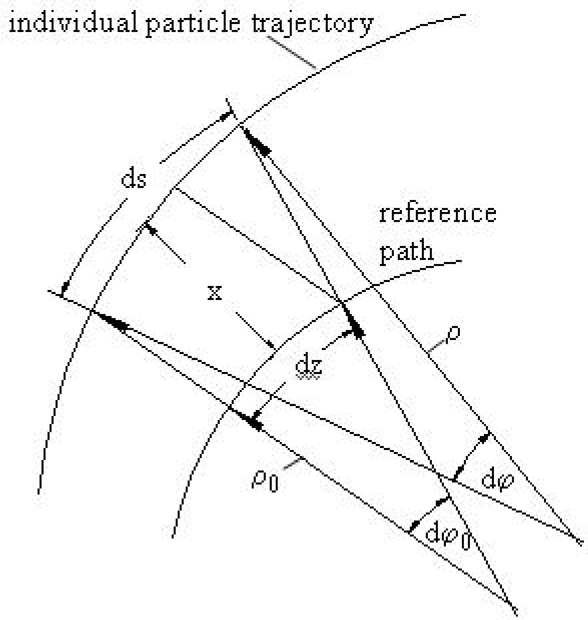
Equations of Motion



Curvilinear coordinate system of beam dynamics (x,y,z)

Origin of coordinate system moves with reference particle

Equations of Motion – 1



with $\kappa = \frac{1}{\rho}$

$$d\varphi_0 = \kappa_0 \, dz$$
$$d\varphi = \kappa \, ds$$
$$ds = (1 + \kappa_0 x) dz$$

flat ring

$$\boxed{u'' = -\frac{d\varphi - d\varphi_0}{dz} = -\kappa(1 + \kappa_0 x) + \kappa_0}$$

$$\kappa_x = \frac{e}{cp} B_y = \frac{e}{cp}\left[B_{y0} + gx + \tfrac{1}{2}s(x^2 - y^2) + \ldots\right]$$

$$\kappa_y = -\frac{e}{cp} B_x = -\frac{e}{cp}\left[B_{x0} + gy + sxy + \ldots\right]$$

$$\frac{e}{cp} = \frac{e}{cp_0(1+\delta)} \approx \frac{e}{cp_0}(1 + \delta + \delta^2 + \ldots)$$

Equations of Motion – 2

equations of motion up to 2nd order and horizontal deflection only:

$$x'' + (k_0 + \kappa_{x0}^2)x = \kappa_0\delta(1-\delta) + (k_0 + \kappa_{x0}^2)\,x\delta - k_0\kappa_{x0}x^2 - \frac{1}{2}m_0(x^2 - y^2) + \ldots$$

$$y'' - k_0 x = +k_0\,y\delta - k_0\kappa_{x0}y^2 + m_0\,xy + \ldots$$

κ_{x0}^2 - terms occur for sector magnets only

Hamiltonian Eq of M - 1

Hamiltonian in curvilinear coordinates of beam dynamics

$$\mathcal{H}(x, p_x, y, p_y, z) = -\frac{e\overline{A}_z}{cp_0} h(1-\delta) - \sqrt{h^2 - h^2 x'^2 - h^2 y'^2}$$

$$= -\frac{e\overline{A}_z}{cp_0} h(1-\delta) - \sqrt{h^2 - p_x^2 - p_y^2} \qquad [h = (1+\kappa x)]$$

vector potential for bending magnet only

K. R. Symon: $\frac{e}{cp} A_{z1} = -\left(\kappa x + \frac{1}{2}\kappa^2 x^2\right)$ I don't think this is correct!

should be: $\frac{e}{cp} A_{z1} = -\left(\frac{1}{2} + \kappa x + \frac{1}{2}\kappa^2 x^2\right)$

or $\frac{e}{cp} \overline{A}_{z1} = -\frac{\frac{1}{2} + \kappa x + \frac{1}{2}\kappa^2 x^2}{1+\kappa x} = -\frac{1}{2}(1+\kappa x)$

vector potential for common beam dynamics

$$\frac{e}{cp} \overline{A}_{z1} = -\frac{1}{2}(1+\kappa x) - \frac{1}{2} k (x^2 - y^2) - \frac{1}{6} m (x^3 - 3xy^2)$$

Hamiltonian Eq of M - 2

first Hamiltonian equation $x' = \frac{\partial\mathcal{H}}{\partial p_x} = -\frac{-p_x}{\sqrt{h^2 - p_x^2}} = \frac{x'}{\sqrt{1-x'^2}} \approx x'$

second Hamiltonian equation

$$\frac{\partial\mathcal{H}}{\partial x} = -(1+\kappa x)\,x'' - \kappa\,x'^2$$

$$= -\frac{e}{cp_0}\frac{\partial\overline{A}_z}{\partial x}(1+\kappa x)(1-\delta) - \frac{e\overline{A}_z}{cp_0}\kappa(1-\delta) - \kappa\sqrt{1-x'^2}$$

$$\frac{e}{cp}\frac{\partial\overline{A}_{z1}}{\partial x} = -\tfrac{1}{2}\kappa - kx - \tfrac{1}{2}m(x^2 - y^2)$$

collect everything

Hamiltonian Eq of M - 3

$$
\begin{aligned}
-(1+\cancel{\kappa x})\,x'' - \cancel{\kappa x'^2} &= \left[\tfrac{1}{2}\kappa + kx - \tfrac{1}{2}m(x^2-y^2)\right](1+\kappa x)(1-\delta) \\
&\quad + \left[\tfrac{1}{2}(1+\kappa x) + \tfrac{1}{2}k(x^2-y^2) + \tfrac{1}{6}m(x^3-3xy^2)\right]\kappa(1-\delta) \\
&\quad - \kappa\sqrt{1-x'^2} \\
&= \tfrac{1}{2}\kappa + \tfrac{1}{2}\kappa^2 x - \tfrac{1}{2}\kappa\delta - \tfrac{1}{2}\kappa^2 x\delta + kx - kx\delta + \cancel{\kappa k x^2} - \tfrac{1}{2}m(x^2-y^2) \\
&\quad + \tfrac{1}{2}\kappa + \tfrac{1}{2}\kappa^2 x - \tfrac{1}{2}\kappa\delta - \tfrac{1}{2}\kappa^2 x\delta + \kappa\cancel{\tfrac{1}{2}}k(x^2-y^2) - \kappa + \tfrac{1}{2}\cancel{\kappa x'^2} + \ldots
\end{aligned}
$$

equation of motion:

$$x'' + (k+\kappa^2)\,x = \kappa\delta + (k+\kappa^2)x\delta - \tfrac{1}{2}m(x^2-y^2) + \ldots$$

focusing of sector magnet

dispersion

chromatic aberration

$$y'' - ky = kx\delta + \tfrac{1}{2}mxy + \ldots$$

Perturbation Hamiltonian

formulating the Hamiltonian with perturbations:

$$\tilde{\mathcal{H}} = \tfrac{1}{2}(k-\kappa^2)x^2 + \tfrac{1}{2}x'^2 - \kappa x\delta - \tfrac{1}{2}(k-\kappa^2)x^2\delta + \tfrac{1}{6}mx^3$$

comparison with the equation of motion

$$x'' + (k+\kappa^2)x = \kappa\delta + (k+\kappa^2)\,\delta x - \tfrac{1}{2}m(x^2-y^2) + \ldots = p_{r,t}(z)x^r y^t$$

gives the more common form of the Hamiltonian

$$\boxed{\tilde{\mathcal{H}} = \tfrac{1}{2}(k+\kappa^2)x^2 + \tfrac{1}{2}x'^2 - \int p_{r,t}(z)x^r y^t \mathrm{d}z}$$

Linear solution

linear equations of motion
$$x'' + k_0 x = 0$$
$$y'' - k_0 x = 0$$
with $k_0 = k_0(z)$

solution
$$\boxed{u = a\sqrt{\beta(z)}\cos[\psi(z) - \psi_0]}$$

where

betatron function $\beta(z)$

betatron phase $\psi(z) = \int_0^z \frac{\mathrm{d}\sigma}{\beta(\sigma)} + \psi_0$

betatron function is solution of $\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 k(z) = 1$

Use: bending magnets, quadrupoles, sextupoles and octupoles

Equation of motion: $$x'' + k(s)x = P(x,y,s)$$

Perturbation terms:

$$P(x,y,s) = \frac{1}{\rho}\delta + k\delta x - \frac{1}{2}m(x^2 - y^2) - \frac{1}{6}r(x^3 - 3xy^2) + \ldots.$$

dispersion

chromatic aberration

sextupole

octupole

General solution: $$x = x_\beta + x_o + x_\delta + \ldots..$$

betatron oscillations

orbit distortion

dispersion function

use uncoupled motion only:

orbit distortion: $x_o'' + k(s)x_o = -\Delta\frac{1}{\rho} - \frac{1}{2}mx_o^2 \ldots.$

dispersion:

$$x_\delta'' + k(s)x_\delta = \tfrac{1}{\rho}\delta - \tfrac{1}{\rho}\delta^2 + k\,\delta\,x_\delta \ldots \qquad \text{where } x_\delta = \eta(s)\,\delta$$

betatron oscillations:

$$x_\beta'' + k(s)x_\beta = (k + m\eta)\,\delta\,x_\beta - \tfrac{1}{2}mx_\beta^2 \ldots.$$

chromatic aberrations, natural chromaticity

chromaticity correction by sextupoles

geometric aberrations

Normalized Coordinates

Equation of motion:

$$x'' + kx = \frac{1}{\rho}\delta + k\delta x - \frac{1}{2}m(x^2 - y^2) + \ldots. = P_{p,q}(z)\, x^{p-1} y^{q-1}.$$

normalized coordinates: $w = \frac{x}{\sqrt{\beta}};\ \varphi = \int \frac{\mathrm{d}s}{\beta}$ (no coupling)

$$\boxed{\begin{aligned}\ddot{w} + \nu^2 w &= \nu_0^2 \beta^{3/2}\left(\frac{1}{\rho}\delta + \sqrt{\beta}\, k\delta w - \frac{1}{2}\beta^{3/2} m x^2 + \ldots\right) \\ &= -p_n(\varphi)\, n \frac{2^{n/2}}{\nu_0^{n/2}} w^{n-1}\end{aligned}}$$

n : order of perturbation

$n = 1$: dipole

$n = 2$: quadrupole, linear perturbation

$n = 3$: sextupole, quadratic terms

....

Action Angle variables

action-angle variables
Liouville's theorem
non-linear tune shift

goal: find cyclic variable

constants of motion

Action-Angle Variables – 1

For oscillators use action-angle variables $(w, \dot{w}, \varphi) \Rightarrow (\psi, J, \varphi)$

generating function: $G = -\frac{1}{2}\nu_0 w^2 \tan(\psi - \vartheta)$

$$\frac{\partial G}{\partial w} = \dot{w} = -\nu_0 w \tan(\psi - \vartheta)$$

$$\frac{\partial G}{\partial \psi} = -J = -\frac{1}{2}\frac{\nu_0 w^2}{\cos^2(\psi - \vartheta)}$$

Action-Angle Variables - 2

transformation equations:

$$\dot{w} = -\sqrt{2\nu_0 J}\,\sin(\psi - \vartheta)$$

$$w = \sqrt{\frac{2J}{\nu_0}}\,\cos(\psi - \vartheta)$$

Hamiltonian in action-angle variables: $\boxed{H = \nu_0 J}$

Courant-Snyder Invariant

$$\frac{\partial H}{\partial J} = \dot{\psi} = \nu \qquad \nu = \nu_0 = \text{const}$$ oscillation frequency is constant

$$\frac{\partial H}{\partial \psi} = 0 = \dot{J} \qquad J = \text{const.} \qquad J = \frac{1}{2}\nu_0 w^2 + \frac{1}{2}\frac{\dot{w}^2}{\nu_0} = \text{const.}$$

go back to practical coordinates:

$$\boxed{J = \frac{1}{2}\nu_0\left(\gamma u^2 + 2\alpha u u' + \beta u'^2\right) = \frac{1}{2}\nu_0 \epsilon}$$

Courant-Snyder Invariant or emittance is a constant of motion

Liouville's Theorem

equation of motion in curvilinear coordinates $x'' + kx = P(x,y,z)$

in normalized coordinates

$$\ddot{w} + \nu_0^2 w = \nu_0^2 \beta^{3/2} P(x,y,z) = \nu_0^2 \beta^{3/2} \sum_{k=2} \overline{p}_k(\varphi)\, \beta^{\frac{k-1}{2}} w^{k-1}$$

Hamiltonian in normalized coordinates

$$\boxed{H = \tfrac{1}{2}\dot{w}^2 + \tfrac{1}{2}\nu_0^2 w^2 + \sum_{k=2} p_k(\varphi)\left(\tfrac{\nu_0}{2}\right)^{k/2} w^k}$$

with $p_k(\varphi)\left(\frac{\nu_0}{2}\right)^{k/2} = \nu_0^2 \overline{p}_k(\varphi)\, \beta^{\frac{k+2}{2}} \frac{1}{k}$

in action-angle coordinates while keeping only n-th order perturbations:

$$\boxed{\overline{H} = \nu_0 J + p_n(\varphi)\, J^{n/2} \cos^n(\psi - \theta)}$$

Hamiltonian in action-angle variables – 2

equation of motion gives oscillator frequency including the effects of perturbations

$$\frac{\partial \overline{H}}{\partial J} = \dot{\psi} = \nu = \nu_0 + \frac{n}{2} p_n(\varphi)\, J^{n/2-1} \cos^n(\psi - \theta)$$

recall unperturbed Hamiltonian

$\overline{H} = \nu_0 J$ coordinate ψ is cyclic ! or J is constant of motion

Non-linear tune shift

general expression for tune: $\nu = \nu_0 + \frac{n}{2} p_n(\varphi)\, J^{n/2-1} \cos^n(\psi - \theta)$

perturbations are periodic with φ:

$$p_n(\varphi) = \sum_q p_{nq} \mathrm{e}^{\mathrm{i}qN\varphi}$$

$$\cos^n\psi = \sum_{|m|\le n} c_{nm} \mathrm{e}^{\mathrm{i}m\psi}$$

N : superperiodicity

with this $p_n(\varphi)\cos^n\psi = c_{n0}p_{n0} + \sum_{\substack{q\ge 0 \\ 0<m\le n}} 2c_{nm}\, p_{nq} \cos(m\psi - qN\varphi)$

and Hamiltonian is

$$\mathcal{H} = \nu_0 J + c_{n0}p_{n0}\, J^{n/2} + J^{n/2} \sum_{\substack{q\ge 0 \\ 0<m\le n}} 2c_{nm}\, p_{nq} \cos(m\psi - qN\varphi)$$

General, nonlinear tune shift

$$\mathcal{H} = \nu_0 J + c_{n0} p_{n0} J^{n/2} + J^{n/2} \sum_{\substack{q\geq 0 \\ 0<m\leq n}} 2c_{nm} p_{nq} \cos(m\psi - qN\varphi)$$

mostly oscillatory terms

Note ! a coherent tune shift for the whole beam exists only for $n=2$
for $n \geq 3$ the tune shift is amplitude dependent and we get therefore a tune spread within the beam

third term on r.h.s. looks oscillatory and therefore ignorable ! ?

not all !

what if $m\psi_0 \approx qN\varphi$!

resonances
resonance patterns
3rd-order resonance
phase space motion
resonance extraction/injection
stop bands and stop band widths

look for slowly varying terms in

$$\mathcal{H} = \nu_0 J + c_{n0} p_{n0} J^{n/2} + J^{n/2} \sum_{\substack{q \geq 0 \\ 0<m\leq n}} 2c_{nm} p_{nq} \cos(m\psi - qN\varphi)$$

$m_r \psi_r \approx qN\varphi$ or with $\psi \approx \nu_0 \varphi$: $m_r \nu_0 \approx rN$

$$\boxed{H = \nu_0 J + c_{n0} p_{n0} J^{n/2} + J^{n/2} \sum_{\substack{r \\ 0<m_r\leq n}} 2c_{nm_r} p_{nr} \cos(m_r \psi_r)}$$

canonical transformation (J,ψ) to (J_1,ψ_1)

$$G_1 = J_1 \left(\psi - \frac{rN}{m_r}\varphi\right)$$

Resonances (cont.)

$$H_1 = (\nu_0 - \tfrac{rN}{m_r}) J_1 + c_{n0}\, p_{n0}\, J_1^{n/2} + \tilde{p}_{nr}\, J_1^{n/2} \cos(m_r \psi_1)$$

resonance, when $\nu_0 \approx \frac{rN}{m_r}$ m_r is order of resonance

all resonances

$$H_1 = \Delta\nu_0\, J_1 + \sum_n c_{n0}\, p_{n0}\, J_1^{n/2} + \sum_n \sum_{\substack{r \\ 0<m_r\le n}} \tilde{p}_{nr}\, J_1^{n/2} \cos(m_r \psi_1)$$

consider only resonances of order n

Start with amplitude J_0 and divide Hamiltonian by $2\, c_{nm_r} p_{nr}\, J_0^{n/2}$

$$R = \frac{J}{J_0}$$

$$\boxed{\Delta \cdot R + \Omega R^2 + R^{n/2} \cos(n\psi_1) = \text{const}}$$

Resonances (cont.)

detuning:

$$\Delta = \frac{\Delta\nu_r}{2\, c_{nmr} p_{nr}\, J_0^{n/2-1}}$$

resonance width

tune spread parameter

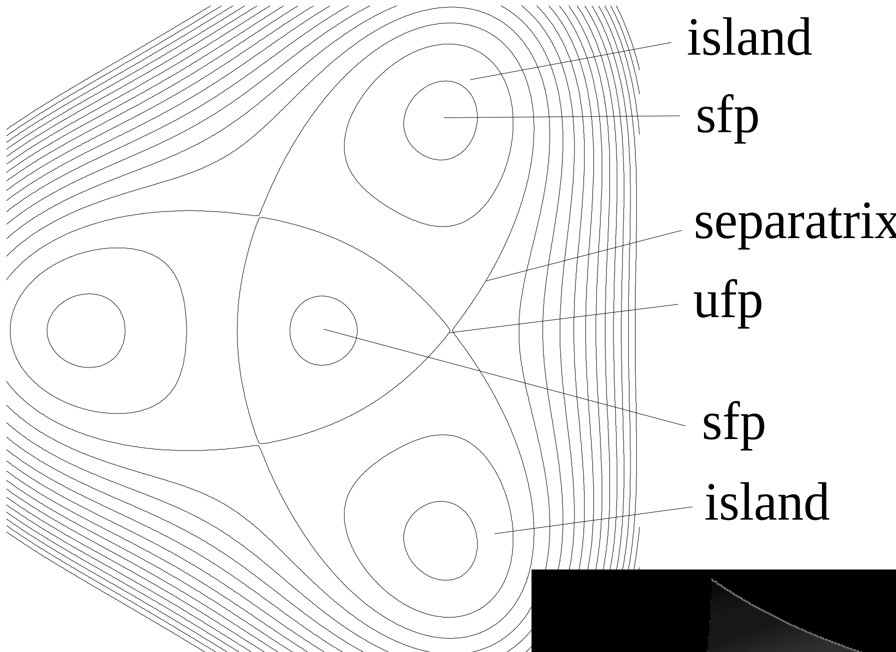
$$\Omega = \frac{c_{40}\, p_{40}}{2\, c_{nmr} p_{nr}\, J_0^{n/2-2}}$$

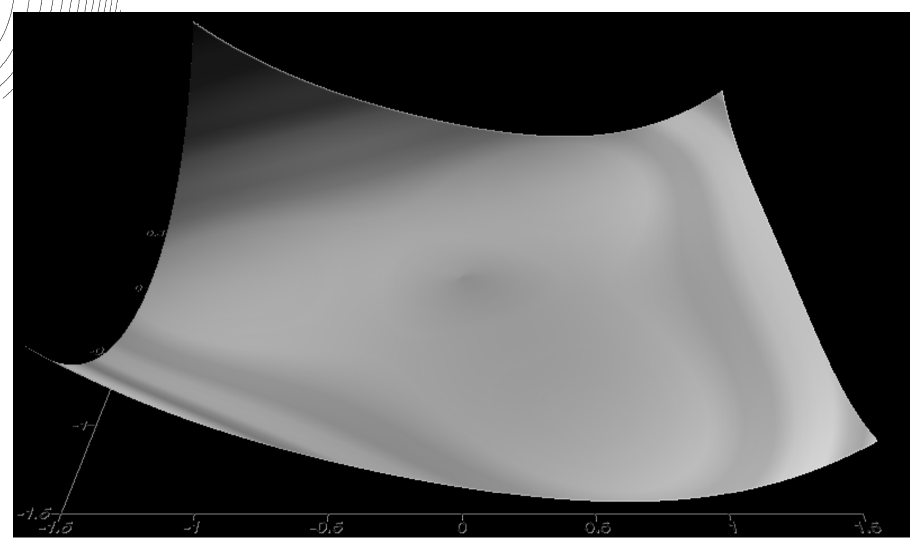
some Landau damping due to octupole field

on resonance $\Delta = 0$ and

$$R^2 \Omega + R^{n/2} \cos n\psi = \mathrm{const}$$

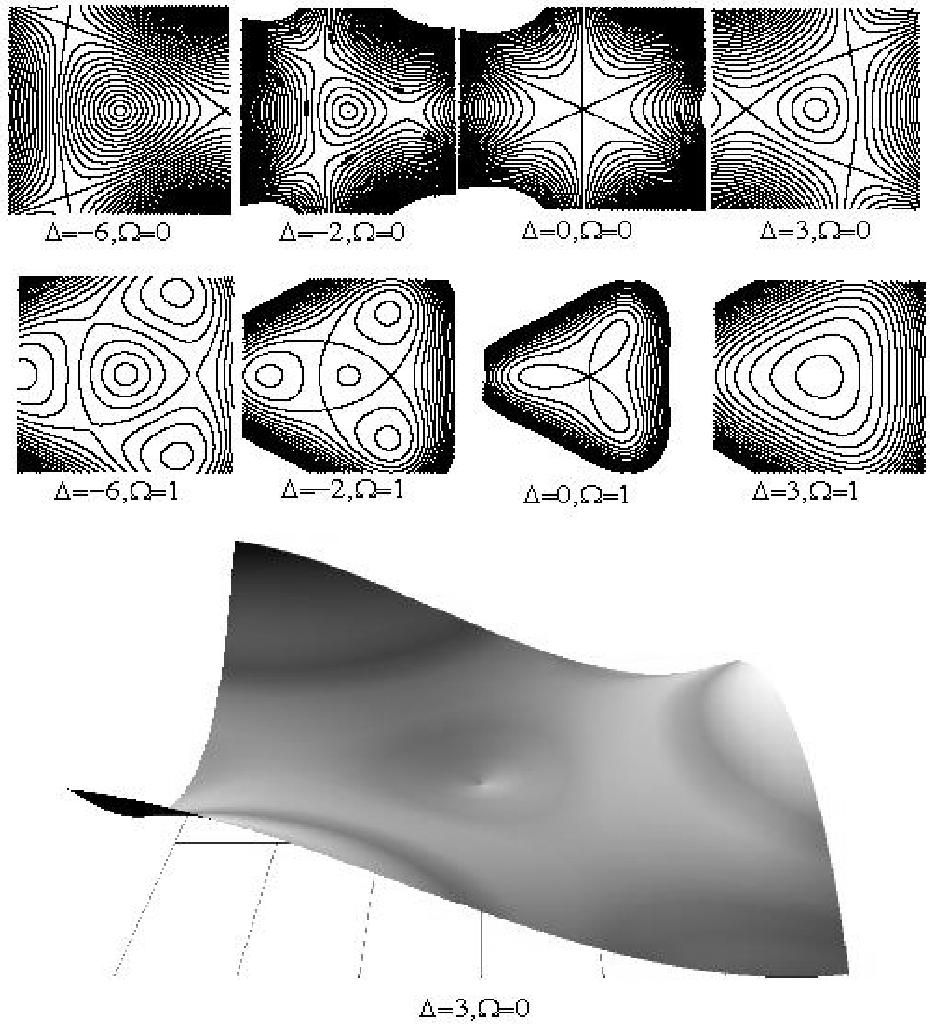
features of resonance patterns



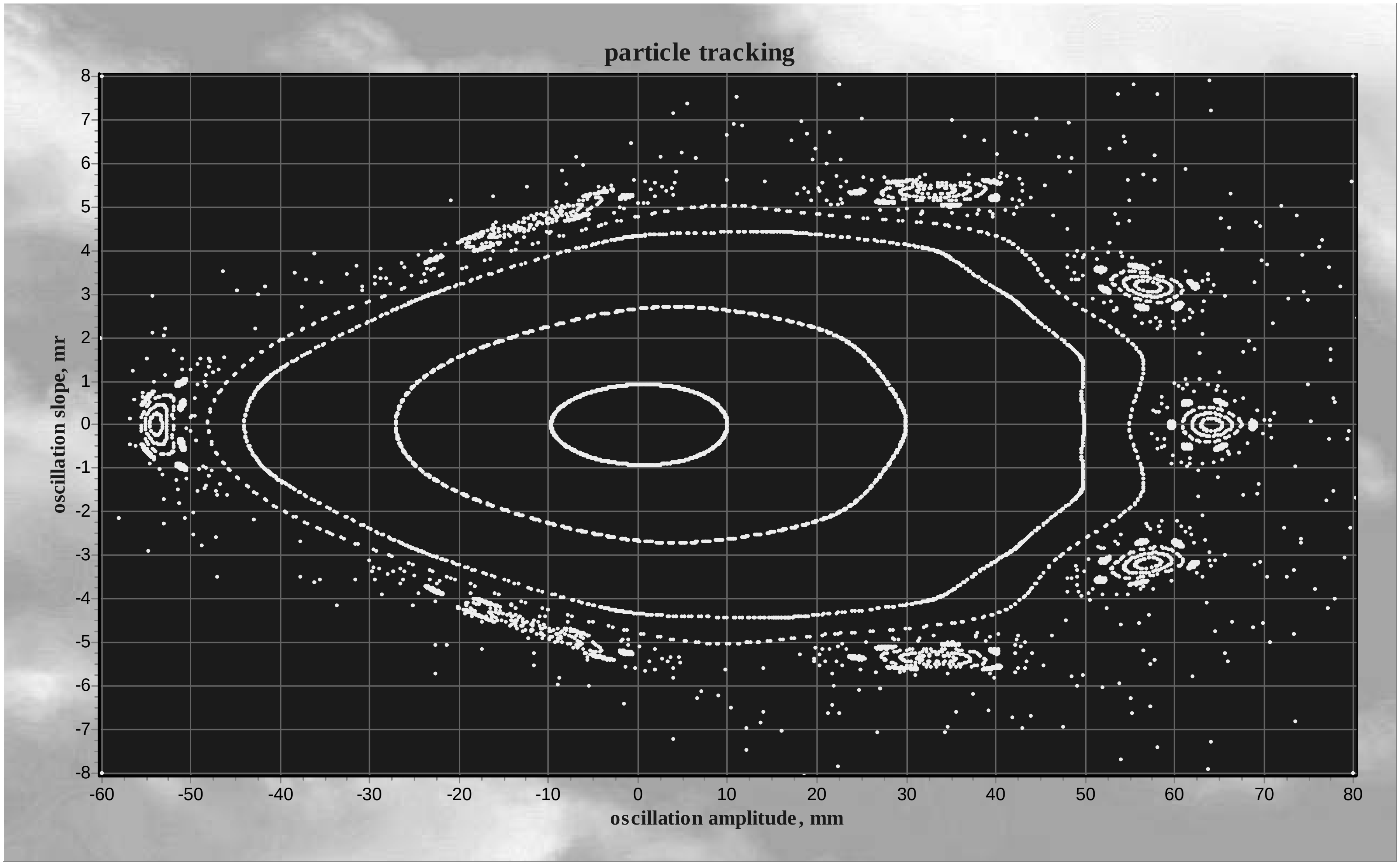


$\Delta=-6.0;\ \Omega=1$

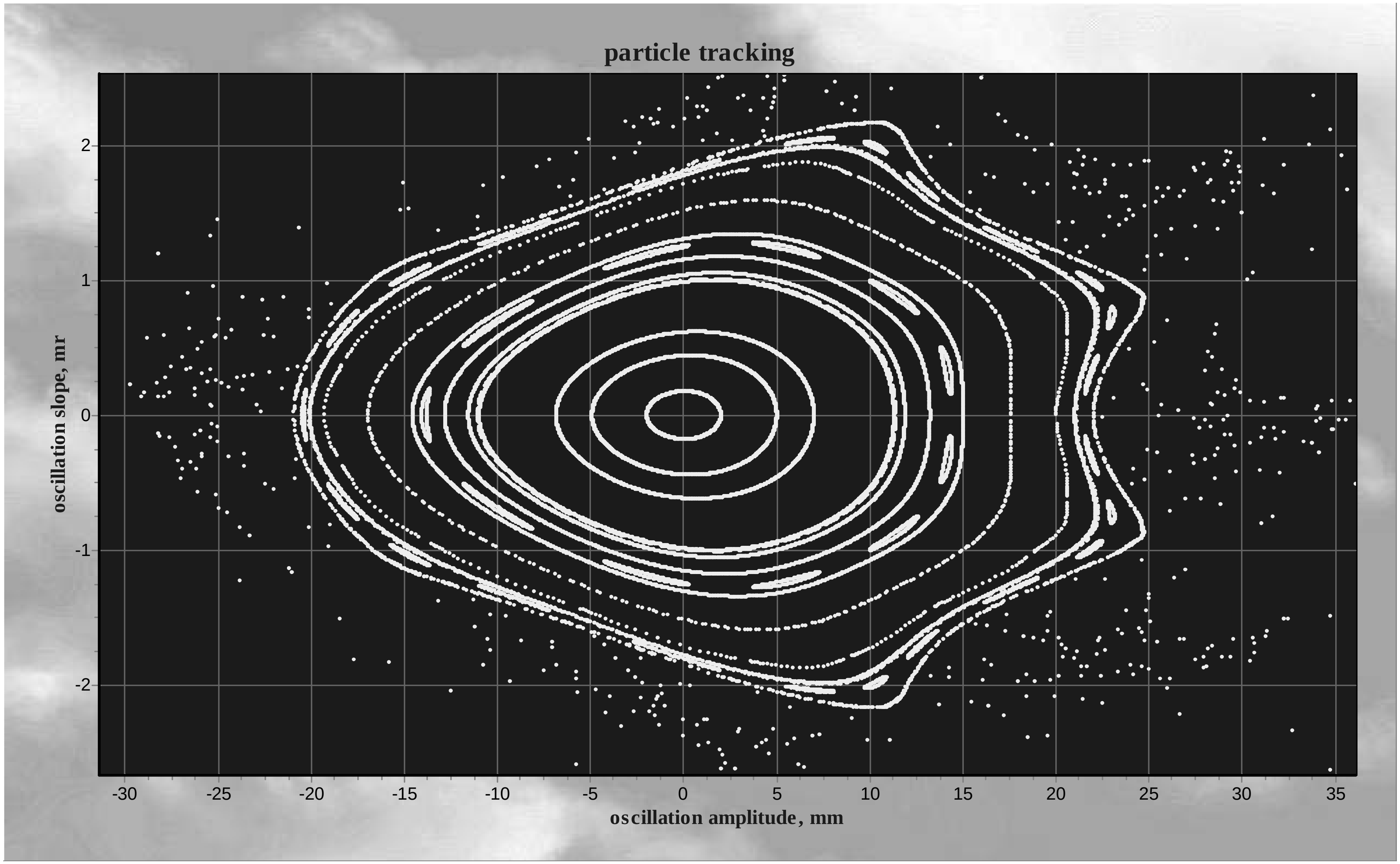
3rd Order Resonance Pattern



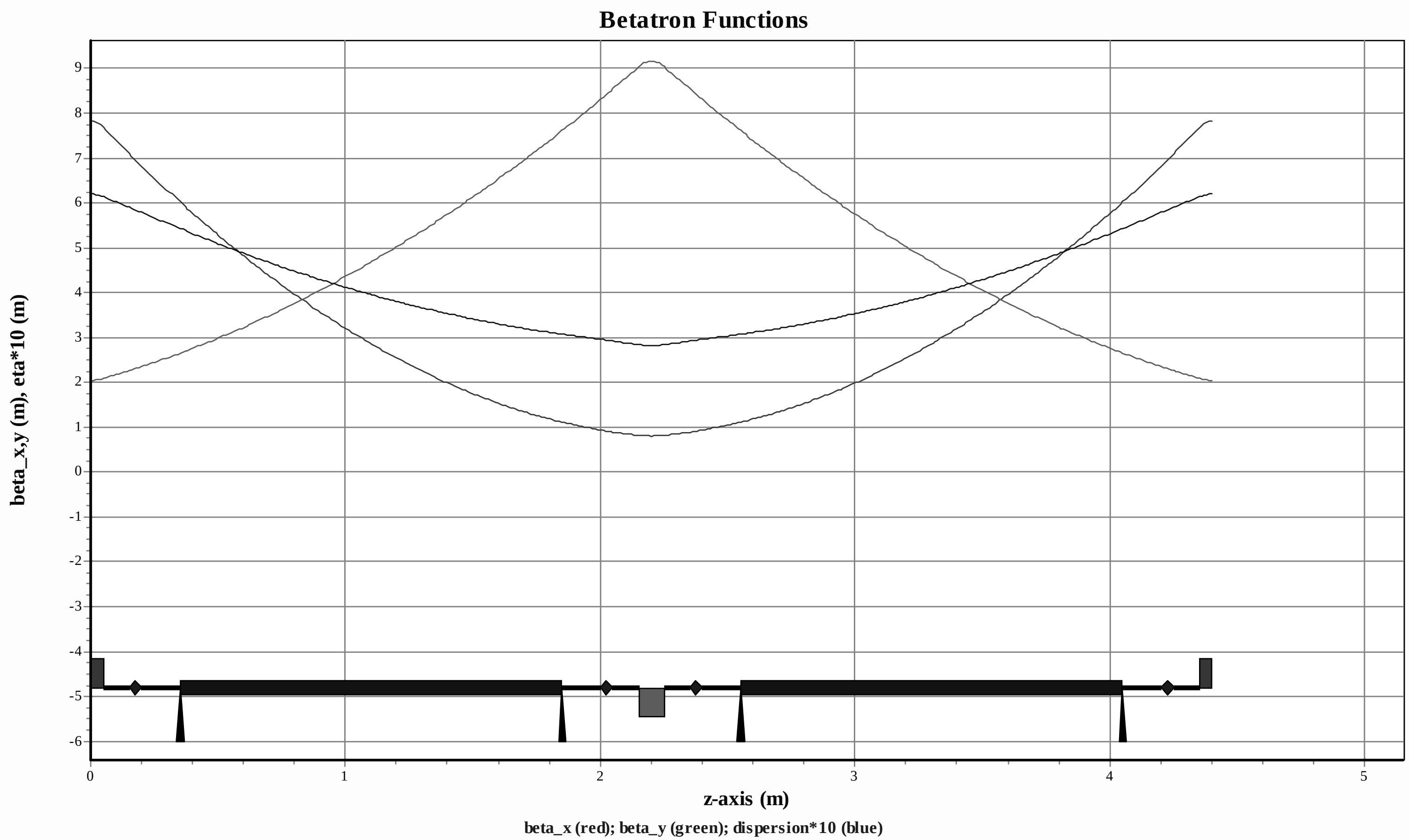
Phase Space Motion in INDUS2



Phase Space Motion in ALS



FODO Lattice



Third-order Resonance-I

Hamiltonian: $H_1 = \Delta\nu_{1/3} J_1 + \widetilde{p}_{3r} J_1^{3/2} \cos(3\psi_1)$

use normalized coordinates:

$$H_1 = ½\Delta\nu_{1/3}\nu_0\left(w^2 + \frac{\dot{w}^2}{\nu_0^2}\right) + \widetilde{p}_{3r}\frac{\nu_0^{3/2}}{2^{3/2}}\left(w^3 - 3w\frac{\dot{w}^2}{\nu_0^2}\right)$$

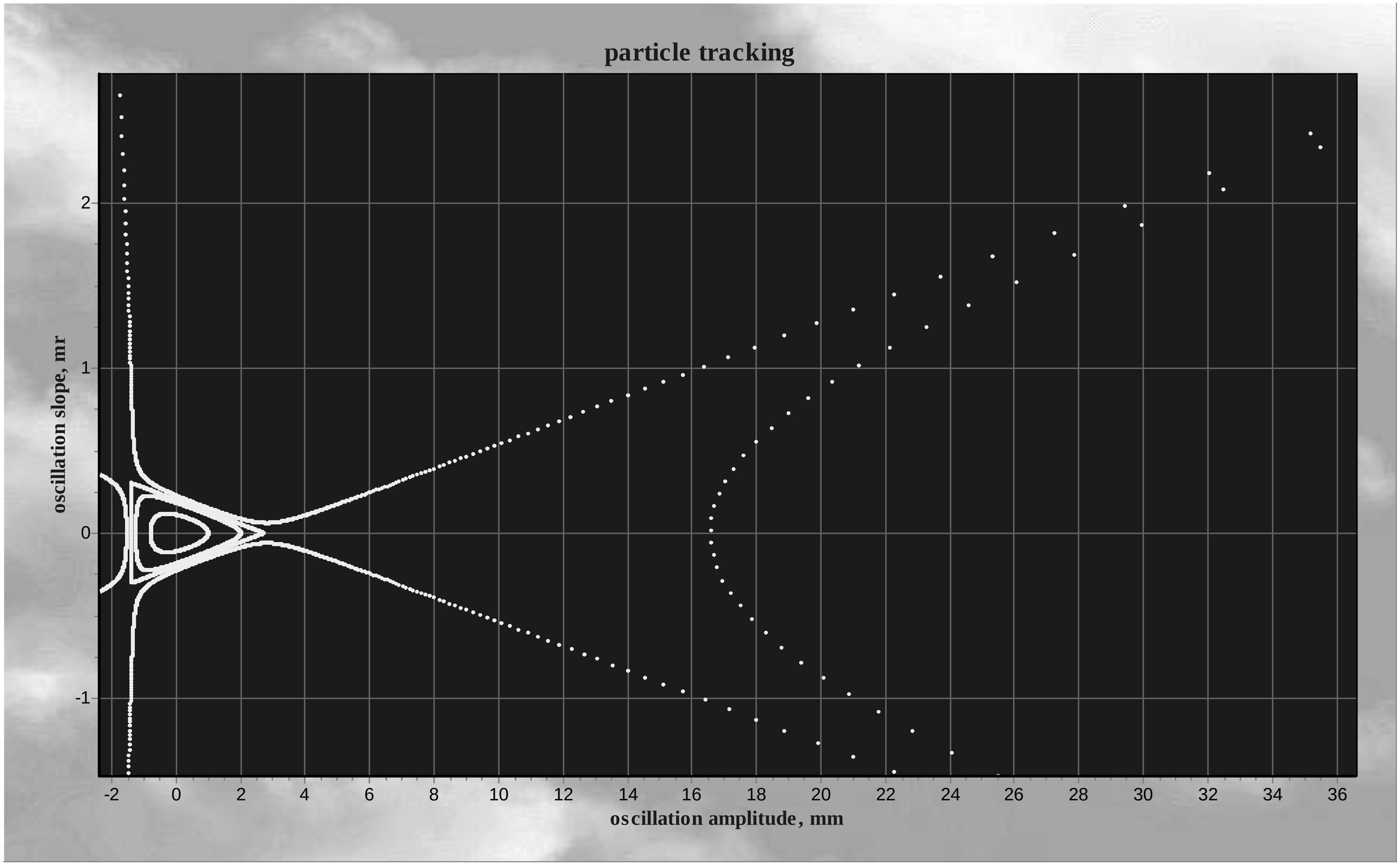
divide by: $\widetilde{p}_{3r}\nu_0^2/4$ and subtract $½W_0^3$

$$\tilde{H}_1 = \frac{3}{2}W_0\left(w^2 + \frac{\dot{w}^2}{\nu_0^2}\right) + \left(w^3 - 3w\frac{\dot{w}^2}{\nu_0^2}\right) - \frac{1}{2}W_0^3$$

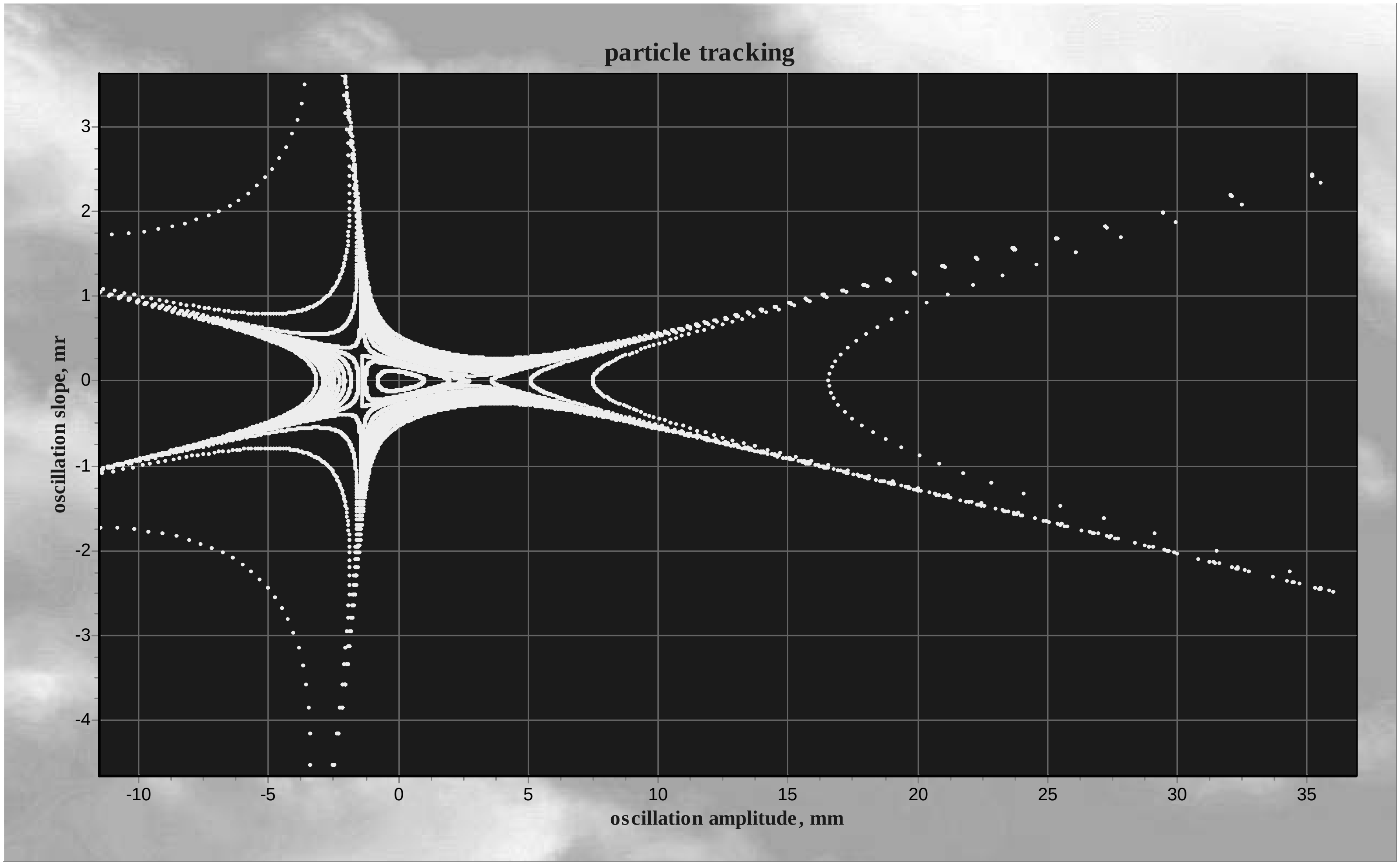
$$= \left(w - \frac{1}{2}W_0\right)\left(w - \sqrt{3}\,\frac{\dot{w}}{\nu_0} + W_0\right)\left(w + \sqrt{3}\,\frac{\dot{w}}{\nu_0} + W_0\right)$$

equations of separatrices

Third-order Resonance-II



Resonance Extraction



Stop band width

we set W=0
next we follow two extreme particles starting at ψ_1=0 and ending at $n\psi_1 = 2\pi$
or starting at $n\psi_1 = \pi$ and going to $n\psi_1 = 3\pi$
starting amplitude R=1

$$R\,\Delta + R^{n/2}\cos n\psi_1 = \Delta \pm 1$$

solving both equations for Δ we get stability for

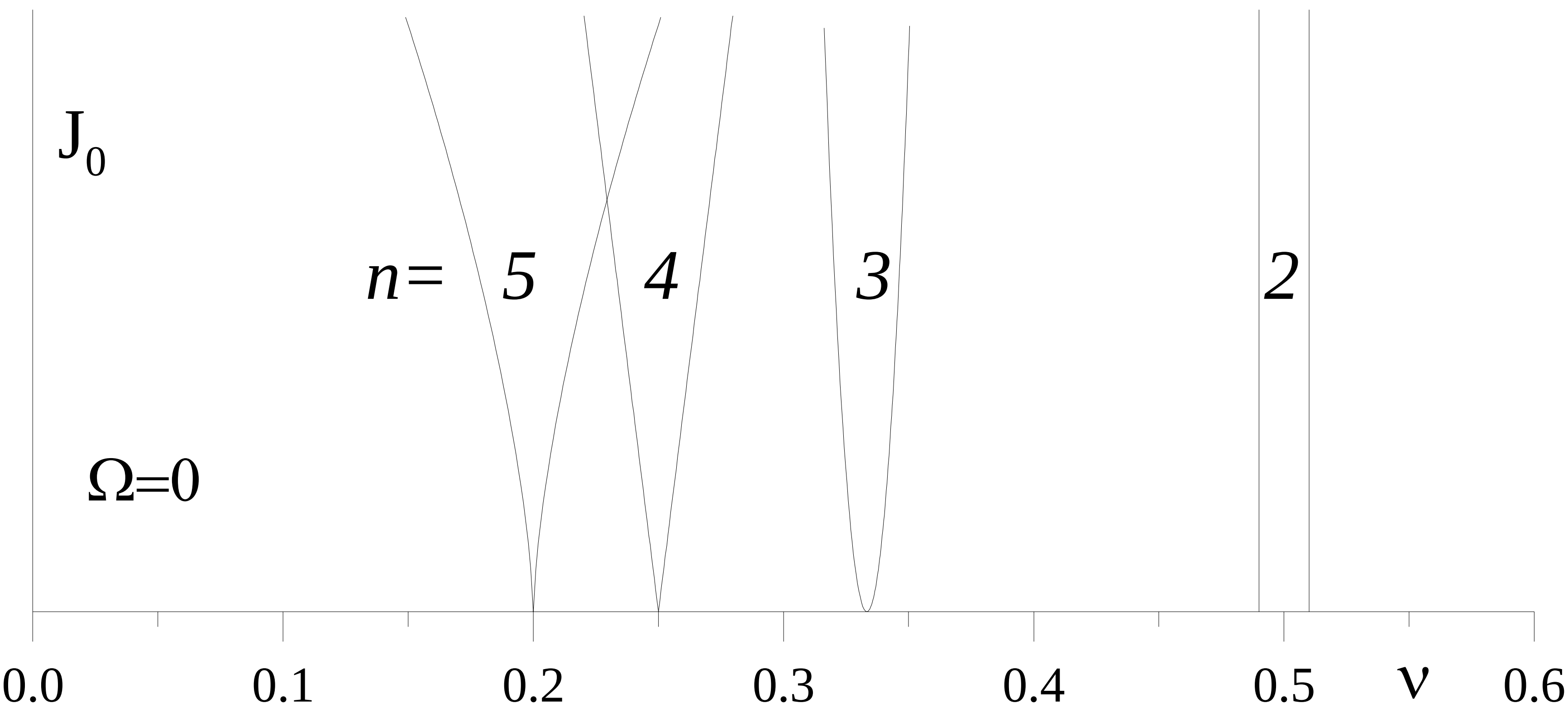
$$\Delta^+ \geq -\frac{R^{n/2-1}}{R-1} = -\frac{n}{2} \qquad \text{and} \qquad \Delta^- \leq \frac{n}{2}$$

total stop band width: $\Delta\nu_{\text{stop}}^{(n)} = 2n\,|c_{nm_r}\,p_{nr}|\,J_0^{n/2-1}$

or

$$\boxed{\Delta\nu_{\text{stop}}^{(n)} = -\frac{n}{2\pi}\left(\frac{\nu_0}{\beta}\right)^{n/2-1} x_0^{n-2}\left|\int_0^{2\pi} p_n(\varphi)\,\mathrm{e}^{-\mathrm{i}rN\varphi}\,\mathrm{d}\varphi\right|}$$

Stop Band Width – graph



particles are unstable between boundaries of stop bands
(perturbation $|2c_{nn}\, p_{nr}|=0.02$)

Hamiltonian and Coupling

Hamiltonian and Coupling – 1

equations of linearly coupled motion

$$x'' + kx = -p(s)\,y$$
$$y'' - ky = -p(s)\,x$$

derived from Hamiltonian $\quad H = \frac{1}{2}x'^2 + \frac{1}{2}y'^2 + \frac{1}{2}kx^2 - \frac{1}{2}ky^2 + p(s)xy$

$H = H_0 + H_1$ where perturbation Hamiltonian is $\quad H_1 = p(s)\,xy$

uncoupled solution

$$u(s) = c_u\sqrt{\beta_u}\,\cos[\psi_u(s) + \phi],$$
$$u'(s) = -\frac{c_u}{\sqrt{\beta_u}}\{\alpha_u(s)\cos[\psi_u(s) + \phi] + \sin[\psi_u(s) + \phi]\},$$

variation of integration constants:

$$u(s) = \sqrt{2a(s)}\,\sqrt{\beta_u}\,\cos[\psi_u(s) + \phi(s)],$$
$$u'(s) = -\frac{\sqrt{2a(s)}}{\sqrt{\beta_u}}\{\alpha_u(s)\cos[\psi_u(s) + \phi(s)] + \sin[\psi_u(s) + \phi(s)]\}$$

Hamiltonian and Coupling – 2

new variables $(u,u') \rightarrow (\phi,a)$

are they canonical?

$$\frac{\partial H}{\partial u'} = \frac{\partial H_0}{\partial u'} + \frac{\partial H_1}{\partial u'} = \frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\partial u}{\partial s} + \frac{\partial u}{\partial a}\frac{\partial a}{\partial s} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial s}$$

$$\frac{\partial H}{\partial u} = \frac{\partial H_0}{\partial u} + \frac{\partial H_1}{\partial u} = -\frac{\mathrm{d}u'}{\mathrm{d}s} = -\frac{\partial u'}{\partial s} - \frac{\partial u'}{\partial a}\frac{\partial a}{\partial s} - \frac{\partial u'}{\partial \phi}\frac{\partial \phi}{\partial s}$$

for uncoupled oscillator $a = \mathrm{const}$ and $\phi = \mathrm{const}$

or $\frac{\partial u}{\partial s} = \frac{\partial H_0}{\partial u'}$ and $\frac{\partial u'}{\partial s} = -\frac{\partial H_0}{\partial u}$

$$-\frac{\partial H_1}{\partial \phi} = -\frac{\partial H_1}{\partial u}\frac{\partial u}{\partial \phi} - \frac{\partial H_1}{\partial u'}\frac{\partial u'}{\partial \phi} = \frac{\partial a}{\partial s} = \frac{\mathrm{d}a}{\mathrm{d}s}$$

$$\frac{\partial H_1}{\partial a} = \frac{\partial H_1}{\partial u}\frac{\partial u}{\partial a} + \frac{\partial H_1}{\partial u'}\frac{\partial u'}{\partial a} = \frac{\partial \phi}{\partial s} = \frac{\mathrm{d}\phi}{\mathrm{d}s}$$

coordinates (a,ϕ) are indeed canonical and

Hamiltonian and Coupling – 3

$$u(s) = \sqrt{2a(s)}\,\sqrt{\beta_u}\,\cos[\psi_u(s) + \phi(s)],$$

$$u'(s) = -\frac{\sqrt{2a(s)}}{\sqrt{\beta_u}}\{\alpha_u(s)\cos[\psi_u(s) + \phi(s)] + \sin[\psi_u(s) + \phi(s)]\}$$

are canonical transformations

insert into perturbation Hamiltonian gives

$$H_1 = 2p(s)\sqrt{\beta_x\beta_y}\,\sqrt{a_x a_y}\,\cos(\psi_x + \phi_x)\cos(\psi_y + \phi_y)$$

and with $\cos(\psi_u + \phi_u) = \frac{1}{2}\left(e^{i(\psi_u+\phi_u)} + e^{-i(\psi_u+\phi_u)}\right)$

$$H_1 = \tfrac{1}{2}p(s)\sqrt{\beta_x\beta_y}\,\sqrt{a_x a_y}\,\sum_{l_x,l_y} e^{i[l_x(\psi_x+\phi_x)+l_y(\psi_y+\phi_y)]}$$

with $l_x, l_y = \pm 1$ and a coupling term like $p(s) = \underline{k}(s)$

Hamiltonian and Coupling – 4

separate slow and fast varying terms

$$\overline{H}_1 = \frac{1}{2} \sum_{l_x, l_y} p(s) \sqrt{\beta_x \beta_y}\, \mathrm{e}^{\mathrm{i}[l_x \psi_x + l_y \psi_y - l_x \nu_{0x} \varphi - l_y \nu_{0y} \varphi]}$$

$$\times \sqrt{a_x a_y}\, \mathrm{e}^{\mathrm{i}[l_x \nu_{0x} \phi_x + l_y \nu_{0y} \phi_y + l_x \varphi_x + l_y \varphi_y]}.$$

factors periodic with lattice: $A(\varphi) = p(s) \sqrt{\beta_x \beta_y}\, \mathrm{e}^{\mathrm{i}[l_x \psi_x + l_y \psi_y - l_x \nu_{0x} \varphi - l_y \nu_{0y} \varphi]}$

Fourier expand $\frac{L}{2\pi} A(\varphi) = \sum_q \kappa_{q l_x l_y}\, \mathrm{e}^{-\mathrm{i} q N \varphi}$

N superperiodicity
L circumference

and define coupling coefficient

$$\begin{aligned} \kappa_{q l_x l_y} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{L}{2\pi} A(\varphi) \mathrm{e}^{\mathrm{i} q N \varphi} \mathrm{d}\varphi \\ &= \frac{1}{2\pi} \int_0^L \sqrt{\beta_x \beta_y}\, \underline{k}(s)\, \mathrm{e}^{\mathrm{i}\left[l_x \psi_x + l_y \psi_y - (l_x \nu_{0x} + l_y \nu_{0y} - qN) 2\pi \frac{s}{L}\right]} \mathrm{d}s. \end{aligned}$$

Hamiltonian and Coupling – 5

or with $l = \pm 1$ the coupling coefficient is

$$\kappa_{ql} = \frac{1}{2\pi} \int_0^L \sqrt{\beta_x \beta_y}\, \underline{k}(s)\, \mathrm{e}^{\mathrm{i}\left[\psi_x + l\psi_y - (\nu_{0x} + l\nu_{0y} - qN) 2\pi \frac{s}{L}\right]} \mathrm{d}s$$

the coupling Hamiltonian becomes now

$$\widetilde{H} = \frac{2\pi}{L} H_1 = \sum_q \kappa_{ql} \sqrt{a_x a_y} \cos(\phi_x + l\phi_y + \Delta\varphi)$$

with $\Delta = \nu_{0x} + l\nu_{0y} - qN$ and the new independent variable $\varphi = \frac{2\pi}{L}$

we keep only slowly varying terms $q = r:\ rN \approx \nu_{0x} + l\nu_{0y}$ or $\Delta_r \approx 0$

one more canonical transformation: $(a_i, \phi_i) \to (\tilde{a}_i, \tilde{\phi}_i)$

from generating function $G = \tilde{a}_x\left(\phi_x + \frac{1}{2}\Delta_r\varphi\right) + \tilde{a}_y\left(\phi_y + l\frac{1}{2}\Delta_r\varphi\right)$

Hamiltonian and Coupling – 6

new variables

$$\tilde{\phi}_x = \frac{\partial G}{\partial \tilde{a}_x} = \phi_x + \tfrac{1}{2}\Delta_r \varphi\,, \qquad a_x = \frac{\partial G}{\partial \phi_x} = \tilde{a}_x\,,$$

$$\tilde{\phi}_y = \frac{\partial G}{\partial \tilde{a}_y} = \phi_y + \tfrac{1}{2}\Delta_r \varphi\,, \qquad a_y = \frac{\partial G}{\partial \phi_x} = \tilde{a}_y\,,$$

resonance Hamiltonian:

$$\widetilde{H}_\mathrm{r} = \widetilde{H} + \frac{\partial G}{\partial \varphi} = \tfrac{1}{2}\Delta_r (a_x + l\,a_y) + \kappa_{rl}\sqrt{a_x a_y}\,\cos(\tilde{\phi}_x + l\,\tilde{\phi}_y)$$

equations of motion:

$$\frac{\partial a_x}{\partial \varphi} = -\frac{\partial \widetilde{H}_\mathrm{r}}{\partial \tilde{\phi}_x} = \kappa_{rl}\sqrt{a_x a_y}\,\sin(\tilde{\phi}_x + l\,\tilde{\phi}_y)\,, \qquad \frac{\partial \tilde{\phi}_x}{\partial \varphi} = \frac{\partial \widetilde{H}_\mathrm{r}}{\partial a_x} = \tfrac{1}{2}\Delta_r + \kappa_{rl}\sqrt{\frac{a_y}{a_x}}\,\cos(\tilde{\phi}_x + l\,\tilde{\phi}_y)\,,$$

$$\frac{\partial a_y}{\partial \varphi} = -\frac{\partial \widetilde{H}_\mathrm{r}}{\partial \tilde{\phi}_y} = l\,\kappa_{rl}\sqrt{a_x a_y}\,\sin(\tilde{\phi}_x + l\,\tilde{\phi}_y)\,, \qquad \frac{\partial \tilde{\phi}_y}{\partial \varphi} = \frac{\partial \widetilde{H}_\mathrm{r}}{\partial a_y} = l\tfrac{1}{2}\Delta_r + \kappa_{rl}\sqrt{\frac{a_x}{a_y}}\,\cos(\tilde{\phi}_x + l\,\tilde{\phi}_y)\,.$$

$l = +1$ sum resonance

$l = -1$ difference resonance

Linear difference and sum resonance

$l = -1$ and $\nu_x - \nu_y = m_r N$ $\Longrightarrow$ $\frac{\mathrm{d}}{\mathrm{d}\varphi}(a_x + a_y) = 0$

since $a_u \propto \epsilon_u$ it follows that $\epsilon_x + \epsilon_y = \text{const}$

no loss of beam!

stop band width $\boxed{\nu_{I,II} = \nu_{x,y} \mp \frac{1}{2}\Delta_r \pm \frac{1}{2}\sqrt{\Delta_r^2 + \kappa^2}}$

for sum resonance $\frac{\mathrm{d}}{\mathrm{d}\varphi}(a_x - a_y) = 0$ or $\epsilon_x - \epsilon_y = \text{const}$

beam size can grow indefinitely, leading to beam loss !

stability criterion: $\Delta_r > \kappa$

Beam Filamentation

Filamentation -1

phase space motion under the influence of non-linear terms

action for unperturbed motion $J = \frac{1}{2}\nu_0 w^2 + \frac{1}{2}\frac{\dot{w}^2}{\nu_0}$

variation of action: $\Delta J_x = \nu_{x0} w\Delta w + \frac{1}{\nu_{x0}}\dot{w}\Delta\dot{w} = \frac{1}{\nu_{x0}}\dot{w}\Delta\dot{w}$

$$\Delta J_y = \nu_{y0} v\Delta v + \frac{1}{\nu_{y0}}\dot{v}\Delta\dot{v} = \frac{1}{\nu_{y0}}\dot{v}\Delta\dot{v}$$

with $\Delta w = \Delta v = 0$ and $\Delta\dot{w} = \nu_{x0}\sqrt{\beta_x}\,\frac{1}{2}m\ell(x^2 - y^2)$

$$\Delta\dot{v} = -\nu_{y0}\sqrt{\beta_y}\; m\ell xy$$

increase of action due to one sextupole

$$\Delta J_x = \frac{m\ell}{4}\sqrt{\frac{2J_x\beta_x}{\nu_{x0}}}\left\{\left(J_x\beta_x - 2J_y\,\beta_y\frac{\nu_x}{\nu_y}\right)\sin\psi_x + J_x\beta_x\sin 3\psi_x \right.$$
$$\left. -J_y\beta_y\frac{\nu_x}{\nu_y}[\sin(\psi_x + 2\psi_y) + \sin(\psi_x - 2\psi_y)]\right\}$$

$$\Delta J_y = \frac{m\ell}{2}\sqrt{\frac{2J_x\beta_x}{\nu_{x0}}}\; J_y\beta_y[\sin(\psi_x + 2\psi_y) - \sin(\psi_x - 2\psi_y)]$$

Filamentation – 2

now we need to sum over all sextupoles and over all turns

from these expressions we can evaluate
the variation of the action (beam emittance)

for the moment we assume that there is only one sextupole

summation over turns n gives sums like

$$\sum_{n=0}^{\infty} \sin[(\psi_{xj} + 2\pi\nu_{x0}n) + 2(\psi_{yj} + 2\pi\nu_{y0}n)]$$

ψ_{xj}, ψ_{yj} are the phases at sextupole locations j

$$\mathcal{I}m\left[e^{i(\psi_{xj}+2\psi_{yj})} \sum_{n=0}^{\infty} e^{i2\pi(\nu_{x0}+2\nu_{y0})n} \right] = \mathcal{I}m \frac{e^{i(\psi_{xj}+2\psi_{yj})}}{1-e^{i2\pi(\nu_{x0}+2\nu_{y0})}}$$

$$\mathcal{I}m \frac{e^{i(\psi_{xj}+2\psi_{yj})}}{1-e^{i2\pi(\nu_{x0}+2\nu_{y0})}} = \frac{\cos[(\psi_{xj}-\pi\nu_{x0})+2(\psi_{yj}-\pi\nu_{y0})]}{2\sin[\pi(\nu_{x0}+2\nu_{y0})]}$$

resonance if $\nu_{x0} + 2\nu_{y0} = p$ (p integer)

Filamentation – 3

there are 4 resonant terms: $\nu_{x0} = p_1$ $\quad \nu_{x0} + 2\nu_{y0} = p_3$

$3\nu_{x0} = p_3$ $\quad \nu_{x0} - 2\,\nu_{y0} = p_4$

summation over all sextupoles

$$\Delta J_{x,\nu_x+2\nu_y} = -\sum_j \frac{m_j \ell_j}{4} \sqrt{\frac{2J_x \beta_{xj}}{\nu_{xo}}}\, J_y \beta_{yj} \frac{\nu_x}{\nu_y} \sin(\psi_{xj} + 2\psi_{yj})$$

to minimize perturbations and emittance dilution distribute sextupoles such that certain harmonics are minimized:

harmonic correction :

$$\sum_j m_j \ell_j \beta_x^{3/2} \mathrm{e}^{\mathrm{i}\psi_{xj}} \quad \rightarrow \quad 0$$

$$\sum_j m_j \ell_j \beta_x^{3/2} \mathrm{e}^{\mathrm{i}3\psi_{xj}} \quad \rightarrow \quad 0$$

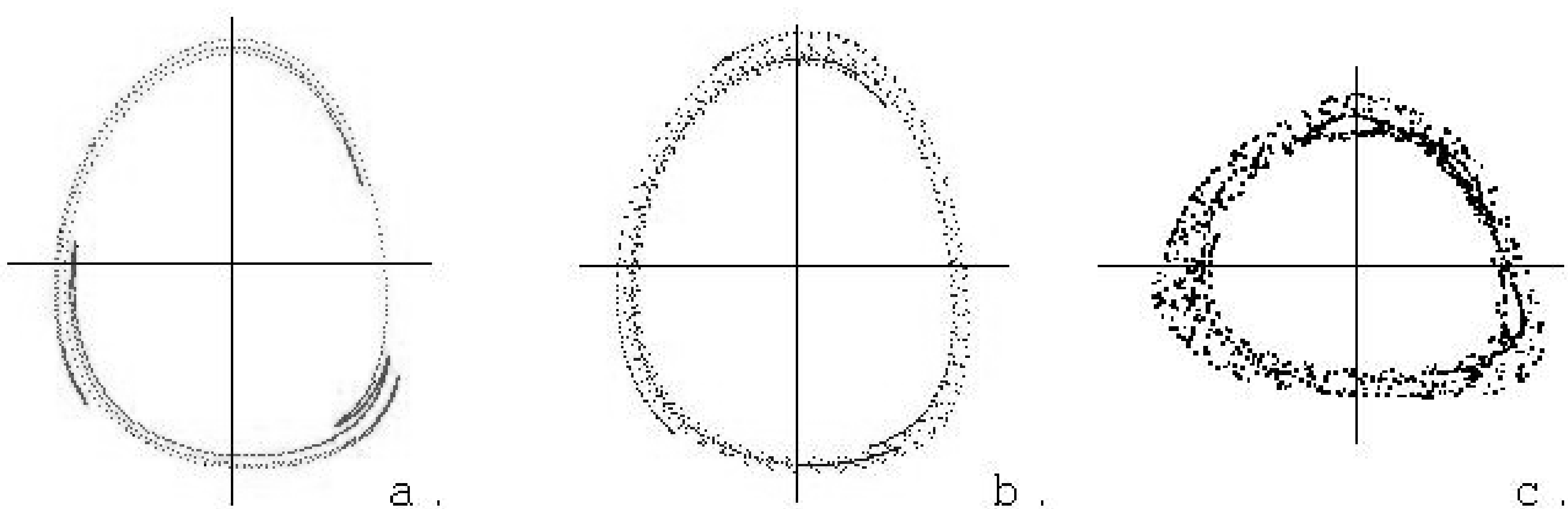
$$\sum_j m_j \ell_j \beta_x^{1/2} \beta_y \mathrm{e}^{\mathrm{i}\psi_{xj}} \quad \rightarrow \quad 0$$

and

$$\sum_j m_j \ell_j \beta_x^{1/2} \beta_y \mathrm{e}^{\mathrm{i}(\psi_{xj}+2\psi_{yj})} \quad \rightarrow \quad 0$$

$$\sum_j m_j \ell_j \beta_x^{1/2} \beta_y \mathrm{e}^{\mathrm{i}(\psi_{xj}-2\psi_{yj})} \quad \rightarrow \quad 0$$

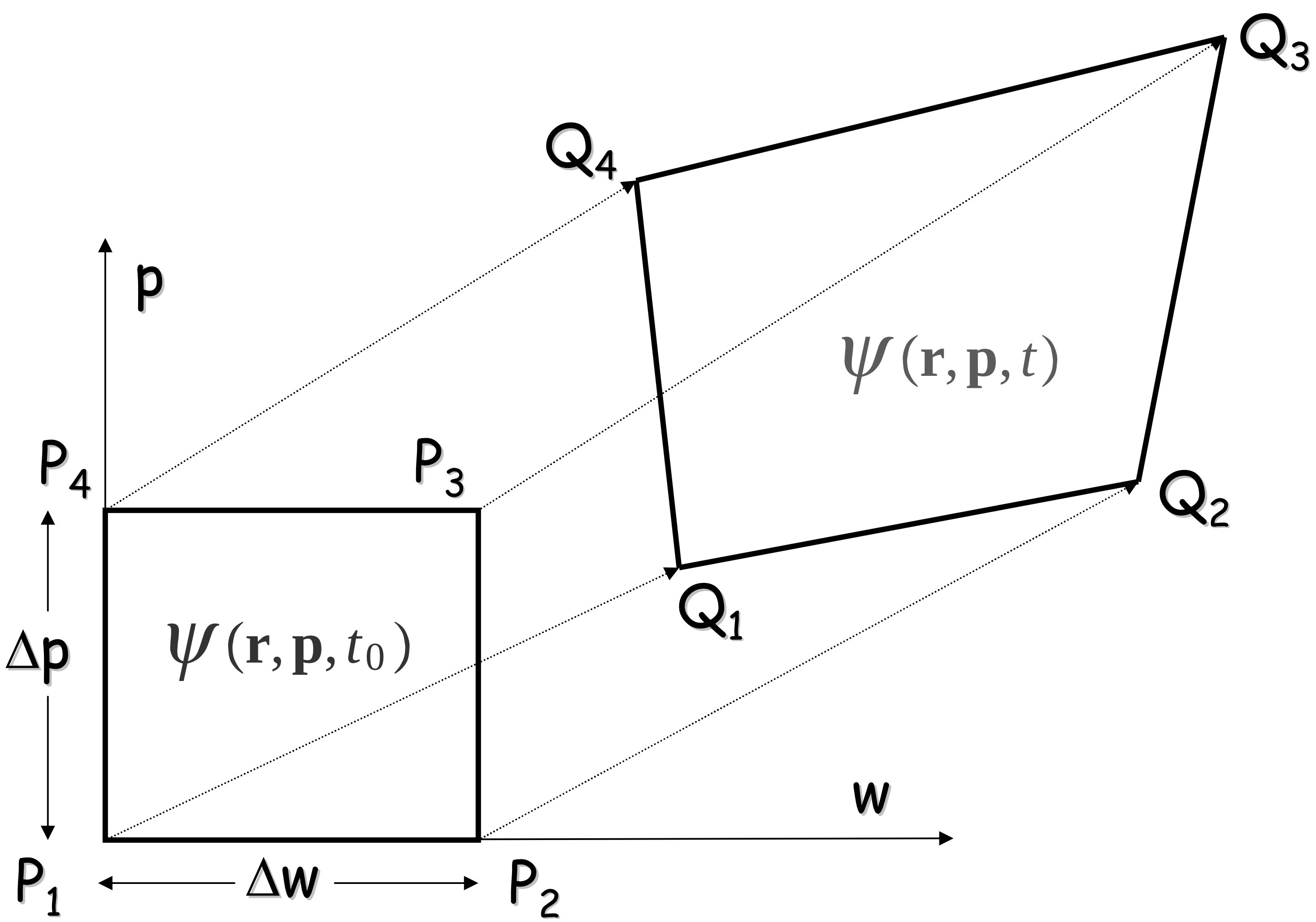
Filamentation – graphs



Filamentation of phase space after passage through an increasing number of FODO cells

Vlasov Equation

Beam Motion in Phase Space



Phase space motion

start at:

$$P_1(w,p) \qquad P_3(w+\Delta w, p+\Delta p)$$
$$P_2(w+\Delta w, p) \qquad P_4(w, p+\Delta p)$$

motion:

$$\dot{w} = f(w,p,t)$$
$$\dot{p} = g(w,p,t)$$

at time $t = t_0 + \Delta t$

$$Q_1(w + f_0\Delta t,\ p + g_0\Delta t)$$
$$Q_2(w + \Delta w + f(w+\Delta w, p, t_0)\Delta t,\ p + g_0\Delta t)$$
$$Q_3(w + \Delta w + f(w+\Delta w, p+\Delta p, t_0)\Delta t, p + \Delta p + g(w+\Delta w, p+\Delta p, t_0)\Delta t)$$
$$Q_4(w + f_0\Delta t, p + \Delta p + g(w+\Delta w, p+\Delta p, t_0)\Delta t)$$

Wronskian

conservation of particles

$$\psi(w,p,t)\Delta A_Q = \psi(w_0,p_0,t_0)\Delta A_P$$

ΔA :phase space area

$$\Delta A_P = |\mathbf{p}_1, \mathbf{p}_2| = \Delta w \Delta p$$

$$\Delta A_Q = |\mathbf{q}_1, \mathbf{q}_2| = \begin{vmatrix} \Delta w + \frac{\partial f}{\partial w}\Delta w \Delta t & \frac{\partial f}{\partial p}\Delta p \Delta t \\ \frac{\partial g}{\partial w}\Delta w \Delta t & \Delta p + \frac{\partial g}{\partial p}\Delta p \Delta t \end{vmatrix} \approx \Delta w \Delta p \left[1 + \left(\frac{\partial f}{\partial w} + \frac{\partial g}{\partial p}\right)\Delta t\right]$$

$1 + \left(\frac{\partial f}{\partial w} + \frac{\partial g}{\partial p}\right)\Delta t$ **Wronskian of system**

Vlasov Equation

$$\psi(w+f_0\Delta t, p+g_0\Delta t, t_0+\Delta t)\left[1+\left(\frac{\partial f}{\partial w}+\frac{\partial g}{\partial p}\right)\Delta t\right]=\psi(w_0,p_0,t_0)$$

use Taylor expansion:

$$\left[\psi_0+\frac{\partial\psi}{\partial w}f_0\Delta t+\frac{\partial\psi}{\partial p}g_0\Delta t+\frac{\partial\psi}{\partial t}\Delta t\right]\left[1+\left(\frac{\partial f}{\partial w}+\frac{\partial g}{\partial p}\right)\Delta t\right]=\psi_0$$

and keep only linear terms

Vlasov equation $$\boxed{\frac{\partial\psi}{\partial t}+f\frac{\partial\psi}{\partial w}+g\frac{\partial\psi}{\partial p}=-\left(\frac{\partial f}{\partial w}+\frac{\partial g}{\partial p}\right)\psi_0}$$

$\left(\frac{\partial f}{\partial w}+\frac{\partial g}{\partial p}\right)$ damping !

if $\left(\frac{\partial f}{\partial w}+\frac{\partial g}{\partial p}\right)=0$

$$\boxed{\frac{\mathrm{d}\psi}{\mathrm{d}t}=\frac{\partial\psi}{\partial t}+f\frac{\partial\psi}{\partial w}+g\frac{\partial\psi}{\partial p}=0}$$

Liouville's Theorem

VE – example-1

harmonic oscillator
in normalized coordinates

$$w = \frac{x}{\sqrt{\beta}}; \ \varphi = \int \frac{\mathrm{d}s}{\nu\beta}$$

$$\ddot{w} + \nu^2 w = 0 \quad \text{or} \quad \frac{\ddot{w}}{\nu} + \nu w = 0$$

with momentum: $p = \frac{\dot{w}}{\nu}$

$$f = \dot{w} = \nu p$$

$$g = \dot{p} = -\nu w$$

$$\left.\begin{aligned} \frac{\partial f}{\partial w} &= 0 \\ \frac{\partial g}{\partial p} &= 0 \end{aligned}\right\}$$

no damping ! and

Vlasov equation is

$$\boxed{\frac{\partial \psi}{\partial \varphi} + \nu p \frac{\partial \psi}{\partial w} - \nu w \frac{\partial \psi}{\partial p} = 0}$$

VE – example-1 cont.

$$\frac{\partial\psi}{\partial\varphi} + \nu p \frac{\partial\psi}{\partial w} - \nu w \frac{\partial\psi}{\partial p} = 0$$

with coordinate transformation

$$w = r\cos\theta$$

$$p = r\sin\theta$$

we get

$$\frac{\partial\psi}{\partial\varphi} - \nu \frac{\partial\psi}{\partial\theta} = 0$$

with solution

$$\psi(w,p,\varphi) = F(r,\theta + \nu\varphi)$$

any function of r and $\theta + \nu\varphi$ is a solution

arbitrary particle distribution rotates with frequency ν in phase space

VE – example-1 cont.

amplitude r of particle is a constant of motion

$$\left.\begin{aligned} r^2 = w^2 + p^2 &= \text{const.} \\ w &= \frac{x}{\sqrt{\beta}} \\ p &= \sqrt{\beta}\, x' + \alpha \frac{x}{\sqrt{\beta}} \end{aligned}\right\} \quad \beta x'^2 + 2\alpha x x' + \gamma x^2 = \text{const.}$$

Courant – Snyder Invariant

tune shift

equation of motion with perturbation terms: $$\ddot{w} + \nu_0^2 w = \nu_0^2 \beta^{\frac{3}{2}} \sum_{n>0} p_n \beta^{\frac{n}{2}} w^n$$

first, use only n=1 or quadrupole terms : $p_1 = -\Delta k$

$$\ddot{w} + \nu_0^2 w = -\nu_0^2 \beta^2 \Delta k\, w \quad \text{or} \quad \ddot{w} + \nu_0^2 (1 + \beta^2 \Delta k) w = 0$$

$$\dot{w} = \nu_0 \sqrt{1 + \beta^2 \Delta k}\; p$$
$$\dot{p} = -\nu_0 \sqrt{1 + \beta^2 \Delta k}\; w$$

$$\nu = \nu_0 \sqrt{1 + \beta^2 \Delta k} \approx \nu_0 \left(1 + \tfrac{1}{2} \beta^2 \Delta k\right)$$

tune shift due to quadrupole field error: $\Delta \nu = \nu_0 \frac{1}{2} \beta^2 \Delta k$

β and Δk are periodic functions in a circular ring and the lowest order Fourier component is: $\Delta \nu = \left(\nu_0 \frac{1}{2} \beta^2 \Delta k\right)_0 = \frac{\nu_0}{4\pi} \oint \beta^2 \Delta k \mathrm{d}\varphi = \frac{1}{4\pi} \oint \beta \Delta k \mathrm{d}s$

tune shift due to quadrupole field error: $\boxed{\Delta \nu = \frac{1}{4\pi} \oint \beta \Delta k \mathrm{d}s}$

tune spread

general perturbation terms: $$\ddot{w} + \nu_0^2 w = \nu_0^2 \beta^{\frac{3}{2}} \sum_{n>0} p_n \beta^{\frac{n}{2}} w^n$$

with same procedure we find tune shifts/spread given by:

$$\nu = \nu_0 \sqrt{1 - \beta^{3/2} \sum_{n>0} p_n \beta^{n/2} w^{n-1}}$$

for small oscillation amplitudes: $w(\varphi) = w_0 \sin(\nu\varphi + \delta)$

and tune variation is: $$\Delta\nu = -\frac{1}{4\pi} \sum_{n>0} \oint p_n \sqrt{\beta^{n+1}}\, w_0^{n-1} \sin^{n-1}(\nu_0 \varphi(s) + \delta) \mathrm{d}s$$

note! tune variation is amplitude dependent for $\mathrm{n} > 1$

Damping - 1

consider damped harmonic oscillator

$$\ddot{w} + 2\alpha_w \dot{w} + \omega_0^2 w = 0 \quad \text{or} \quad \begin{aligned} \dot{w} &= f_w = \omega_0 p_w \\ \dot{p}_w &= g_w = -\omega_0 w - 2\alpha_w p_w \end{aligned}$$

Vlasov equation $\frac{\partial \psi}{\partial t} + f\frac{\partial \psi}{\partial w} + g\frac{\partial \psi}{\partial p} = -\left(\frac{\partial f}{\partial w} + \frac{\partial g}{\partial p}\right)\psi_0$

becomes:

$$\frac{\partial \psi}{\partial t} + \omega_0 p_w \frac{\partial \psi}{\partial w} - (\omega_0 w + 2\alpha_w p_w)\frac{\partial \psi}{\partial p} = 2\alpha_w \psi$$

for weak damping we try the solution of a damped harmonic oscillator:

$$w = w_0 e^{-\alpha_w t} \cos\sqrt{\omega_0^2 - \alpha_w^2}\, t = r\, e^{-\alpha_w t} \cos\theta$$

$$\frac{\omega_0 p_w + \alpha_w w}{\sqrt{\omega_0^2 - \alpha_w^2}} = -w_0 e^{-\alpha_w t} \sin\sqrt{\omega_0^2 - \alpha_w^2}\, t = -r\, e^{-\alpha_w t} \sin\theta$$

Damping – 2

keeping only linear damping terms, we get a quasi invariant:

$$r\,e^{-2\alpha_w t} = w^2 + p_w^2 + 2\frac{\alpha_w}{\omega_0} w p_w$$

solution for the phase space density is now

$$\boxed{\psi(w, p_w, t) = e^{2\alpha_w t} F(r, \phi)}$$

with $\phi = \theta + \sqrt{\omega_0^2 - \alpha_w^2}\, t$ and $F(r,\phi)$ an arbitrary function of r and ϕ

adiabatic damping

consider particle dynamics within a bunch

distance of a particle from bunch center be $w = \tau$

conjugate momentum $p_w = \epsilon = E - E_s$

$$f = \dot{\tau} = -\eta_c \beta^2 \frac{\epsilon}{E_s} \qquad \xrightarrow{\text{linac}} \qquad f = \dot{\tau} = \frac{1}{\beta^2\gamma^2}\frac{\epsilon}{E_s}$$

$$g = \dot{\epsilon} = \frac{1}{T}\left[eV_{rf}(\tau_s + \tau) - U(E_s + \epsilon)\right] \qquad \qquad g = \dot{\epsilon} = \frac{1}{T} e V_{rf}(\tau_s + \tau)$$

damping ? $\frac{\partial f}{\partial w} + \frac{\partial g}{\partial p} = -2\alpha_w = 0$ no ? where did adiabatic damping go ?

must consider relative energy spread $p_w = \frac{\epsilon}{E_s} = \frac{E - E_s}{E_s} = \delta$

$$g = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\epsilon}{E_s} = \frac{\frac{\epsilon}{E_s} - \frac{\epsilon}{E_0}}{\Delta t} = -\delta \frac{\dot{E}}{E_s}$$

$$\frac{\partial f}{\partial w} + \frac{\partial g}{\partial p} = -\frac{\dot{E}}{E_s} = -2\alpha_w = -2\frac{1}{\delta}\frac{\mathrm{d}\delta}{\mathrm{d}t}$$

$$\int \frac{\mathrm{d}\delta}{\delta} = \ln\frac{\delta}{\delta_0} = -\frac{1}{2}\int\frac{\dot{E}}{E_s}\mathrm{d}t = -\frac{1}{2}\ln\frac{E_s}{E_o}$$

or $\delta = \delta_0 \sqrt{\frac{E_0}{E_0 + \dot{E}t}}$

longitudinal damping

$$f = \dot{\tau} = \frac{1}{\beta^2\gamma^2}\frac{\epsilon}{E_s}$$

$$g = \dot{\epsilon} = \frac{1}{T}\left[eV_{rf}(\tau_s + \tau) - U(E_s + \epsilon)\right]$$

Taylor expansion

$$\left.\begin{aligned} eV_{rf}(\tau_s + \tau) &= eV_{rf}(\tau_s) + e\frac{V_{rf}}{\partial \tau}\tau, \\ -U(E_s + \epsilon) &= -U(E_s) - \left.\frac{\partial U}{\partial E}\right|_{E_s}\epsilon. \end{aligned}\right\} \quad \dot{\epsilon} = \frac{1}{T}\left[e\dot{V}_{rf}(\tau_s)\tau - \left.\frac{\partial U}{\partial E}\right|_{E_s}\epsilon\right]$$

damping decrement $\alpha_\epsilon = +\frac{1}{2}\frac{1}{T}\frac{\partial U}{\partial E}$

$U = \frac{1}{c}\int P_\gamma \mathrm{d}\sigma$

integrate along actual path

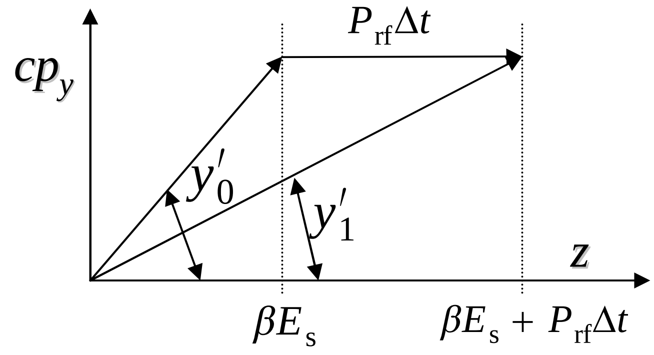
$$\left.\frac{\partial U}{\partial E}\right|_{E_s} = \frac{U_s}{E_s}(2 + \vartheta) \quad \text{with} \quad \vartheta = \frac{\int_L \eta\left(\frac{1}{\rho^3} + 2\frac{k}{\rho}\right)\mathrm{d}s}{\int_L \frac{1}{\rho^2}\mathrm{d}s}$$

damping decrement

$$\boxed{\alpha_\epsilon = \frac{U_s}{2TE_s}(2 + \vartheta) = \frac{U_s}{2TE_s}J_\epsilon = \frac{\langle P_\gamma \rangle}{2E_s}J_\epsilon}$$

vertical damping

accelerating cavity



$$y'_0 = \frac{cp_\perp}{\beta E_{\rm s}}$$

$$y'_1 = \frac{cp_\perp}{\beta E_{\rm s} + P_\gamma \frac{\Delta s}{c}} \approx \frac{cp_\perp}{\beta E_{\rm s}}\left(1 - \frac{P_\gamma}{\beta E_{\rm s}}\frac{\Delta s}{c}\right)$$

$$f = \frac{\Delta w}{\Delta\varphi} = \frac{y_1 - y_0}{\sqrt{\beta_y}\,\Delta\varphi} = \frac{y'_0}{\sqrt{\beta_y}}\frac{\Delta s}{\Delta\varphi} = \nu\sqrt{\beta_y}\,y'_0$$

$$g = \frac{\Delta p}{\Delta\varphi} = \frac{\frac{\mathrm{d}w_1}{\mathrm{d}\varphi} - \frac{\mathrm{d}w_0}{\mathrm{d}\varphi}}{\nu\Delta\varphi} = \frac{-\sqrt{\beta_y}\,\frac{P_\gamma}{\beta c E_{\rm s}}\Delta s y'_{\rm o} + F(y)}{\nu\Delta\varphi}$$

$$\frac{\partial g}{\partial p} = \nu\frac{\partial g}{\partial\frac{\mathrm{d}w}{\mathrm{d}\varphi}} = -\frac{P_\gamma}{E_{\rm s}}\frac{T_{\rm rev}}{2\pi} = -2\alpha_y\frac{T_{\rm rev}}{2\pi}$$

$$\frac{\Delta s}{\beta c\Delta\varphi} = \frac{T_{\rm rev}}{2\pi}$$

$$\boxed{\alpha_y = \frac{\langle P_\gamma\rangle}{2E_{\rm s}}}$$

Robinson Criterion – 1

3-dim Vlasov equation $$\frac{\partial \Psi}{\partial t} + \mathbf{f}\,\nabla_r \Psi + \mathbf{g}\nabla_p \Psi = -(\nabla_r \mathbf{f} + \nabla_p \mathbf{g})\Psi$$

total damping decrement $$\nabla_r \mathbf{f} + \nabla_p \mathbf{g} = -2(\alpha_x + \alpha_y + \alpha_\epsilon)$$

we observe particle trajectory along segment Δs including synchrotron radiation and acceleration in rf-cavity

$$\left.\begin{aligned} x &= x_0 + x_0' \Delta s, \\ y &= y_0 + y_0' \Delta s, \\ \tau &= \tau_0 + \eta_c \frac{\epsilon_0}{E_s} \frac{\Delta s}{\beta c}. \end{aligned}\right\} \qquad \mathbf{f} = \dot{\mathbf{r}} = \beta c \left(x_0', y_0', \eta_c \frac{\epsilon}{E_s} \right)$$

$$\left.\begin{aligned} x' &= x_0' - \frac{P_{\mathrm{rf}}}{E_s} \frac{\Delta s}{\beta c} x_0', \\ y' &= y_0' - \frac{P_{\mathrm{rf}}}{E_s} \frac{\Delta s}{\beta c} y_0'. \\ \epsilon &= \epsilon_0 - P_\gamma \frac{\Delta s}{\beta c} + P_{\mathrm{rf}} \frac{\Delta s}{\beta c} \end{aligned}\right\} \qquad \mathbf{g} = \dot{\mathbf{p}} = \left(-\frac{P_{\mathrm{rf}}}{E_s} x_0', -\frac{P_{\mathrm{rf}}}{E_s} y_0', -P_\gamma + P_{\mathrm{rf}} \right)$$

Robinson Criterion - 2

$$\mathbf{f} = \dot{\mathbf{r}} = \beta c\left(x_0', y_0', \eta_c \frac{\epsilon}{E_s}\right)$$

$$\mathbf{g} = \dot{\mathbf{p}} = \left(-\frac{P_{rf}}{E_s}x_0', -\frac{P_{rf}}{E_s}y_0', -P_\gamma + P_{rf}\right)$$

$$\nabla_r \mathbf{f} = 0$$

$$\nabla_p \mathbf{g} = -\frac{P_{rf}}{E_s} - \frac{P_{rf}}{E_s} - 2\frac{P_\gamma}{E_s} = -4\frac{P_\gamma}{E_s}$$

$$\boxed{\nabla_r \mathbf{f} + \nabla_p \mathbf{g} = -2(\alpha_x + \alpha_y + \alpha_\epsilon) = -4\frac{P_\gamma}{E_s}}$$

horizontal damping decrement $\alpha_x = \frac{P_\gamma}{2E_s}(1 - \vartheta)$

Damping times

$$\alpha_x = \frac{\langle P_\gamma \rangle}{2E_s}(1-\vartheta) = \frac{\langle P_\gamma \rangle}{2E_s} J_x$$

$$\alpha_y = \frac{\langle P_\gamma \rangle}{2E_s} = \frac{\langle P_\gamma \rangle}{2E_s} J_y$$

$$\alpha_\epsilon = \frac{\langle P_\gamma \rangle}{2E_s}(2+\vartheta) = \frac{\langle P_\gamma \rangle}{2E_s} J_\epsilon$$

$$\vartheta = \frac{\int_L \eta\left(\frac{1}{\rho^3}+2\frac{k}{\rho}\right)\mathrm{d}s}{\int_L \frac{1}{\rho^2}\mathrm{d}s}$$

$$\langle P_\gamma \rangle = \frac{2}{3} r_c c\, mc^2 (\beta\gamma)^4 \left\langle \frac{1}{\rho^2} \right\rangle$$

damping partition numbers:

$$J_\epsilon = 2+\vartheta, \quad J_y = 1, \quad J_x = 1-\vartheta$$

or

$$\sum J_i = 4$$

Fokker-Planck equation

statistical processes

Vlasov equation includes only differentiable functions, but no statistical processes we introduce the modification:

$$\dot{w} = f_w(w, p_w, t) + \sum \xi_i \, \delta(t - t_i) \qquad \Delta w_i = \int \xi_i \delta(t - t_i) \mathrm{d}t = \xi_i,$$

$$\dot{p}_w = g_w(w, p_w, t) + \sum \pi_i \, \delta(t - t_i) \qquad \Delta p_{wi} = \int \pi_i \delta(t - t_i) \mathrm{d}t = \pi_i.$$

where ξ_i and π_i are statistical processes occurring at times $t = t_i$

look at evolution of phase space

$$\Psi(w + f_w \Delta t, p_w + g_w \Delta t, t + \Delta t) \Delta A_Q$$

$$= \Delta A_P \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(w - \xi, p_w - \pi, t) P_w(\xi) P_p(\pi) \mathrm{d}\xi \mathrm{d}\pi.$$

$P_w(\xi), P_p(\pi)$ are the probabilities that amplitude $(w - \xi)$ or momentum $(p_w - \pi)$ be changed by a statistical process to become w or p_w

Evolution of phase space

statistical processes can change amplitude and momentum by arbitrary large values, probability is large only for small changes

quadratic Taylor's expansion

$$\Psi(w-\xi, p_w-\pi, t) =$$

$$\Psi_o - \xi\frac{\partial\Psi_o}{\partial w} - \pi\frac{\partial\Psi_o}{\partial p_w} + \frac{1}{2}\xi^2\frac{\partial^2\Psi_o}{\partial w^2} + \frac{1}{2}\pi^2\frac{\partial^2\Psi_o}{\partial p_w^2} + \xi\pi\frac{\partial^2\Psi_o}{\partial w\partial p_w},$$

and $$\int\int\Psi(w-\xi, p_w-\pi, t)P_w(\xi)P_p(\pi)\mathrm{d}\xi\mathrm{d}\pi =$$

$$\Psi_o + \frac{1}{2}\frac{\partial^2\Psi_o}{\partial w^2}\int\xi^2P_w(\xi)\mathrm{d}\xi + \frac{1}{2}\frac{\partial^2\Psi_o}{\partial p_w^2}\int\pi^2P_p(\pi)\mathrm{d}\pi.$$

with N statistical processes per unit time:

$$\tfrac{1}{2}\int\xi^2P_w(\xi)\mathrm{d}\xi = \tfrac{1}{2}\langle\mathcal{N}_\xi\xi^2\rangle\Delta t$$

$$\tfrac{1}{2}\int\pi^2P_p(\pi)\mathrm{d}\pi = \tfrac{1}{2}\langle\mathcal{N}_\pi\pi^2\rangle\Delta t$$

Fokker - Planck equation

similar to derivation of Vlasov equation we get

$$\frac{\partial \Psi_o}{\partial t} + f_w \frac{\partial \Psi_o}{\partial w} + g_w \frac{\partial \Psi_o}{\partial p_w} =$$
$$-\left(\frac{\partial f_w}{\partial w} + \frac{\partial g_w}{\partial p_w}\right)\Psi_o + \frac{1}{2}\langle \mathcal{N}_\xi \xi^2 \rangle \frac{\partial^2 \Psi_o}{\partial w^2} + \frac{1}{2}\langle \mathcal{N}_\pi \pi^2 \rangle \frac{\partial^2 \Psi_o}{\partial p_w^2}.$$

with diffusion coefficients

$$D_\xi = \frac{1}{2}\langle \mathcal{N}_\xi \xi^2 \rangle,$$
$$D_\pi = \frac{1}{2}\langle \mathcal{N}_\pi \pi^2 \rangle,$$

we get the general **Fokker - Planck equation**

$$\boxed{\frac{\partial \Psi}{\partial t} + f_w \frac{\partial \Psi}{\partial w} + g_w \frac{\partial \Psi}{\partial p_w} = 2\,\alpha_w \Psi + D_\xi \frac{\partial^2 \Psi}{\partial w^2} + D_\pi \frac{\partial^2 \Psi}{\partial p_w^2}}$$

or for damped oscillator

$$\boxed{\frac{\partial \Psi}{\partial t} + \omega_0 p_w \frac{\partial \Psi}{\partial w} - (\omega_0 w + 2\alpha_w p_w)\frac{\partial \Psi}{\partial p_w} = 2\alpha_w \Psi + D_\xi \frac{\partial^2 \Psi}{\partial w^2} + D_\pi \frac{\partial^2 \Psi}{\partial p_w^2}}$$

Solution of F-P equation - 1

use cylindrical coordinates $(w, p_w) \to (r, \theta)$ with $\begin{aligned} w &= r\cos\theta \\ p_w &= r\sin\theta \end{aligned}$

define total diffusion $D = \frac{1}{2}(D_\xi + D_\pi)$

$$\frac{\partial \Psi}{\partial t} = 2\alpha_w \Psi + (\alpha_w r + \tfrac{D}{r})\frac{\partial \Psi}{\partial r} + D\frac{\partial^2 \Psi}{\partial r^2}$$

try separation of variables $\Psi(r,t) = \sum_n F_n(t) G_n(r)$

$$\dot{F}_n G_n = 2\alpha_w F_n G_n + (\alpha_w r + \tfrac{D}{r}) F_n G_n' + D F_n G_n''$$

$$\frac{\dot{F}_n}{F_n} = 2\alpha_w + (\alpha_w r + \tfrac{D}{r})\frac{G_n'}{G_n} + D\frac{G_n''}{G_n} = -\alpha_n$$

$$F_n(t) = \text{const.}\, \mathrm{e}^{-\alpha_n t}$$

$$\Psi(r,t) = \sum_{n \ge 0} c_n G_n(r) \mathrm{e}^{-\alpha_n t}$$

Solution of F-P equation - 2

initial particle distribution: $\Psi_o(r, t=0) = \sum_{n \geq 0} c_n G_{no}(r)$

$$\frac{\partial^2 G_n}{\partial r^2} + \left(\frac{1}{r} + \frac{\alpha_w}{D} r\right)\frac{\partial G_n}{\partial r} + \frac{\alpha_w}{D}\left(2 + \frac{\alpha_n}{\alpha_w}\right) G_n = 0$$

assume a wall at $r = R$

all terms with $\alpha_n > 0$ vanish after a while because of damping

$\alpha_n < 0$ define instabilities which we do not consider here

for stationary solution only terms with $\alpha_n = 0$ contribute

for $R \to \infty$

solution is a Gaussian: $\Psi(r,t) = \sum_{\substack{n \geq 0 \\ \alpha_n = 0}} c_n G_n(r) \propto \exp\left(-\frac{\alpha_w}{2D} r^2\right)$

with standard width $\boxed{\sigma_r = \sqrt{\frac{D}{\alpha_w}} \quad \text{and} \quad \Psi(r) = \frac{1}{\sqrt{2\pi}\,\sigma_r} \mathrm{e}^{-r^2/2\sigma_r^2}}$

Solution of F-P equation - 3

distribution in (w, p_w) space with $r^2 = w^2 + p_w^2$ and $\sigma_w = \sigma_{p_w} = \sqrt{\frac{D}{\alpha_w}}$

$$\Psi(w, p_w) = \frac{1}{2\pi\sigma_w\sigma_{pw}} \mathrm{e}^{-w^2/2\sigma_w^2} \mathrm{e}^{-p_w^2/2\sigma_{pw}^2}$$

one step more to real space: $u = x$ or $u = y$

$$u = \sqrt{\beta_u}\, w$$

$$p_w = \frac{\dot{w}}{\nu} = \sqrt{\beta_u}\, u' - \frac{\beta'_u}{\sqrt{\beta_u}} u$$

$$\Psi(u, u') \propto \exp\left(-\frac{\gamma_u u^2 - \beta'_u u u' + \beta_u u'^2}{2\sigma_w^2}\right)$$

integrating over all u' , for example, gives with $\int_{\infty}^{\infty} \mathrm{e}^{-p^2x^2 \pm qx} \mathrm{d}x = \frac{\sqrt{\pi}}{p} \mathrm{e}^{q^2/(4p^2)}$

the spatial particle distribution:

$$\Psi(u) = \frac{1}{\sqrt{2\pi}\sqrt{\beta_u}\,\sigma_w} \mathrm{e}^{-u^2/2\sigma_u^2}$$ with $$\sigma_u = \sqrt{\beta}\,\sigma_w = \sqrt{\beta}\sqrt{\tfrac{1}{2}\tau_u D_u}$$

finite aperture

Gaussian is infinitely wide, but real vacuum chamber is not !

Gaussian tails get cut off

cutting off the tails will not alter core particle distribution much

try $\Psi(r,t) = \mathrm{e}^{-\frac{\alpha_w}{2D}r^2} g(r)\mathrm{e}^{-\alpha t}$ with $\Psi(A,t) = 0$ at aperture

lifetime $\tau = 1/\alpha$ due to limited Gaussian

$$g'' + \left(\frac{1}{r} - \frac{r}{\sigma^2}\right)g' + \frac{\alpha}{\alpha_w \sigma^2} g = 0$$

$$g(r) = 1 + \sum\nolimits_{k\geq 1} C_k x^k \quad \text{with} \quad x = \frac{r^2}{2\sigma^2}$$

$$C_k = \frac{1}{(k!)^2} \prod\nolimits_{p=1}^{p=k} (p-1-X) \approx -\frac{(k-1)!}{(k!)^2} X$$

$$g(r) = 1 - \frac{\alpha}{2\alpha_w} \sum\nolimits_{k\geq 1} \frac{1}{kk!} x^k \approx 1 - \frac{\alpha}{2\alpha_w} \frac{\mathrm{e}^x}{x} \quad \text{for} \quad x = \frac{A^2}{2\sigma^2} \gg 1$$

quantum lifetime

imposing condition $g(A) = 0$

we get from $g(r) \approx 1 - \frac{\alpha}{2\alpha_w} \frac{e^x}{x} = 0$ with $x = \frac{A^2}{2\sigma^2} \gg 1$

quantum lifetime with $\tau_q = 1/\alpha$

$$\boxed{\tau_q = \frac{1}{2} \tau_w \frac{e^x}{x}}$$

for good lifetime, apertures should be 7-8 sigma's: $A \approx (7-8)\sigma$

$7\sigma_w \Longrightarrow x = 24.5 \Longrightarrow e^x/x = 1.8 \cdot 10^9 \Longrightarrow \tau \gtrsim 10\,\mathrm{h}$

$8\sigma_w \Longrightarrow x = 32 \Longrightarrow e^x/x = 2.47 \times 10^{12} \Longrightarrow \tau \gtrsim 10{,}000\,\mathrm{h}$

no damping

consider very high energy electron beam lines (linear collider final focus lines)

we have potential synchrotron radiation, but no acceleration in rf-cavities $\alpha_w = 0$

$$\frac{\partial^2 G_n}{\partial r^2} + \left(\frac{1}{r} + \frac{\alpha_w}{D} r\right)\frac{\partial G_n}{\partial r} + \frac{\alpha_w}{D}\left(2 + \frac{\alpha_n}{\alpha_w}\right) G_n = 0 \qquad \frac{\partial^2 G_n}{\partial r^2} + \frac{1}{r}\frac{\partial G_n}{\partial r} + \frac{\alpha_n}{D} G_n = 0$$

look for solutions $\Psi_n(r,t) = c_n G_n(r) \mathrm{e}^{-\alpha_n t}$ with $G_n(r) = \mathrm{e}^{-r^2/2\sigma_0^2}$ and $\sigma(t=0) = \sigma_0$

$$\alpha_n = \frac{2D}{\sigma_0^2} - \frac{D}{\sigma_0^4} r^2$$

and

$$\Psi(r,t) = A \exp\left(-\frac{2D}{\sigma_0^2} t\right) \exp\left[\left(-\frac{r^2}{2\sigma_0^2}\right)\left(1 - \frac{2D}{\sigma_0^2} t\right)\right]$$

particle distribution is an expanding Gaussian

$$\sigma^2(t) = \frac{\sigma_0^2}{1 - \frac{2D}{\sigma_0^2} t} \approx \sigma_0^2\left(1 + \frac{2D}{\sigma_0^2} t\right)$$

with

$\sigma^2 = \sigma_u^2 = \epsilon_u \beta_u$ we find a beam emittance increasing linearly with time

$$\epsilon_u = \epsilon_{u0} + \frac{2D}{\beta_u} t$$ at a rate $$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \frac{2D}{\beta} = \frac{1}{\beta}\left(\mathrm{D}_\xi + \mathrm{D}_\pi\right)$$

Diffusion Coefficient

particles with different energies travel along different paths:

$$\Delta x = \eta(s)\,\frac{cp_1 - cp_0}{cp_0}$$

emission of a photon :

$$\Delta w = \xi = -\,\frac{\eta(s)}{\sqrt{\beta_x}}\,\frac{\epsilon_\gamma}{E_\mathrm{o}}$$

$$\Delta \dot{w} = \pi = -\,\sqrt{\beta_x}\,\eta'(s)\,\frac{\epsilon_\gamma}{E_\mathrm{o}} - \frac{\alpha_x}{\sqrt{\beta_x}}\,\eta(s)\,\frac{\epsilon_\gamma}{E_\mathrm{o}}$$

negative signs because of energy loss ϵ_γ

emission of photon occurs $\mathcal{N} = \mathcal{N}_\xi = \mathcal{N}_\pi$ per unit time, and

$$\xi^2 + \pi^2 = \left(\frac{\epsilon_\gamma}{E_\mathrm{o}}\right)^2 \left[\frac{\eta^2}{\beta_x} + \left(\sqrt{\beta_x}\,\eta' + \frac{\alpha_x}{\sqrt{\beta_x}}\,\eta\right)^2\right] = \left(\frac{\epsilon_\gamma}{E_\mathrm{o}}\right)^2 \mathcal{H}$$

total diffusion coefficient: $D = \frac{1}{2}\langle \mathcal{N}(\xi^2 + \pi^2)\rangle_s = \frac{1}{2E_\mathrm{o}^2}\langle \mathcal{N}\langle \epsilon_\gamma^2\rangle \mathcal{H}\rangle_z$

Photon Emission

emission probability of photon with energy $\hbar\omega$

$$\frac{\mathrm{d}n(\omega)}{\mathrm{d}\omega} = \frac{1}{\hbar\omega}\frac{\mathrm{d}P(\omega)}{\mathrm{d}\omega} = \frac{P_\gamma}{\hbar\omega_c^2}\frac{9\sqrt{3}}{8\pi}\int_\zeta^\infty K_{5/3}(x)\,\mathrm{d}x \qquad \text{with} \qquad \zeta = \omega/\omega_c$$

total photon flux $\mathcal{N} = \frac{P_\gamma}{\hbar\omega_c}\frac{9\sqrt{3}}{8\pi}\int_0^\infty\int_\zeta^\infty K_{5/3}(x)\,\mathrm{d}x\mathrm{d}\zeta = \frac{15\sqrt{3}}{8}\frac{P_\gamma}{\hbar\omega_c}$

photon energy $\mathcal{N}\langle\epsilon_\gamma^2\rangle = \hbar^2\int_0^\infty \omega^2\, n(\omega)\,\mathrm{d}\omega = \frac{9\sqrt{3}\,P_\gamma\,\hbar\omega_c}{8\pi}\int_0^\infty\zeta^2\int_\zeta^\infty K_{5/3}(x)\,\mathrm{d}x\,\mathrm{d}\zeta = \frac{55}{24\sqrt{3}}P_\gamma\,\hbar\omega_c$

total diffusion coefficient

$$\boxed{D = \frac{1}{2}\langle\mathcal{N}(\xi^2+\pi^2)\rangle_s = \frac{55}{48\sqrt{3}}\frac{\langle P_\gamma\,\hbar\omega_c\,\mathcal{H}\rangle_s}{E_0^2}}$$

with

$$P_\gamma = \frac{2}{3}r_c mc^2\frac{c\gamma^4}{\rho^2}$$

$$\hbar\omega_c = \frac{3}{2}\hbar c\frac{\gamma^3}{\rho}$$

$$D = \frac{55}{48\sqrt{3}}\frac{r_c\hbar c}{mc^2}c\gamma^5\left\langle\frac{\mathcal{H}}{\rho^3}\right\rangle$$

Equilibrium emittances

$$\sigma_u = \sqrt{\beta}\,\sigma_w = \sqrt{\beta}\sqrt{\tfrac{1}{2}\tau_u D_u}$$

$$D = \frac{55}{48\sqrt{3}}\,\frac{r_c \hbar c}{mc^2}\,c\gamma^5 \left\langle \frac{\mathcal{H}}{\rho^3} \right\rangle$$

horizontal equilibrium beam emittance

$$\boxed{\epsilon_x = \frac{\sigma_x^2}{\beta_x} = C_q \gamma^2 \frac{\langle \mathcal{H}/|\rho^3| \rangle_z}{J_x \langle 1/\rho^2 \rangle_z}}$$

with $C_q = \frac{55}{32\sqrt{3}}\,\frac{\hbar c}{mc^2} = 3.84 \cdot 10^{-13}\,\mathrm{m}$

vertical emittance seems to vanish because $\mathcal{H} = 0$!? (if $\eta_y = 0$)

in this case, we cannot ignore anymore recoil from photon emission

photons are emitted within rms angle $\pm 1/\gamma$ $\Rightarrow$ $\delta y' = \frac{1}{\gamma}\frac{\epsilon_\gamma}{E_0}$ and $\delta y = 0$

$$\xi^2 = 0,$$

$$\pi^2 = \beta_y \frac{1}{\gamma^2}\left(\frac{\epsilon_\gamma}{E_0}\right)^2.$$

$\Rightarrow$

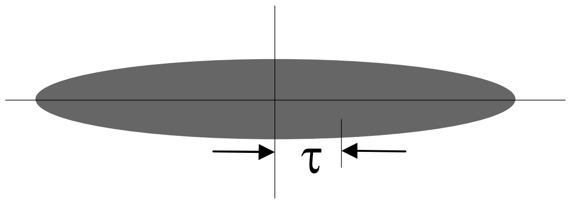
$$\boxed{\epsilon_y = C_q \frac{\langle \beta_y/|\rho^3| \rangle_s}{J_y \langle 1/\rho^2 \rangle_s} \approx 10^{-13}\,\mathrm{m}}$$

Energy spread and bunch length – 1

longitudinal dynamics is expresses by the coordinates:

$$w = -\frac{\Omega_s}{\eta_c}\tau$$

$$p = \frac{\epsilon}{E_0}$$



$\eta_c = \frac{1}{\gamma^2} - \alpha_c$ momentum compaction

synchrotron oscillation frequency

$$\Omega_s^2 = \omega_0^2 \frac{\eta_c heV_0 \cos\varphi_s}{2\pi E_0}$$

$$\dot{p} = \frac{\dot{\epsilon}}{E_0} = \frac{1}{T_0 E_0}\left[e\frac{\partial}{\partial t}V(\varphi_s + \omega\tau)\tau - \frac{\partial U}{\partial \epsilon}\epsilon\right] = -\Omega_s w - 2\alpha_\epsilon p$$

in circular accelerator $\dot{\tau} = -\eta_c \frac{\epsilon}{E_0}$ $\dot{w} = \Omega_s p$

$$\left.\begin{aligned} f &= \dot{w} = \Omega_s p \\ g &= \dot{p} = -\Omega_s w - 2\alpha_\epsilon p \end{aligned}\right\}$$ expressions are similar to transverse case

$$\frac{\sigma_\epsilon}{E_0} = \sqrt{\frac{1}{2}\tau_\epsilon D_\epsilon}$$

Energy spread and bunch length - 2

photon emission does not change position $\xi = 0$

$$\pi = \epsilon_\gamma / E_0$$

similar to transverse case with $\mathcal{H} = 0$

$$D_\epsilon = \frac{1}{2}\langle \mathcal{N}(\xi^2 + \pi^2)\rangle_s = \frac{55}{48\sqrt{3}} \frac{\langle P_\gamma \hbar\omega_c\rangle_s}{E_0^2}$$

equilibrium energy spread

$$\boxed{\frac{\sigma_\epsilon^2}{E_0^2} = \mathrm{C_q}\ \gamma^2 \ \frac{\langle |1/\rho^3|\rangle_s}{J_\epsilon \langle 1/\rho^2\rangle_s}}$$

bunch length from energy spread: $\sigma_s = \frac{|\eta_c|}{\Omega_\mathrm{s}} \frac{\sigma_\epsilon}{E_0}$

with synchrotron frequency $\Omega_\mathrm{s}^2 = \omega_0^2 \frac{\eta_\mathrm{c} heV_0 \cos\varphi_\mathrm{s}}{2\pi E_0}$

bunch length

$$\boxed{\sigma_s^2 = \frac{2\pi C_\mathrm{q}}{(mc^2)^2} \frac{\eta_c E_0^3 R^2}{J_\epsilon h e\widehat{V}_0 \cos\psi_s} \frac{\langle |1/\rho^3|\rangle_s}{\langle 1/\rho^2\rangle_s}}$$

Particle-Photon Interaction

particle-photon interaction
Cherenkov radiation
Compton effect
Poynting vector
energy transmission

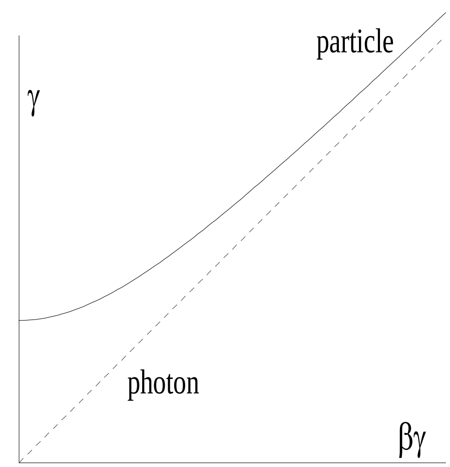
Photon Absorption and Emission

	electron	photon
energy, E/mc^2	γ	$\hbar\omega/mc^2$
momentum, cp/mc^2	$\beta\gamma$	$\hbar k/mc$

we plot energy vs momentum for both electron and photon: dispersion relation

for electron: $\gamma^2 = (\beta\gamma)^2 + 1$

for photon: $\frac{\hbar\omega}{mc^2} = \frac{\hbar k}{mc} = \lambda_c k$

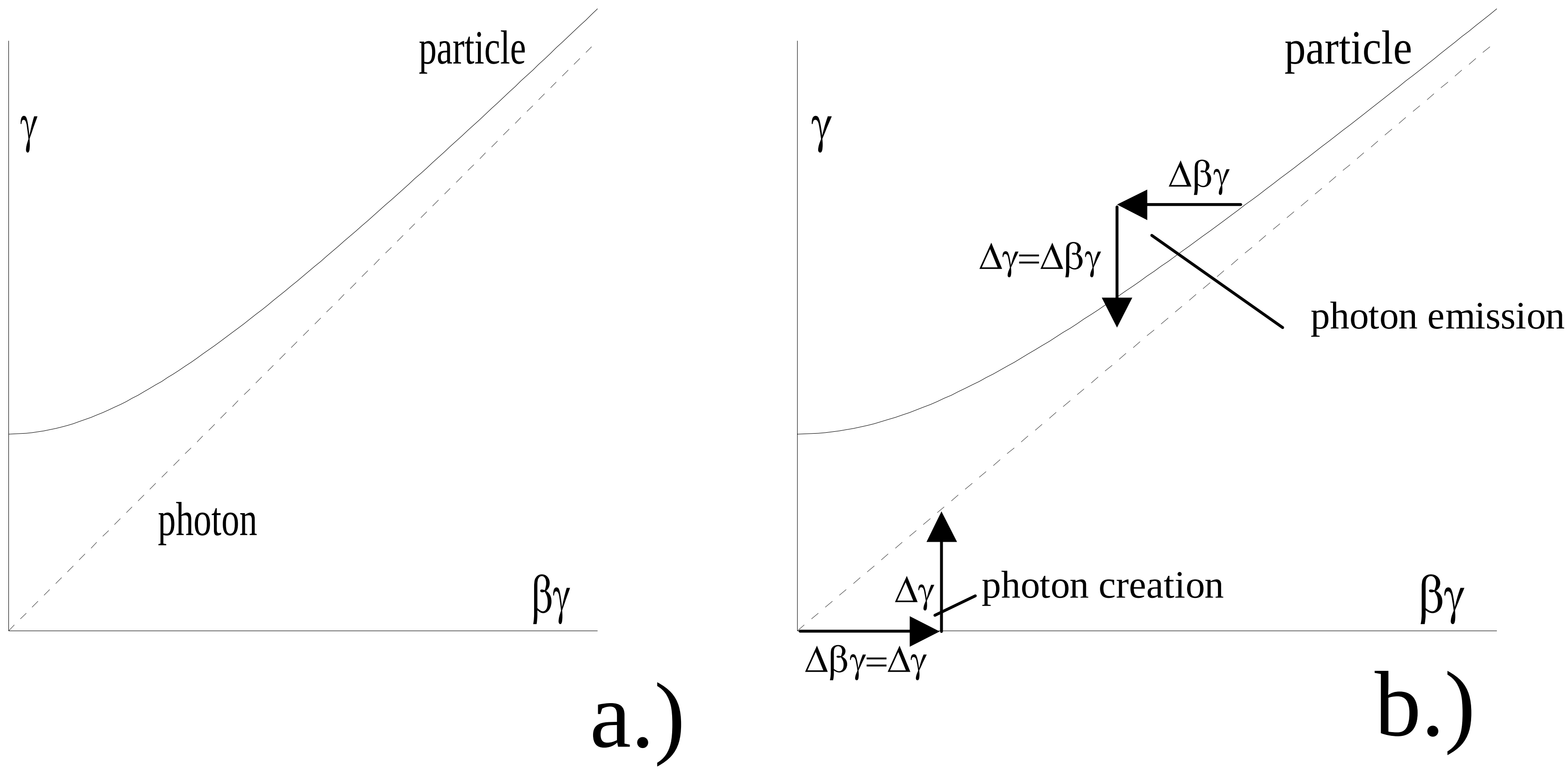


a.)

Compton wavelength: $\lambda_c = 3.86\ 10^{-13}$m

Spontaneous emission or absorption in vacuum violates energy and/or momentum conservation

E-cp graph


particle
γ
photon
βγ
a.)
particle
γ
Δβγ
Δγ=Δβγ
photon emission
Δγ
photon creation
βγ
Δβγ=Δγ
b.)

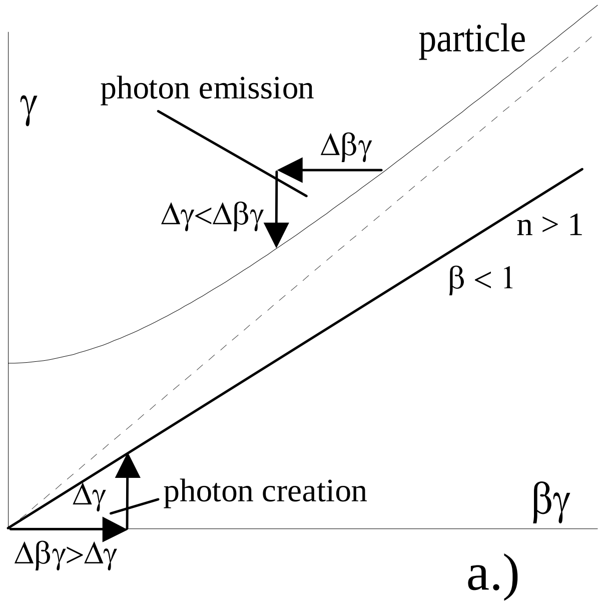
consider medium with refractive index: n > 1

c $\longrightarrow$ c/n

from phase of EM wave: $\varphi = \omega t - kz$

we get $\dot{\varphi} = \omega = k v_{\varphi}$

and phase velocity: $v_{\varphi} = \frac{\omega}{k} = \frac{c}{n}$



with: $\mathrm{E} = \hbar\omega, \omega = \frac{c}{n}k$, and $\mathrm{p} = \hbar \mathrm{k}$

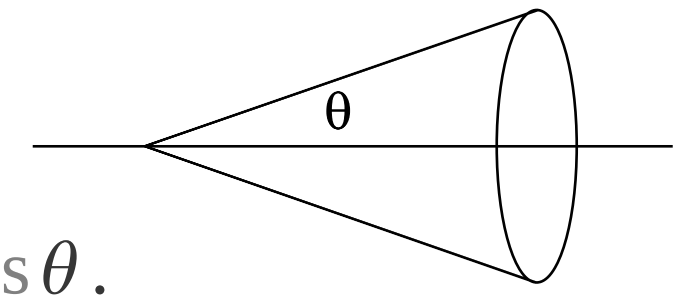
$$\frac{dE}{dcp} = \frac{dE}{d\omega}\frac{d\omega}{dk}\frac{dk}{dcp} = \hbar\frac{c}{n}\frac{1}{c\hbar} = \frac{1}{n} < 1$$

Cherenkov Condition

energy conservation: $\Delta\gamma_{\mathrm{p}} = \Delta\gamma = \beta\Delta\beta\gamma = \frac{1}{n}\Delta(\beta\gamma)_{\mathrm{p}}$

momentum conservation:

from symmetry: $\Delta(\beta\gamma)_{\mathrm{p}\perp} = 0$



$$\Delta\beta\gamma = \Delta(\beta\gamma)_{\mathrm{p}\parallel} = \Delta(\beta\gamma)_{\mathrm{p}} \cos\theta\,.$$

$$\boxed{n\beta\cos\theta = 1}$$

Cherenkov angle, θ

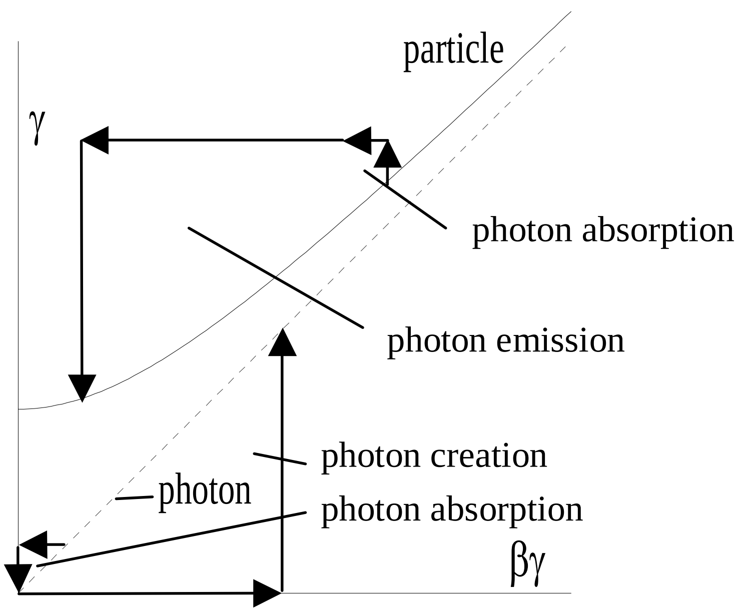
An electron can spontaneously emit photons in a material environment where n>1

But, SR is emitted in vacuum !??

To meet momentum conservation we need <u>3 body event</u>

Compton Effect:

electron absorbs photon then emits another photon



Incoming photon can be:

static electric fields
static magnetic fields
EM field, laser

Examples of Cherenkov/Compton Effects

Cherenkov Effect
- Cherenkov Radiation
- Particle Acceleration

Compton Effect
- Synchrotron Radiation
- Undulator/wiggler Radiation
- Free Electron Laser
- Thomson Scattering

Energy Conservation

kinetic energy change:

$$\Delta E_{\mathrm{kin}} = \int \mathbf{F}\, \mathrm{d}\mathbf{s}$$

$$\Delta E_{\mathrm{kin}} = \int \mathbf{E}\, \mathrm{d}\mathbf{s} + e\frac{[c]}{c}\int \underbrace{[\mathbf{v}\times\mathbf{B}]\mathbf{v}}_{=0}\, \mathrm{d}t$$

work done by EM-fields:

$$\frac{\mathrm{d}E_{\mathrm{kin}}}{\mathrm{d}t} = \mathbf{v}\mathbf{F}_{\mathrm{L}} = \int \rho\mathbf{v}\mathbf{E}\, \mathrm{d}V = \int \mathbf{j}\mathbf{E}\, \mathrm{d}V$$

from Maxwell's eq.

$$4\pi\mathbf{j} = c\nabla\times\mathbf{B} - \frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t}$$

$$\int \mathbf{j}\mathbf{E}\mathrm{d}V = \frac{c}{4\pi}\int\left[\mathbf{B}\ \underbrace{\nabla\times\mathbf{E}}_{=-\frac{1}{c}\dot{B}}\ -\nabla(\mathbf{E}\times\mathbf{B}) - \frac{1}{c}\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t}\mathbf{E}\right]\mathrm{d}V$$

field energy:

$$u = \frac{1}{8\pi}[\mathbf{E}^2 + \mathbf{B}^2]$$

Poynting Vector

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int u\,\mathrm{d}V}_{\substack{\text{change of}\\ \text{field energy}}} + \underbrace{\int \mathbf{jE}\,\mathrm{d}V}_{\substack{\text{particle energy}\\ \text{loss or gain}}} + \underbrace{\oint \mathbf{Sn}\,\mathrm{d}s}_{\substack{\text{radiation loss through}\\ \text{closed surface } \mathbf{s}}} = 0$$

Poynting Vector: $\mathbf{S} = \frac{c}{4\pi}[\mathbf{E} \times \mathbf{B}]$

radiation per unit surface area

$$\mathbf{S} \parallel \mathbf{n}$$
$$\mathbf{S} \perp \mathbf{E}$$
$$\mathbf{S} \perp \mathbf{B}$$

EM Field Vectors

from Maxwell's equation: $\nabla \times \mathbf{E} + \frac{1}{c}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = \mathbf{0}$

EM waves:

$$\mathbf{E} = \mathbf{E}_0 \mathrm{e}^{\mathrm{i}(\omega t - k\mathbf{nr})}$$
$$\mathbf{B} = \mathbf{B}_0 \mathrm{e}^{\mathrm{i}(\omega t - k\mathbf{nr})}$$

$$\nabla(-\cancel{\mathrm{i}}\cancel{k}\mathbf{nr}) \times \mathbf{E} + \cancel{\tfrac{1}{c}}\cancel{\mathrm{i}}\cancel{\omega}B = 0$$
$$\nabla(\mathbf{nr}) = \nabla(n_x x + n_y y + n_z z) = (n_x, n_y, n_z) = \mathbf{n}$$

$$(\mathbf{n} \times \mathbf{E}) = \mathbf{B} \quad \Longrightarrow \quad \mathbf{E} \perp \mathbf{B}$$

$$\boxed{\mathbf{S} = [4\pi\epsilon_0]\frac{c}{4\pi}\mathbf{E}^2\mathbf{n}}$$

Examples

1.) crossed static electric and magnetic field

$$\oint \mathbf{nS}\,\mathrm{d}s = \frac{c}{4\pi}\int \nabla[\mathbf{E}\times\mathbf{B}]\,\mathrm{d}V = 0$$: no radiation

2.) charge at rest:

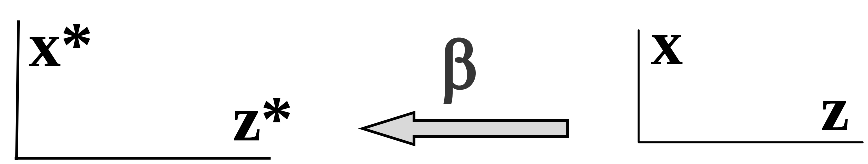
$$\mathbf{E} \neq \mathbf{0}$$
$$\mathbf{B} = \mathbf{0}$$

$$\mathbf{S} = \mathbf{0}$$

3.) charge in uniform motion:

in charge rest frame: see 2.)

Lorentz transformation to lab system:



uniformly moving charge

Lorentz transformation of fields:

$$\mathbf{E} = \gamma \mathbf{E}^* - \frac{\gamma^2}{\gamma+1} \boldsymbol{\beta}(\boldsymbol{\beta}\mathbf{E}^*)$$

$$\mathbf{B} = -\gamma(\boldsymbol{\beta} \times \mathbf{E}^*)$$

$$\left.\begin{array}{l} \boldsymbol{\beta} = (0, 0, -\beta) \\ \mathbf{E}^* \neq 0 \\ \mathbf{B}^* = 0 \end{array}\right\} \quad \begin{array}{ll} E_x = \gamma E_x^* & B_x = +\gamma\beta E_y^* \\ E_y = \gamma E_y^* & B_y = +\gamma\beta E_x^* \\ E_z = E_z^* & B_z = 0 \end{array}$$

$$\mathbf{S} = \frac{c}{4\pi}[4\pi\epsilon_0][-\gamma\beta E_x^* E_z^*, -\gamma\beta E_y^* E_z^*, \gamma^2\beta(E_x^{*2} + E_y^{*2})]$$

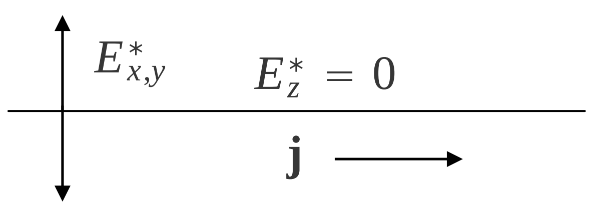
Energy transmission

moving line charge, current: $E^*_{x,y} = \frac{q}{r};\ E^*_z = 0$

$$\mathbf{S} = \frac{c}{4\pi}[4\pi\epsilon_0]\left[0, 0, \propto \gamma^2\beta\frac{q^2}{r^2}\right]$$

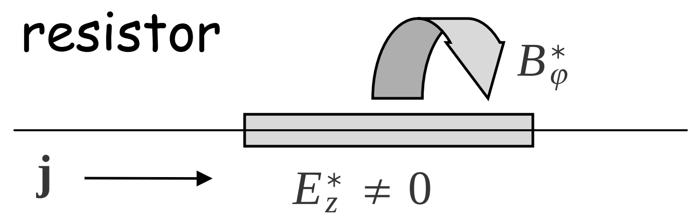
this “radiation” is confined to vicinity of electric current

responsible for transmission of electrical energy
- along wires
- transmission lines



$\mathbf{S} = S_z = \gamma^2\beta(E_x^{*2} + E_y^{*2})$

resistor



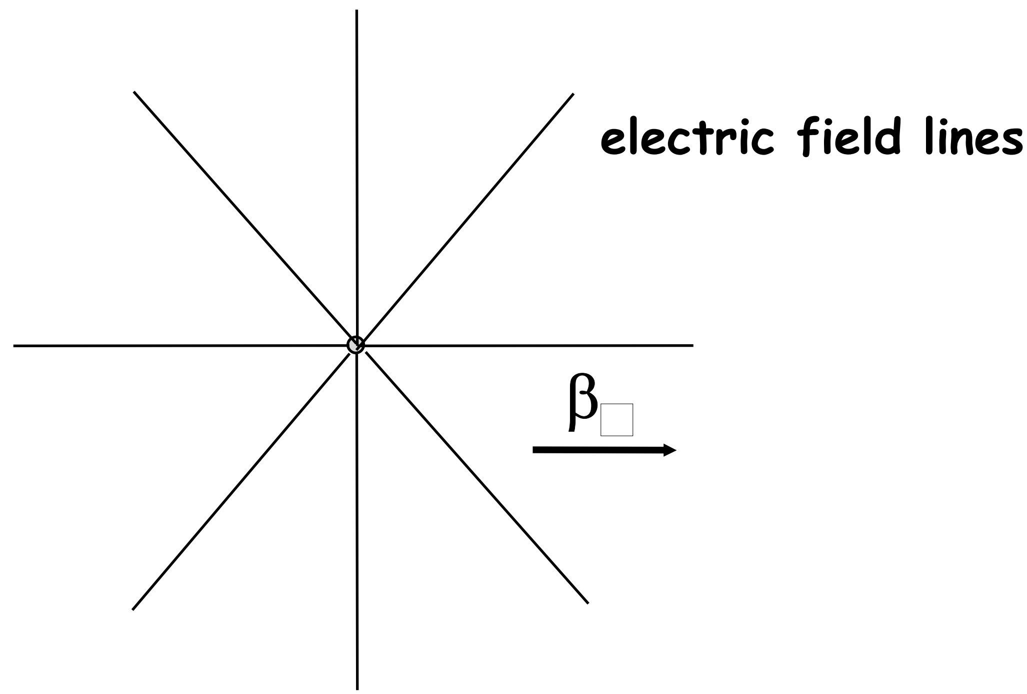
$\mathbf{S} = [<0, <0, >0]$

losses

Synchrotron Radiation

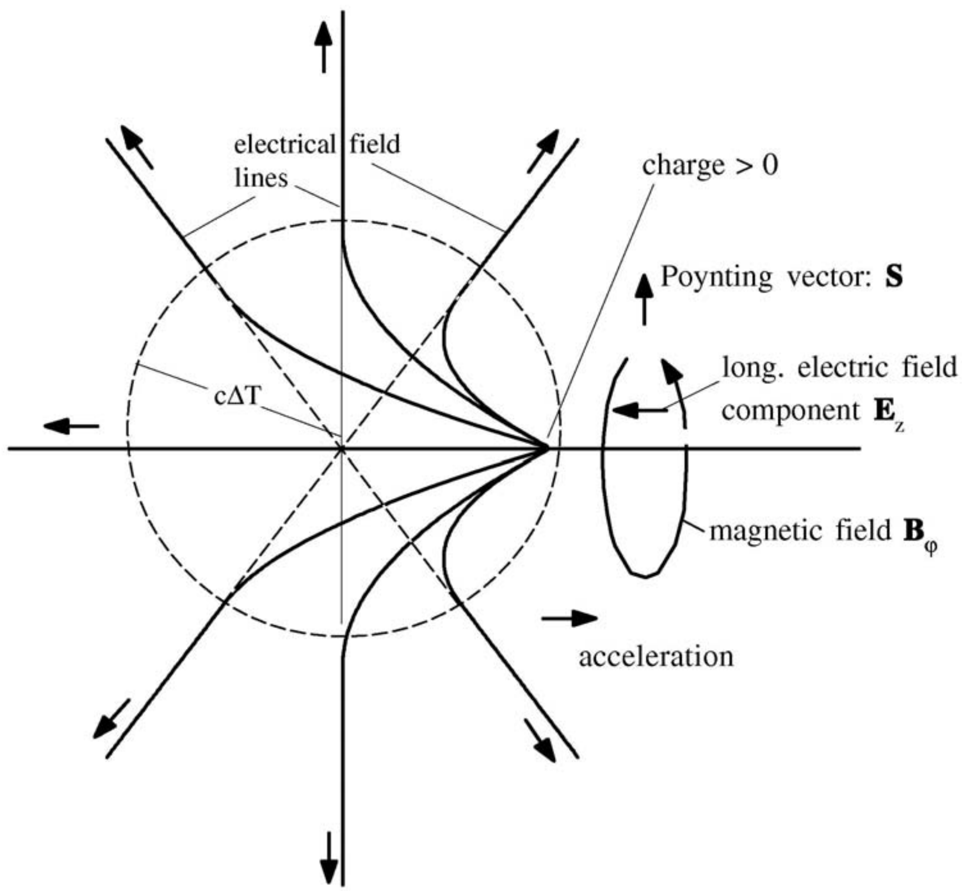
Accelerated Charges

charge moving uniformly at velocity β



Longitudinal acceleration

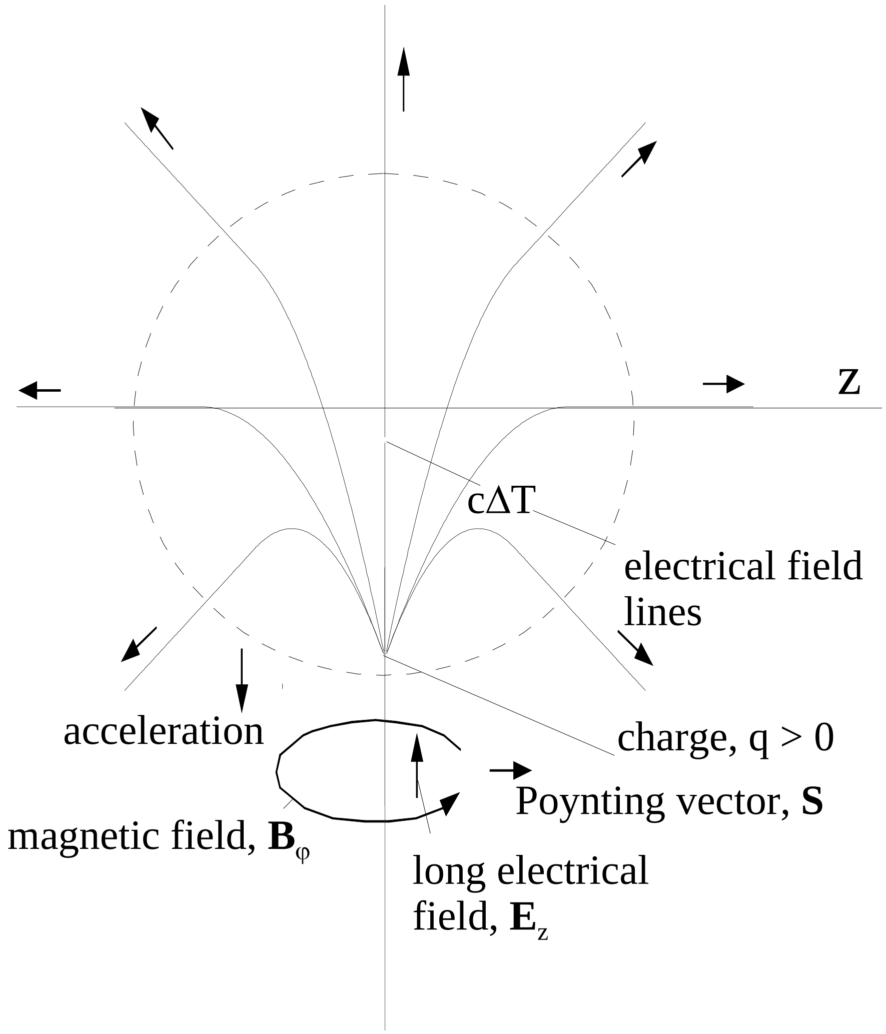
- reference system continues to move with velocity β
- accelerate charge for a time ΔT



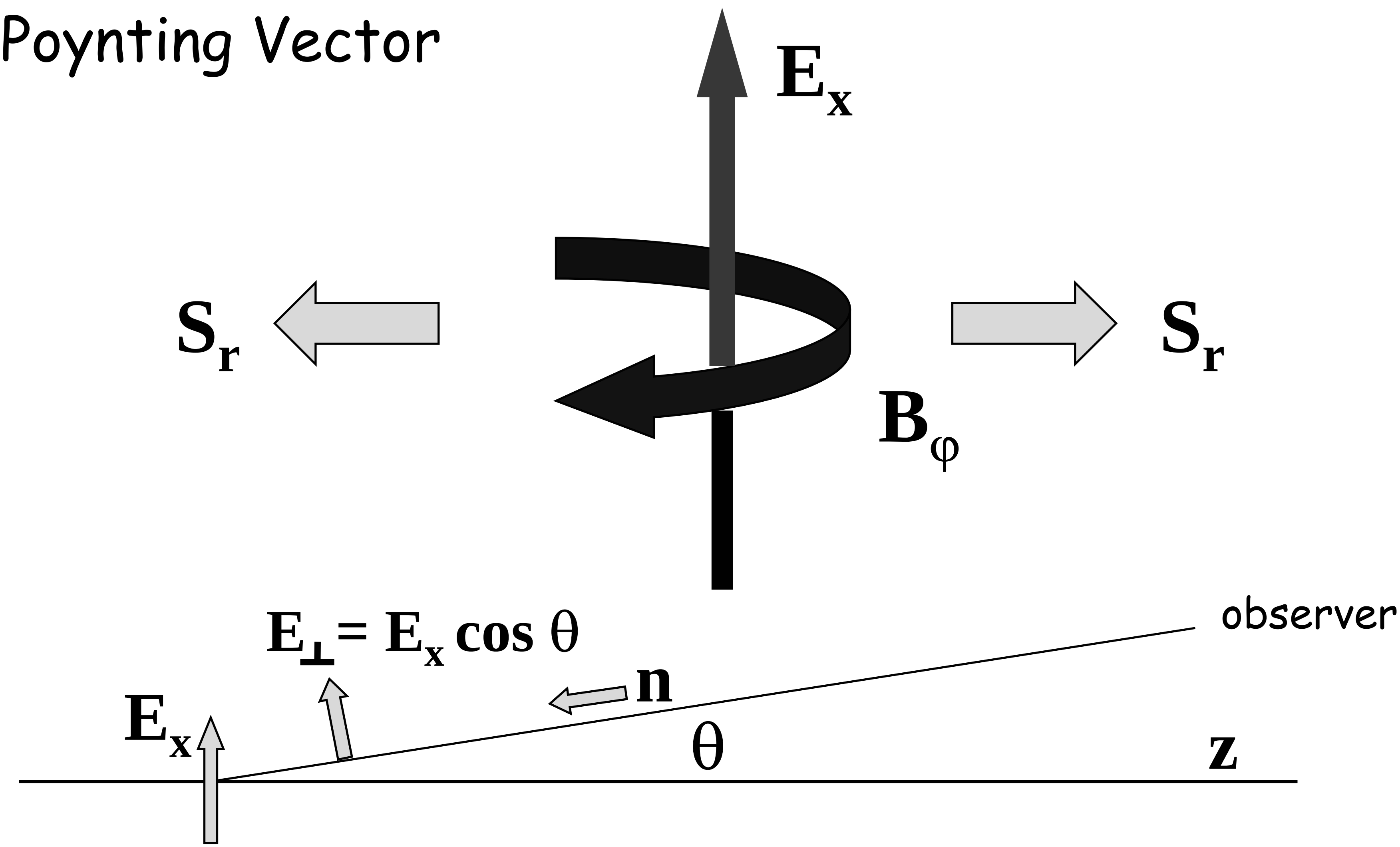
acceleration causes perturbation of electric field lines E_z because c is finite

motion of charge generates current mag. field B_j

Transverse acceleration



Poynting Vector

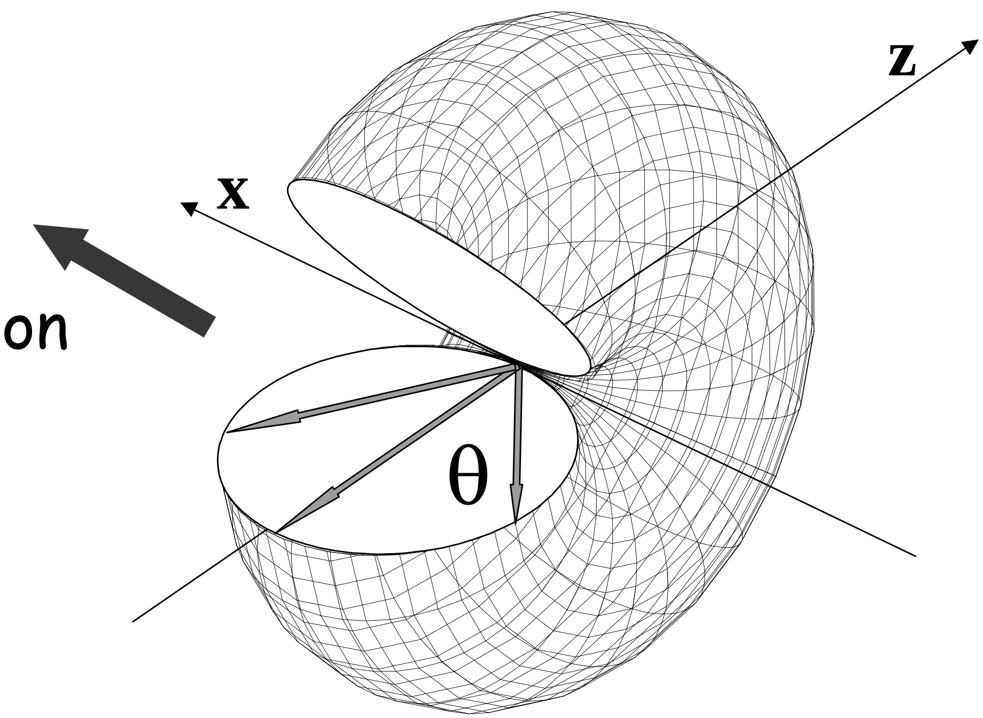


Poynting vector

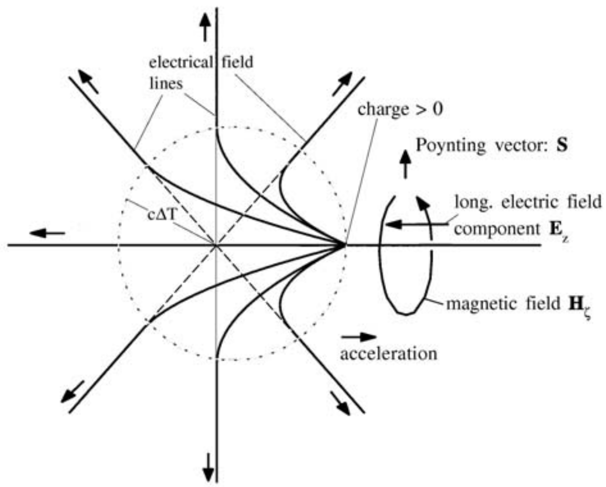
$$\left.\begin{aligned}\mathbf{S} &= \frac{c}{4\pi}[\mathbf{E}\times\mathbf{B}] \\ \mathbf{B} &= \mathbf{n}\times\mathbf{E}\end{aligned}\right\} \quad \mathbf{S} = \frac{c}{4\pi}|\mathbf{E}_\perp|^2\mathbf{n} = \frac{c}{4\pi}|\mathbf{E}|^2\cos^2\theta\,\mathbf{n}$$

radiation lobe

acceleration



Electric field



formulate expression for field:

quantities available:

q: charge
a: acceleration
r: distance from source

$$\mathbf{E} \propto q\,\mathbf{a}\,\frac{1}{r}$$

all radiation within spherical layer of thickness $c\Delta T$
total radiation energy is constant, $\sim \mathbf{E}^2 V$
$V \sim r^2 \quad \Rightarrow \quad \mathbf{E} \sim \mathbf{1}/r$

Expression for Poynting vector

with correct dimensions,
radiation field becomes:

$$\mathbf{E} = -\frac{q\mathbf{a}}{c^2 r}$$

Poynting vector:

$$\mathbf{S} = \frac{c}{4\pi}\left(\frac{q\mathbf{a}}{c^2 r}\right)^2 \cos^2\theta\, \mathbf{n} = \frac{e^2 \mathbf{a}^2}{4\pi c^3 r^2} \cos^2\theta\, \mathbf{n}$$

radiation power in electron system:

$$P = \oint \mathbf{S}^* \mathbf{n}^* \,\mathrm{d}\sigma = \frac{2}{3}\frac{e^2}{c}\dot{\beta}^{*2}$$

(use * for electron reference system)

Radiation power

radiation power in electron system

$$P^* = \frac{2}{3}\frac{e^2}{c}\beta^{*2}$$

in laboratory system?

4-acceleration

recall invariance properties of 4-vectors !

$$\tilde{a}^2 = \gamma^6 \left\{ \mathbf{a}^2 - [\beta \times \mathbf{a}]^2 \right\} = \tilde{a}^{*2}$$

evaluate 4-acceleration in particle system:

$\beta = 0$ and $\gamma = 1$

$$\tilde{a}^{*2} = a^{*2}$$

Radiation power in lab system

$$P^* = \frac{2}{3}\frac{e^2 a^{*2}}{c^3} = \frac{2}{3}\frac{e^2}{c^3}\gamma^6\left\{\mathbf{a}^2 - [\beta \times \mathbf{a}]^2\right\} = P$$

parallel vs transverse acceleration

$$\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$$

$\mathbf{a}_{\parallel}$ is $\parallel$ to β

$\mathbf{a}_{\perp}$ is $\perp$ to β

$$\mathbf{a}^2 - [\beta \times \mathbf{a}]^2 = \mathbf{a}_{\parallel}^2 + \mathbf{a}_{\perp}^2 - \beta^2 \mathbf{a}_{\perp}^2$$

$$\boxed{\begin{aligned} P_{\parallel} &= \frac{2}{3}\frac{e^2}{c}\gamma^6 \dot{\beta}_{\parallel}^2 \\ P_{\perp} &= \frac{2}{3}\frac{e^2}{c}\gamma^4 \dot{\beta}_{\perp}^2 \end{aligned}}$$

Radiation power cont.

find more practical formulation by introducing forces:

$$\mathrm{d}c\mathbf{p} = \beta mc^2 \mathrm{d}\gamma + \gamma mc^2 \mathrm{d}\beta$$
$$= \gamma mc^2 \left(\gamma^2\beta^2 + 1\right)\mathrm{d}\beta \ = \gamma^3 mc^2 \mathrm{d}\beta$$

radiation power for long. acceleration:

$$P_{\|} = \frac{2}{3}\frac{e^2}{c}\gamma^6 \frac{1}{\gamma^6\left(mc^2\right)^2}\left(\frac{\mathrm{d}c\mathbf{p}}{\mathrm{d}t}\right)_{\|}^2 = \frac{2}{3}\frac{r_c}{mc^3}\left(\frac{\mathrm{d}c\mathbf{p}}{\mathrm{d}t}\right)_{\|}^2$$

linear accelerator: $\left(\frac{\mathrm{d}c\mathrm{p}}{\mathrm{d}t}\right)_{\|} = ec\mathbf{E}$

very small!

Radiation Power transverse acceleration

transverse acceleration:

$$\mathrm{d}\gamma = 0 \qquad \Rightarrow \qquad \mathrm{d}c\mathbf{p}_{\perp} = \gamma mc^2 \mathrm{d}\beta_{\perp}$$

radiation power for transverse acceleration:

$$P_{\perp} = \frac{2}{3}\frac{e^2}{c}\gamma^4 \frac{1}{\gamma^2 \left(mc^2\right)^2} \left(\frac{\mathrm{d}c\mathbf{p}}{\mathrm{d}t}\right)_{\perp}^2 = \frac{2}{3}\frac{r_{\mathrm{c}}}{mc^3}\gamma^2 \left(\frac{\mathrm{d}c\mathbf{p}}{\mathrm{d}t}\right)_{\perp}^2$$

Final radiation power

$$F_\perp = \left(\frac{\mathrm{d}\mathbf{p}_\perp}{\mathrm{d}t}\right) = \frac{\gamma m v^2}{\rho} = [c]e\beta B$$

$$\frac{\mathrm{d}c\mathbf{p}_\perp}{\mathrm{d}t} = [c]ec\beta B$$

instantaneous radiation power from one electron

$$P_\perp = \frac{cC_\gamma}{2\pi}\frac{E^4}{\rho^2}$$

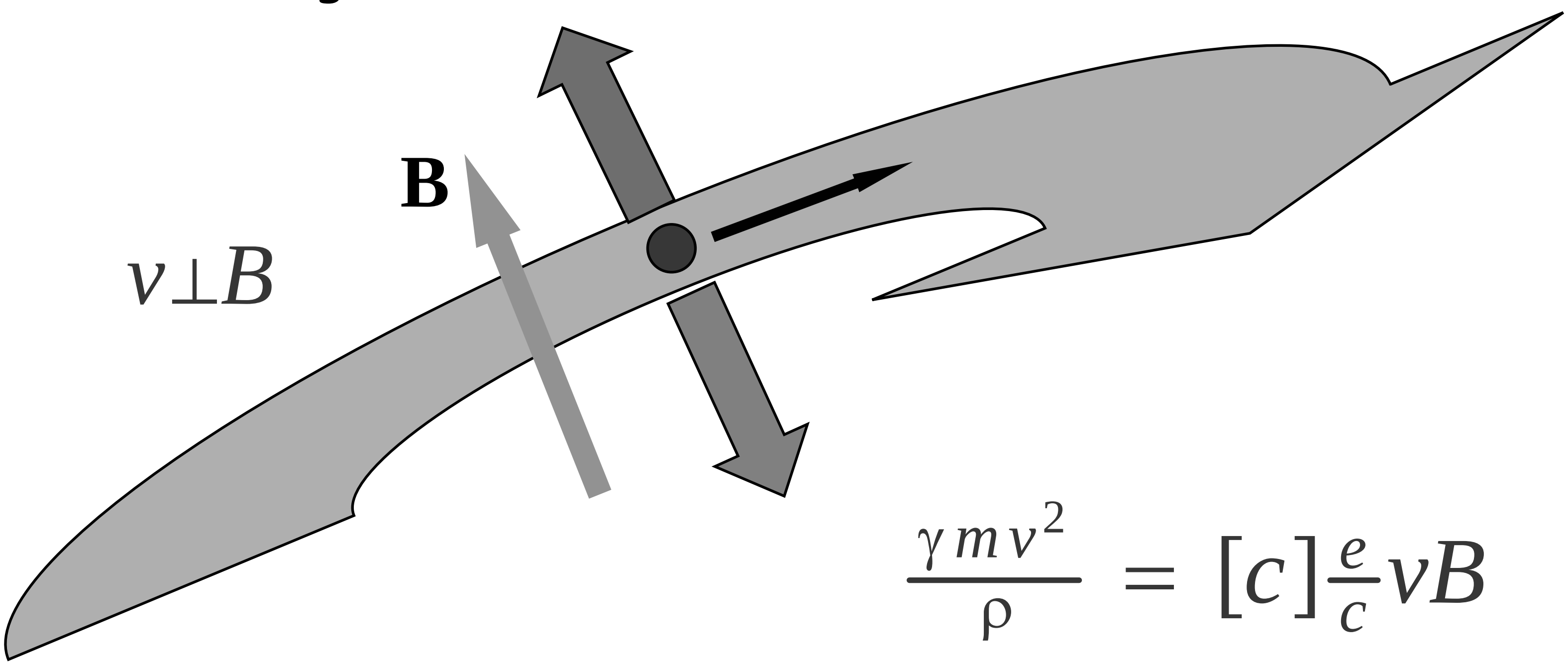
with

$$C_\gamma = \frac{4\pi}{3}\frac{r_\mathrm{c}}{\left(mc^2\right)^3} = 8.85 \cdot 10^{-5}\,\frac{\mathrm{m}}{\mathrm{GeV}^3}$$

Transverse acceleration

Lorentz force from magnetic fields

centrifugal force = Lorentz force



$$\frac{\gamma m v^2}{\rho} = [c]\frac{e}{c} v B$$

bending radius: $$\frac{1}{\rho}\left[\mathrm{m}^{-1}\right] = [c]\frac{eB}{\beta E} = 0.3\frac{B(\mathrm{T})}{E(\mathrm{GeV})}$$

Energy loss

energy loss per turn:

$$\Delta E = \oint P_\gamma \, \mathrm{d}t = C_\gamma \frac{E^4}{\rho^2}$$

avg. radiation power (isomagnetic ring)

$$P_\gamma = \Delta E \cdot \frac{I}{e} = C_\gamma \frac{E^4}{\rho} I$$

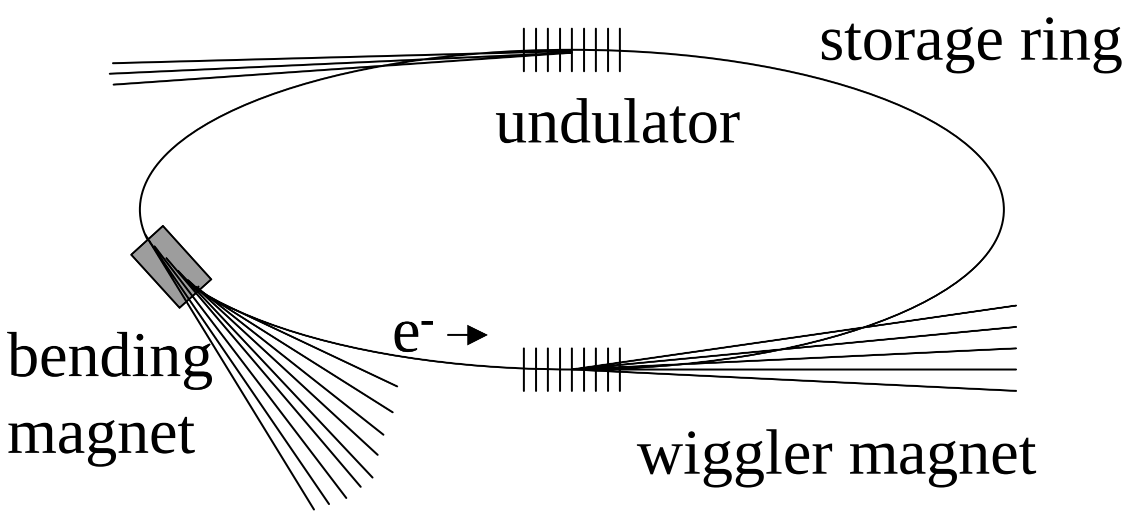
I circulating current

do protons radiate?

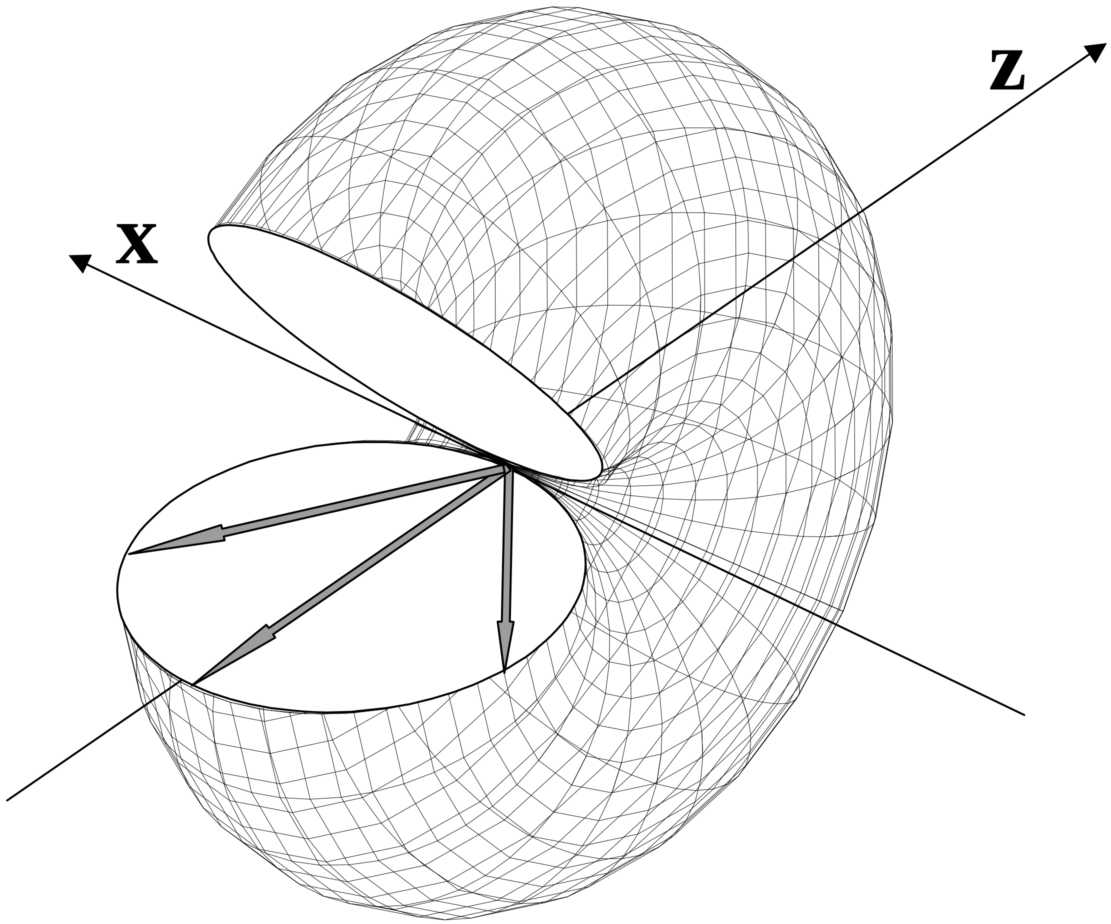
$$\frac{P_\mathrm{p}}{P_\mathrm{e}} = \left(\frac{m_\mathrm{e}}{m_\mathrm{p}}\right)^4 \approx 10^{-13} \quad !$$

Synchrotron Light Source

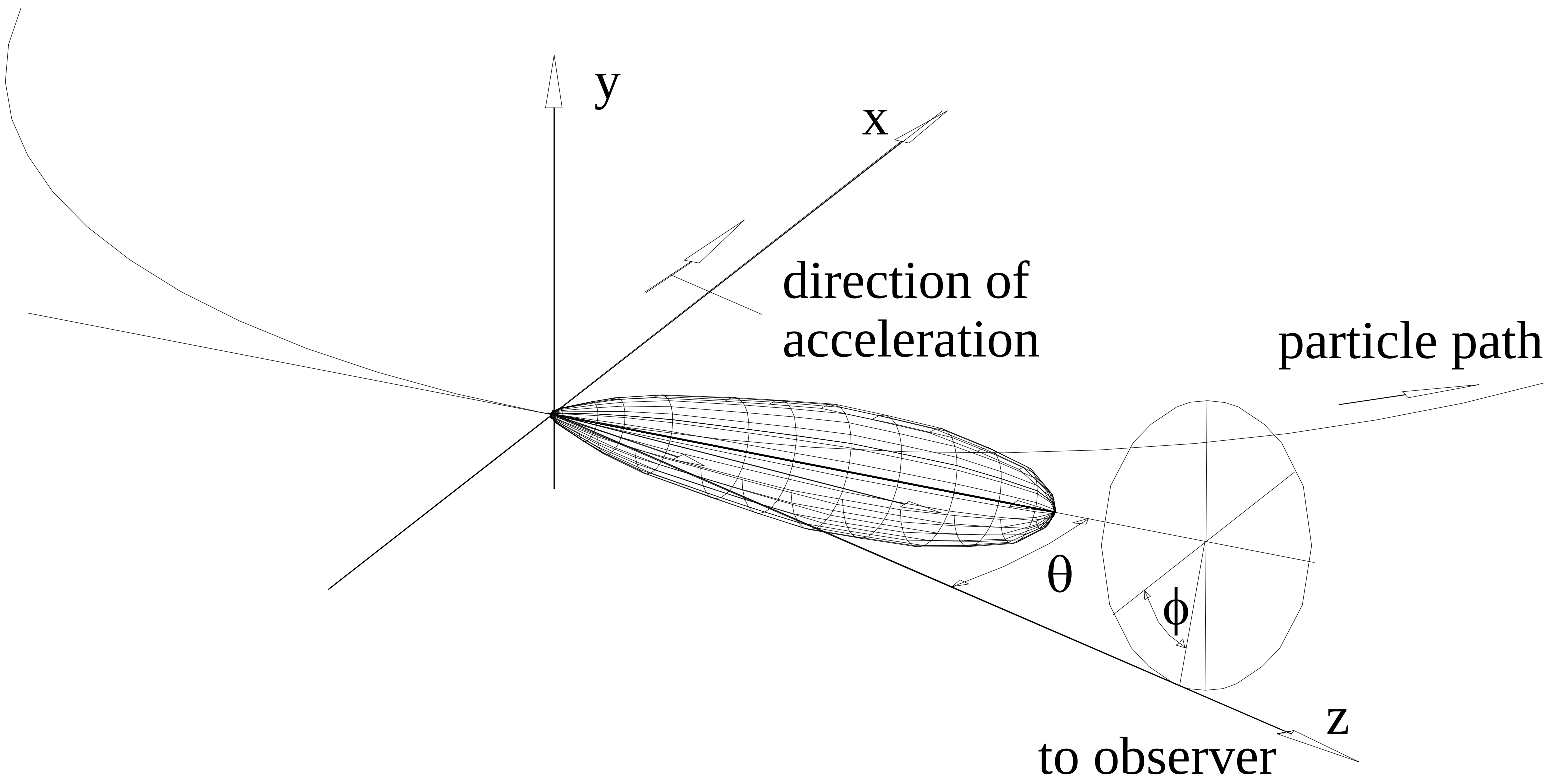
Synchrotron light source - schematic



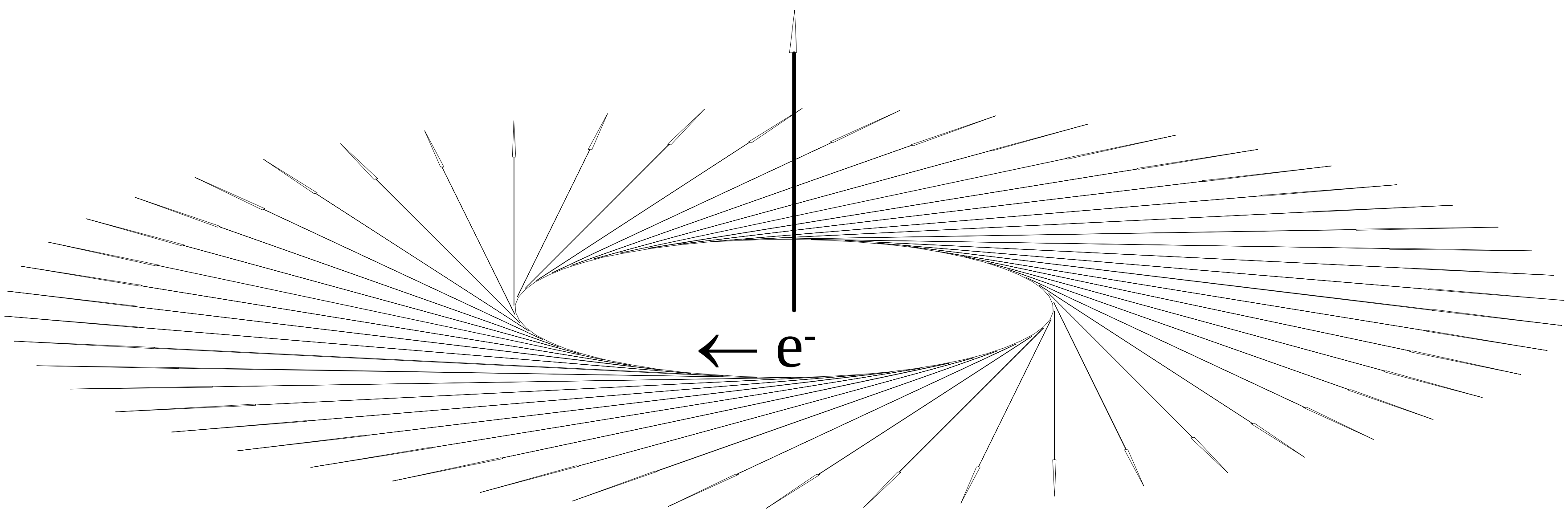
doughnut



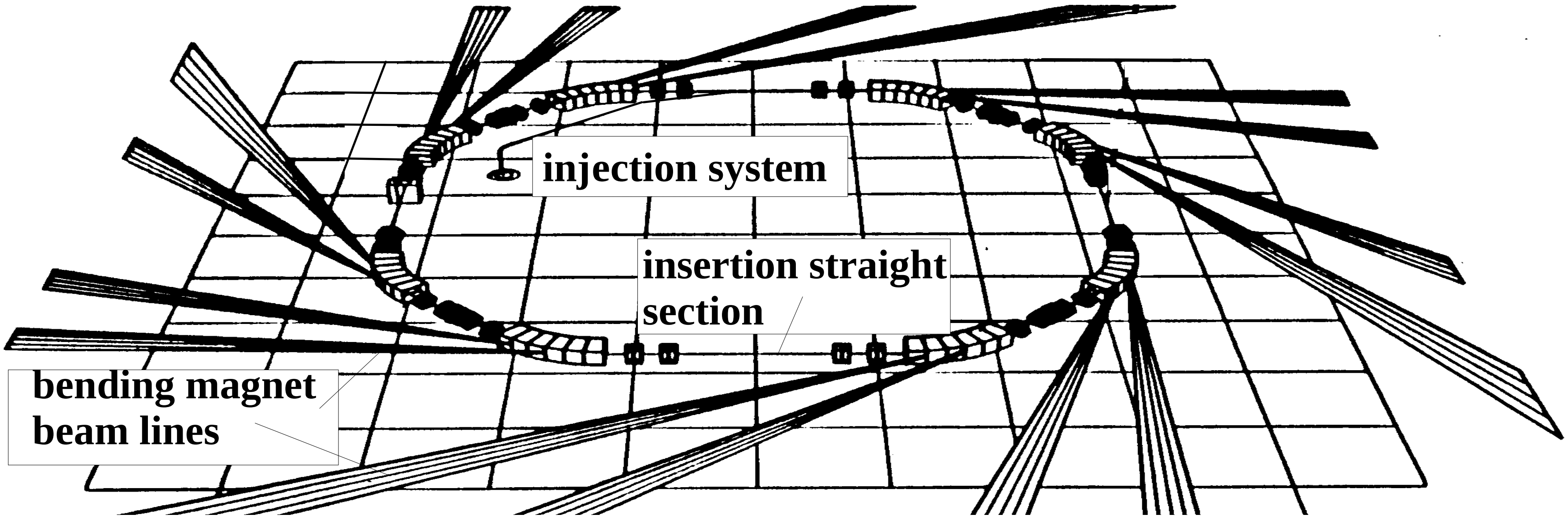
Transformation of doughnut to lab. system



Radiation swath



CAMD



injection system
insertion straight section
bending magnet beam lines

synchrotron radiation power

$$P_\gamma(\mathrm{GeV/sec}) = \frac{cC_\gamma}{2\pi}\frac{E^4(\mathrm{GeV})}{\rho^2(\mathrm{m})}$$

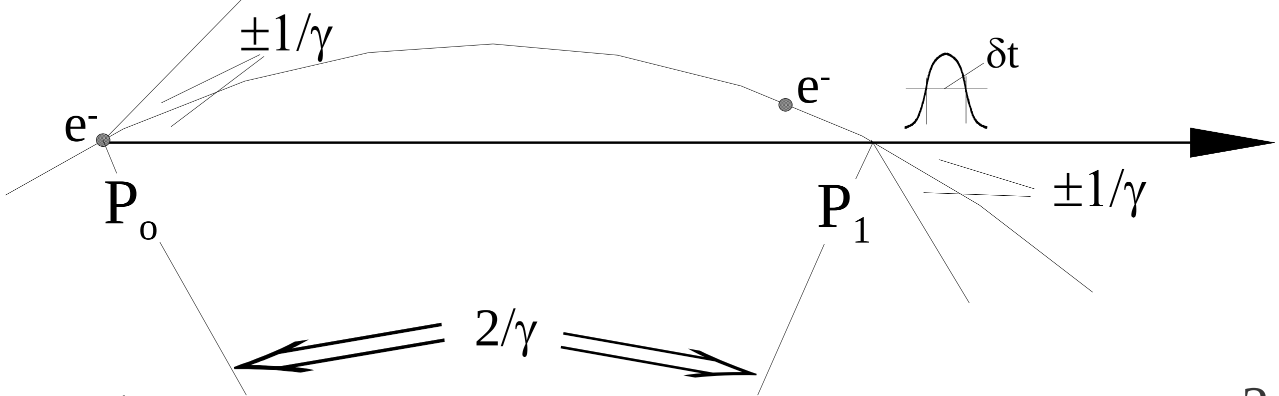
$$C_\gamma = \frac{4\pi}{3}\frac{r_c}{(mc^2)^3} = 8.8575 \cdot 10^{-5}\,\mathrm{m/GeV^3}$$

energy loss per turn

$$U_o = \oint P_\gamma \mathrm{d}t = \oint P_\gamma \mathrm{d}s/c$$

$$U_o(\mathrm{GeV}) = C_\gamma \frac{E^4(\mathrm{GeV})}{\rho(\mathrm{m})}$$

Synchrotron radiation spectrum



$$t_\gamma = \frac{2\rho \sin\frac{1}{\gamma}}{c} \qquad t_e = \frac{2\rho}{\beta c \gamma}$$

$$\delta t = t_e - t_\gamma = \frac{2\rho}{\beta c \gamma} - \frac{2\rho \sin\frac{1}{\gamma}}{c}$$

$$\omega_{\max} \approx \frac{\pi}{\delta t} \approx \frac{3\pi}{4} c \frac{\gamma^3}{\rho}$$

$$\boxed{\omega_c = \frac{3}{2} c \frac{\gamma^3}{\rho}}$$

critical photon frequency/energy

Differential photon flux

Spectral and spatial photon flux:

$$\Delta\dot{N}_{ph} = C_\Omega E^2 I \frac{\Delta\omega}{\omega} \frac{\omega^2}{\omega_c^2} K_{2/3}^2(\xi) F(\xi,\theta) \Delta\theta \Delta\psi$$

$$F(\xi,\theta) = (1+\gamma^2\theta^2)^2 \left[1 + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} \frac{K_{1/3}^2(\xi)}{K_{2/3}^2(\xi)} \right]$$

$$C_\Omega = \frac{3\alpha}{4\pi^2 e (mc^2)^2} = 1.3255 \cdot 10^{22} \frac{\text{photons}}{\sec \text{rad}^2 \, \text{GeV}^2 \, \text{A}}$$

$$\xi = \frac{1}{2} \frac{\omega}{\omega_c} (1+\gamma^2\theta^2)^{3/2}$$

θ: vertical observation angle

ψ: horizontal observation angle

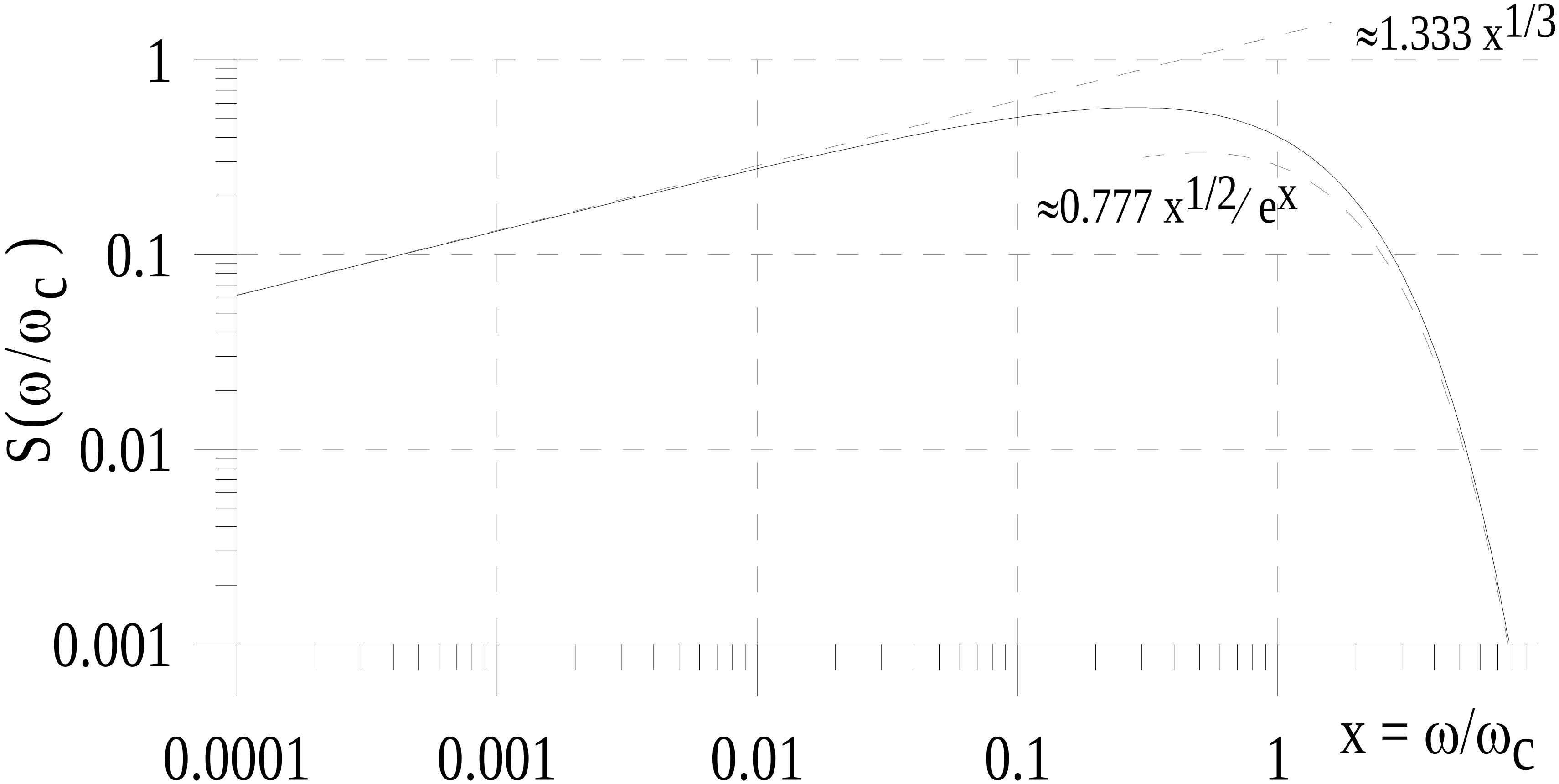
Angle integrated photon flux

$$\frac{d\dot{N}_{ph}}{d\psi} = C_\psi E I \frac{\Delta\omega}{\omega} S\left(\frac{\omega}{\omega_c}\right)$$

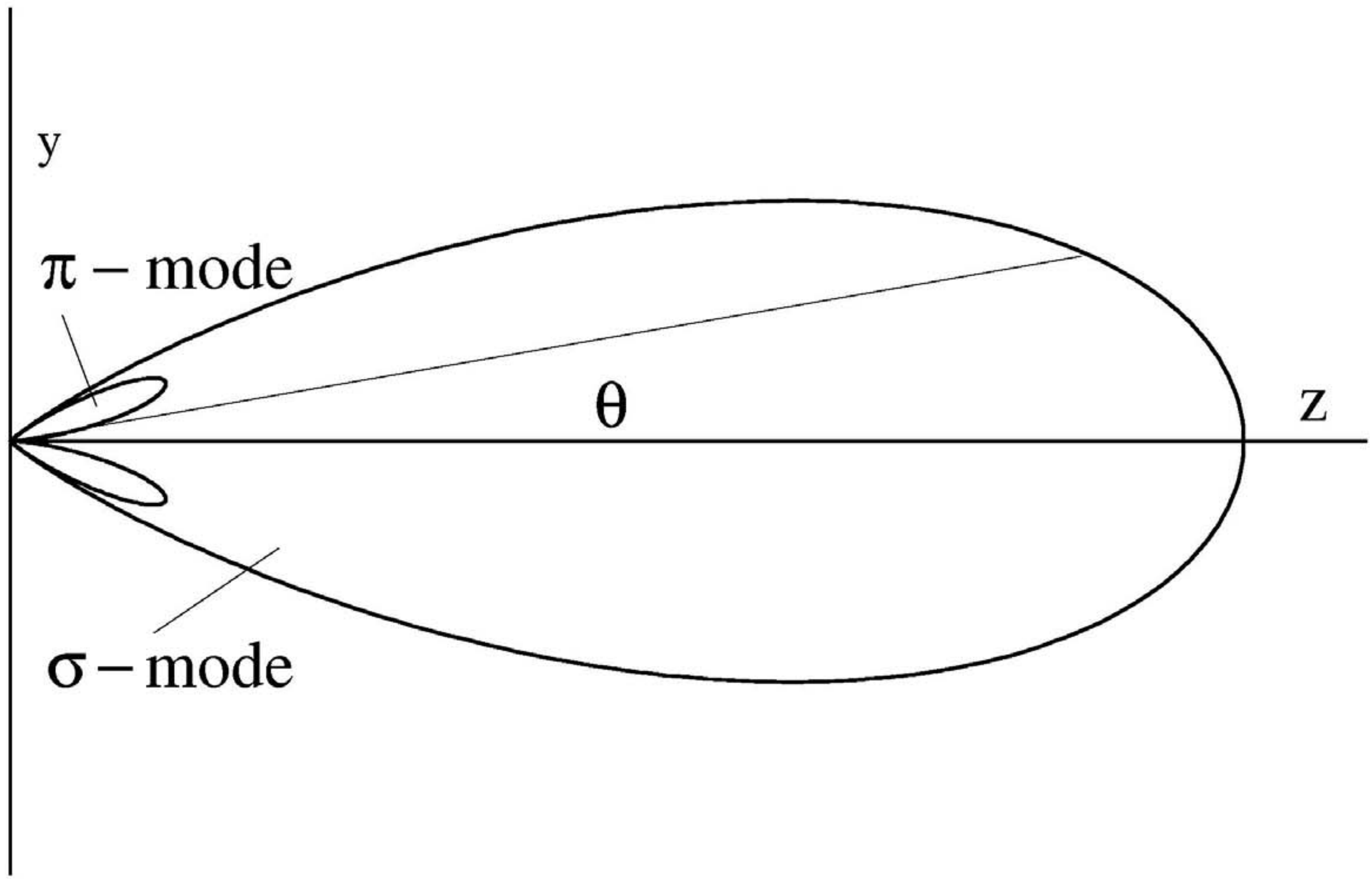
$$C_\psi = \frac{4\alpha}{9 e m c^2} = 3.9614 \cdot 10^{19} \frac{\text{photons}}{\sec \text{rad A GeV}}$$

$$S(\omega/\omega_c) = \frac{9\sqrt{3}}{8\pi} \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{5/3}(x)\, dx$$

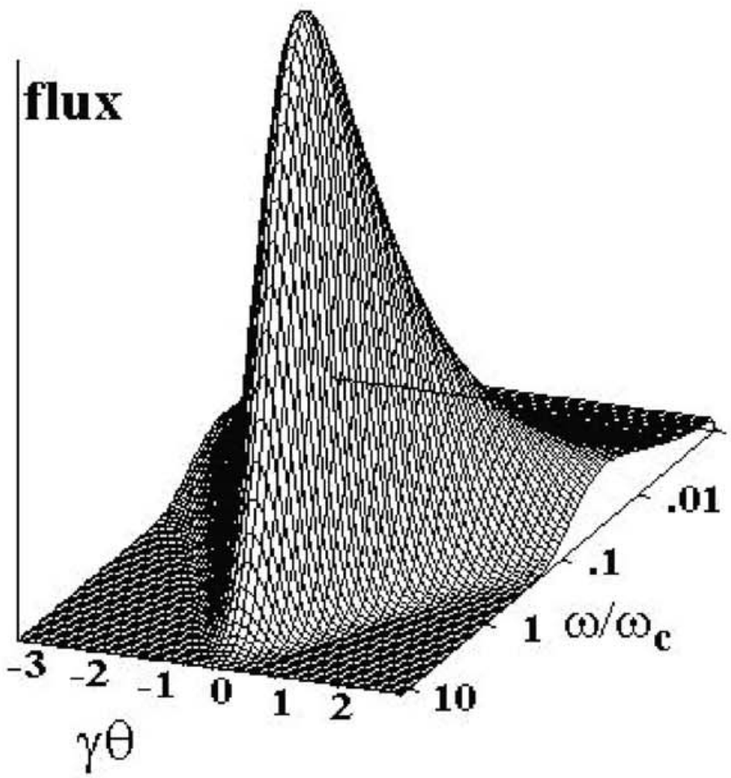
Universal function

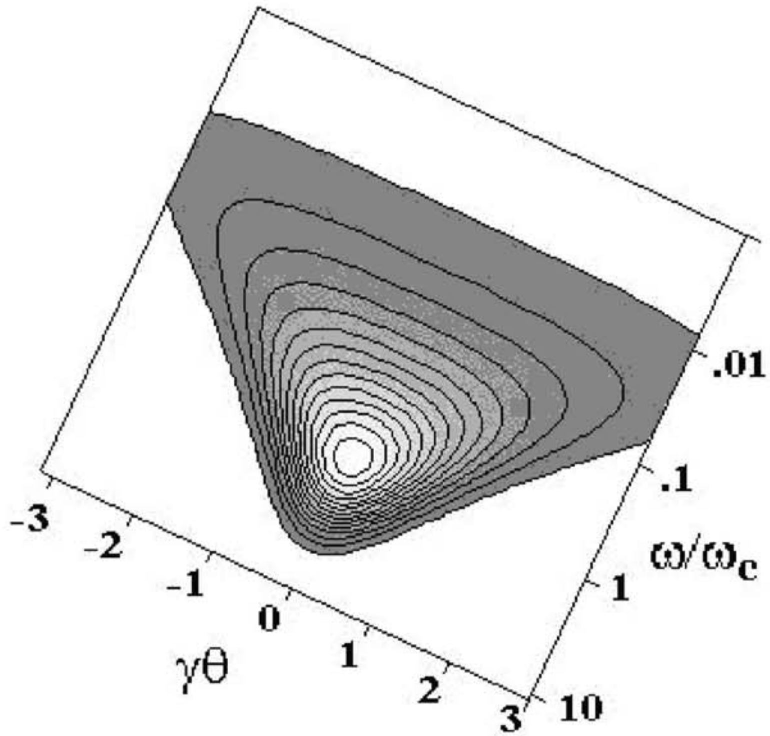


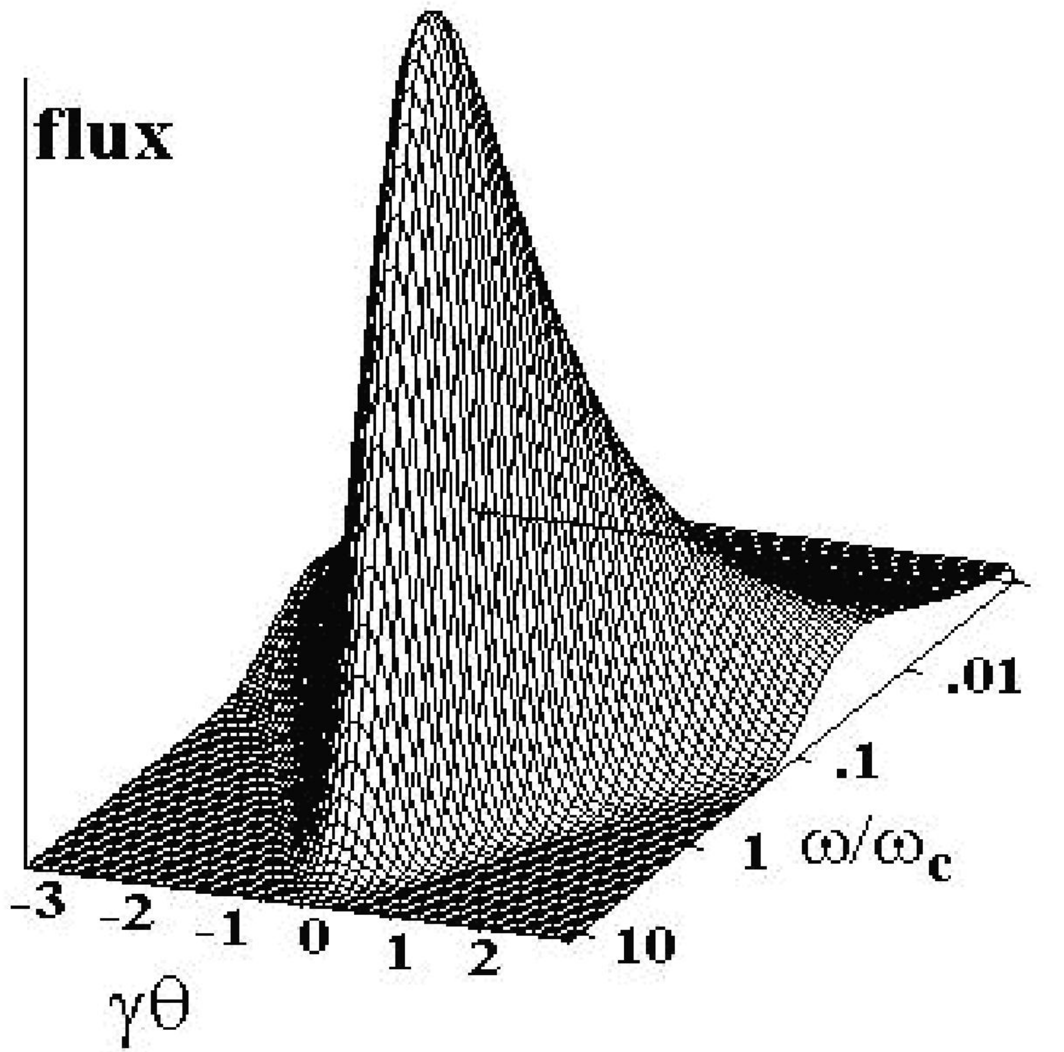
σ–π Intensity



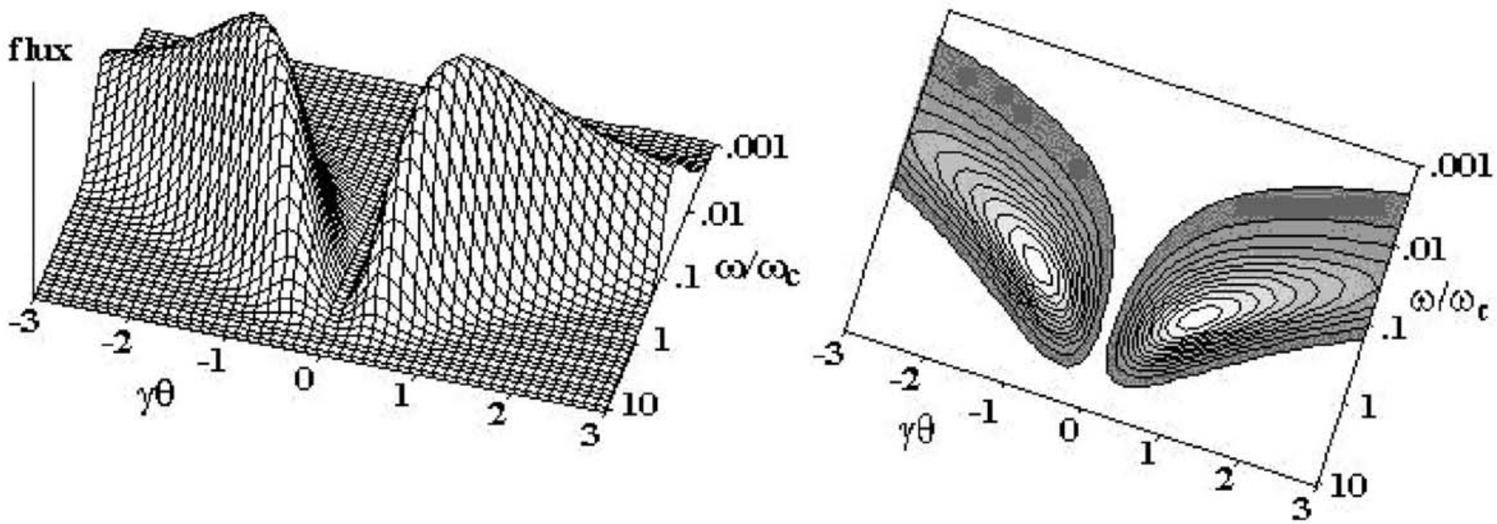
σ-mode radiation



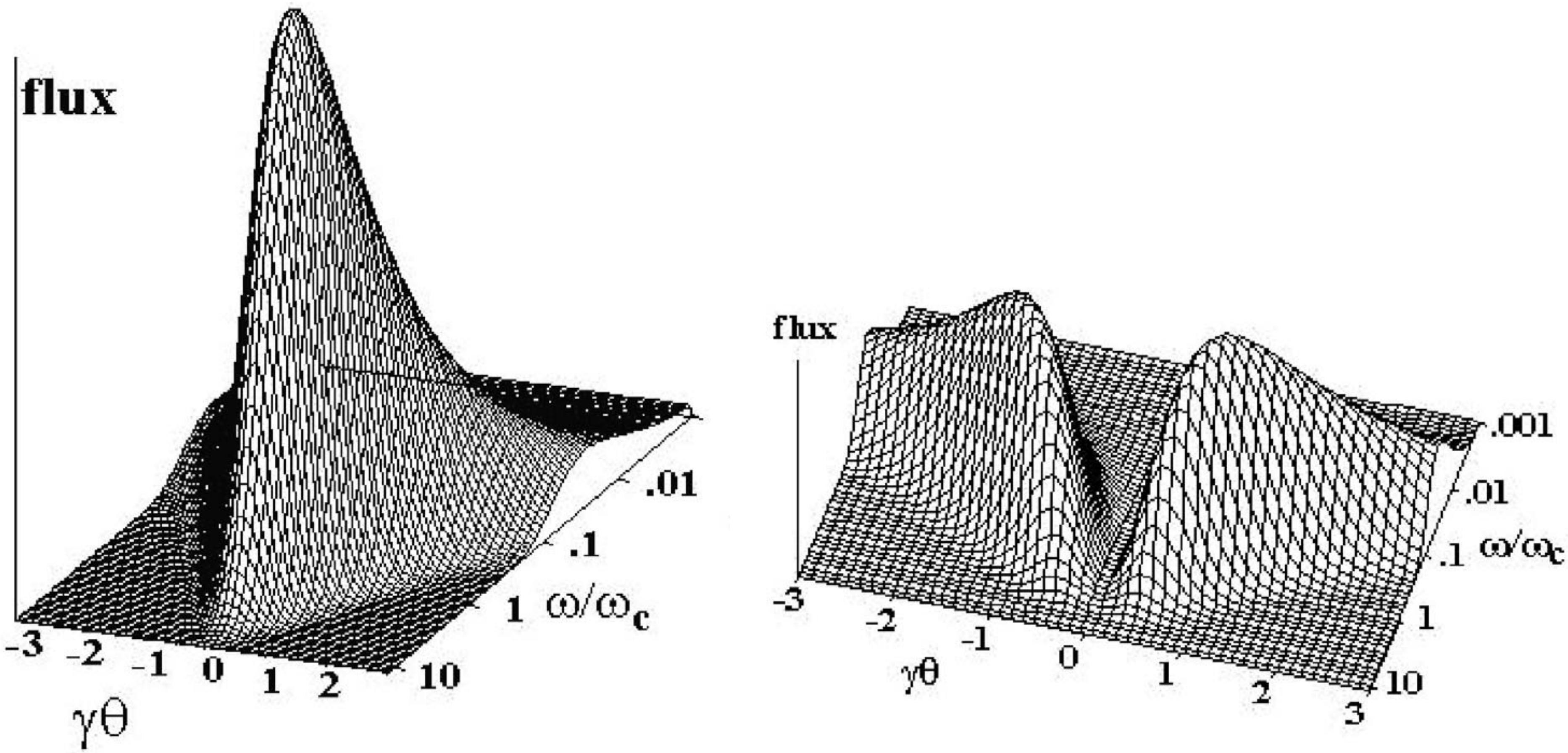



flux
.01
.1
1 ω/ω_c
10
-3 -2 -1 0 1 2
γθ

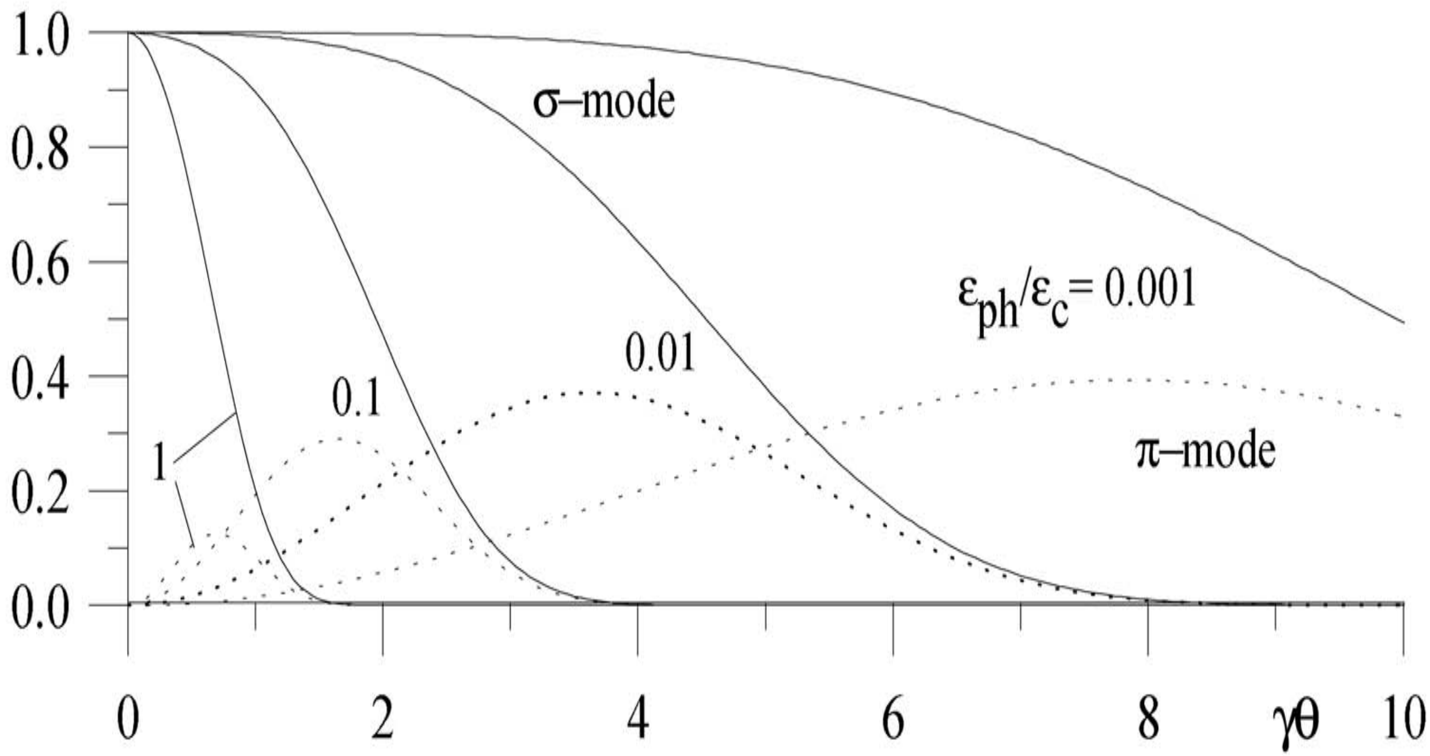
π-mode radiation



Compare σ–π mode radiation



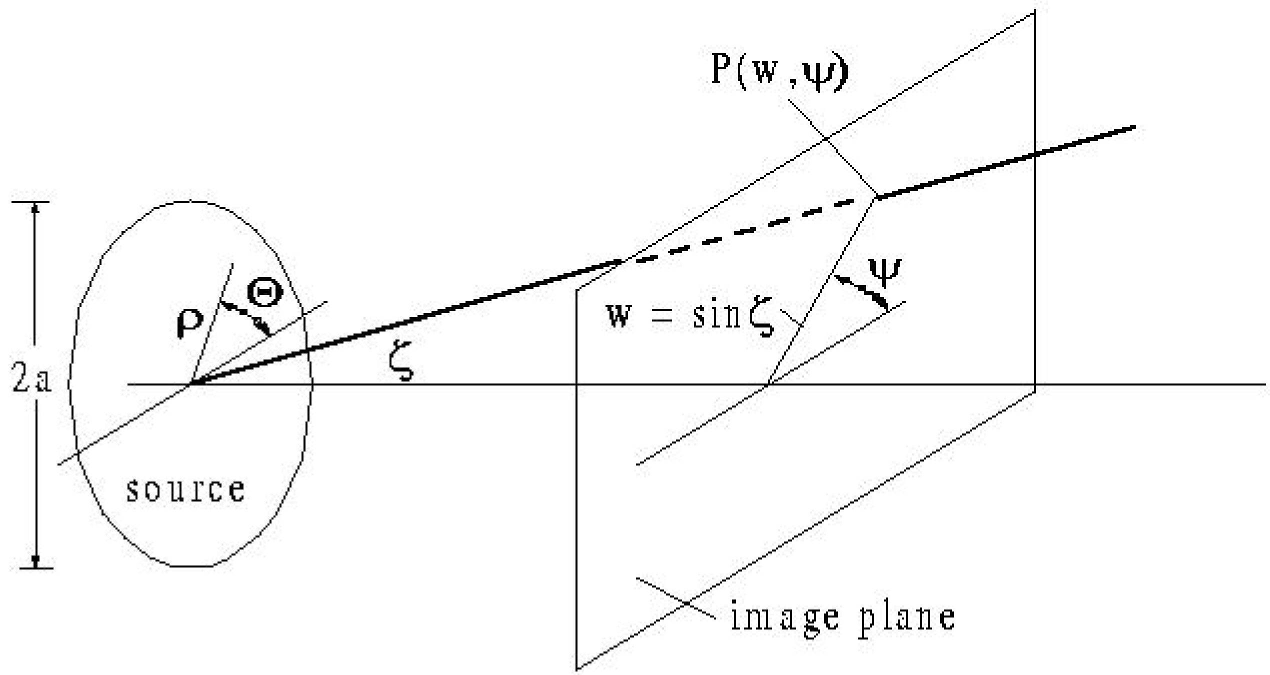
relative photon flux for σ– and π-mode radiation



Coherent Radiation

Spatial coherence

Fraunhofer diffraction integral



Fraunhofer diffraction integral:

$$U(P) = C\int_0^a \int_0^{2\pi} \exp[-ik\rho w\cos(\Theta - \psi)]\, d\Theta\, \rho\, d\rho$$

we set $\alpha = \Theta - \psi$ and get with $J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp[-\mathrm{i}x\cos\alpha]\,\mathrm{d}\alpha$

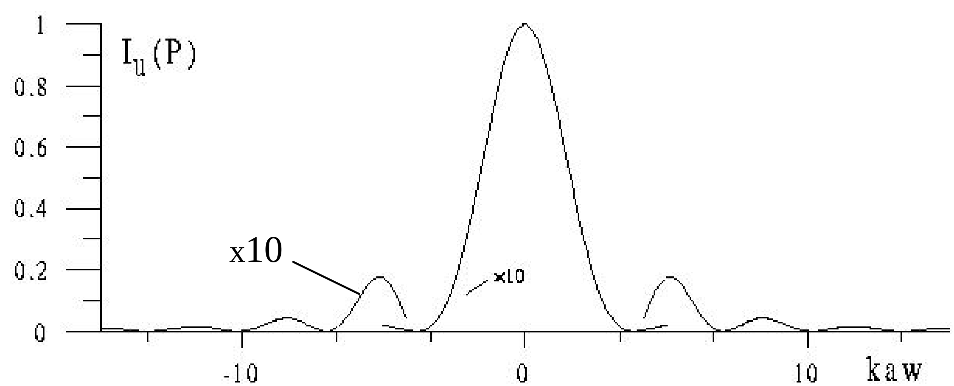
$$\mathrm{U(P)} = 2\pi\mathrm{C}\int_0^a J_0(k\rho w)\,\rho\,\mathrm{d}\rho$$

$J_0(x)$ 0^{th}- order Bessel's function

with $\int_0^x J_0(y)\,y\,\mathrm{d}y = x J_1(x)$ and $\mathrm{I(P)} = \mathrm{U}^2\mathrm{(P)}$

$$\boxed{I(P) = I_0 \frac{4J_1^2(k\rho w)}{(k\rho w)^2}}$$

$$I_0 = I(w \to 0)$$



Gaussian source

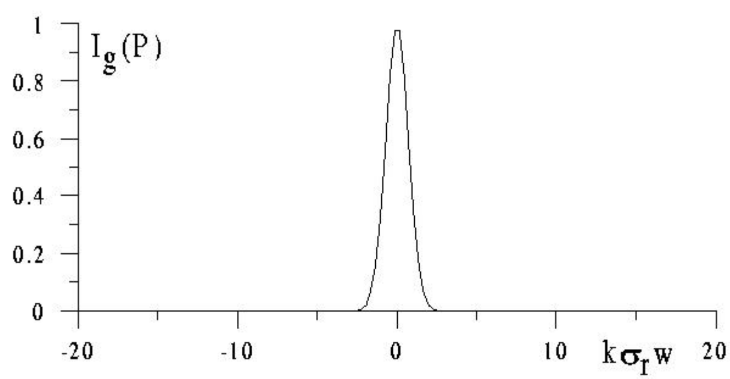
for Gaussian source distribution: $\mathrm{U}_G(\mathrm{P}) = \mathrm{C}\int_0^\infty \exp\left(-\frac{\rho^2}{2\sigma_r^2}\right) J_0(k\rho w)\,\rho\,\mathrm{d}\rho$

with $x = \frac{\rho}{\sqrt{2}\sigma_r}$ and $k\rho w = \sqrt{2}\,x\,k\sigma_r\,w = 2x\sqrt{z}$

$$\mathrm{U}_G(\mathrm{P}) = \mathrm{C}\int_0^\infty \mathrm{e}^{-x^2} x\,J_0(2x\sqrt{z}\,)\,\mathrm{d}x = C\exp\left[-\frac{1}{2}(k\sigma_r w)^2\right]$$

no ring structure !

Gaussian distribution with $w = \sigma_{r'} = \frac{1}{k\sigma_r}$



diffraction limited source emittance

$$\boxed{\sigma_r\sigma_{r'} = \frac{\lambda}{2\pi}}$$

diffraction limited source emittance

$$\sigma_r \sigma_{r'} = \frac{\lambda}{2\pi} \quad \Longrightarrow \quad \sigma_{x,y}\sigma_{x',y'} = \frac{\lambda}{4\pi}$$

electron beam with emittance

$$\epsilon_x \leq \frac{\lambda}{4\pi}$$

$$\epsilon_y \leq \frac{\lambda}{4\pi}$$

is source of spatially coherent radiation

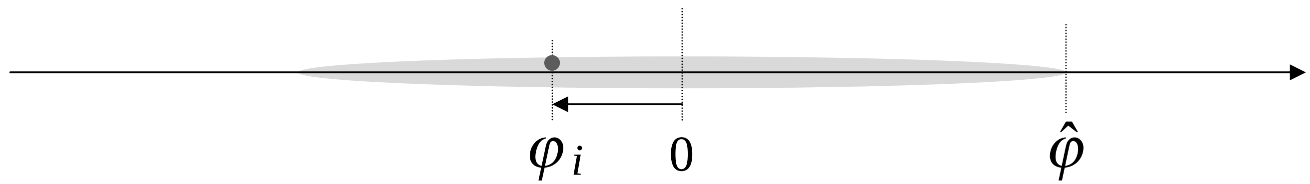
for wavelengths λ and longer

Temporal coherence

coherent radiation - 1

temporal coherence properties of radiation depends on the temporal distribution of electrons

each electron emits a field at frequency ω given by: $E_i = E_0\,\mathrm{e}^{-\mathrm{i}(\omega t - \varphi_i)}$

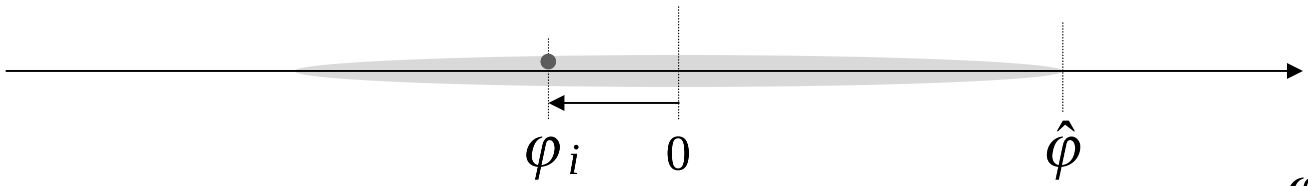


total radiation field is: $E = \sum_i E_0\,\mathrm{e}^{-\mathrm{i}(\omega t - \varphi_i)}$

total radiation power:

$$P \propto E_0^2 \sum_i^N e^{-\mathrm{i}(\omega t - \varphi_i)} \sum_j^N e^{\mathrm{i}(\omega t - \varphi_j)} = E_0^2 \sum_{i,j}^N e^{\mathrm{i}(\varphi_i - \varphi_j)} = E_0^2 \left(N + \sum_{i \neq j}^N e^{\mathrm{i}(\varphi_i - \varphi_j)} \right)$$

short bunches



$$\varphi = \frac{2\pi}{\lambda} s$$

if bunch length $s \ll \lambda$ then $\varphi_i - \varphi_j \ll 2\pi$ for all i,j

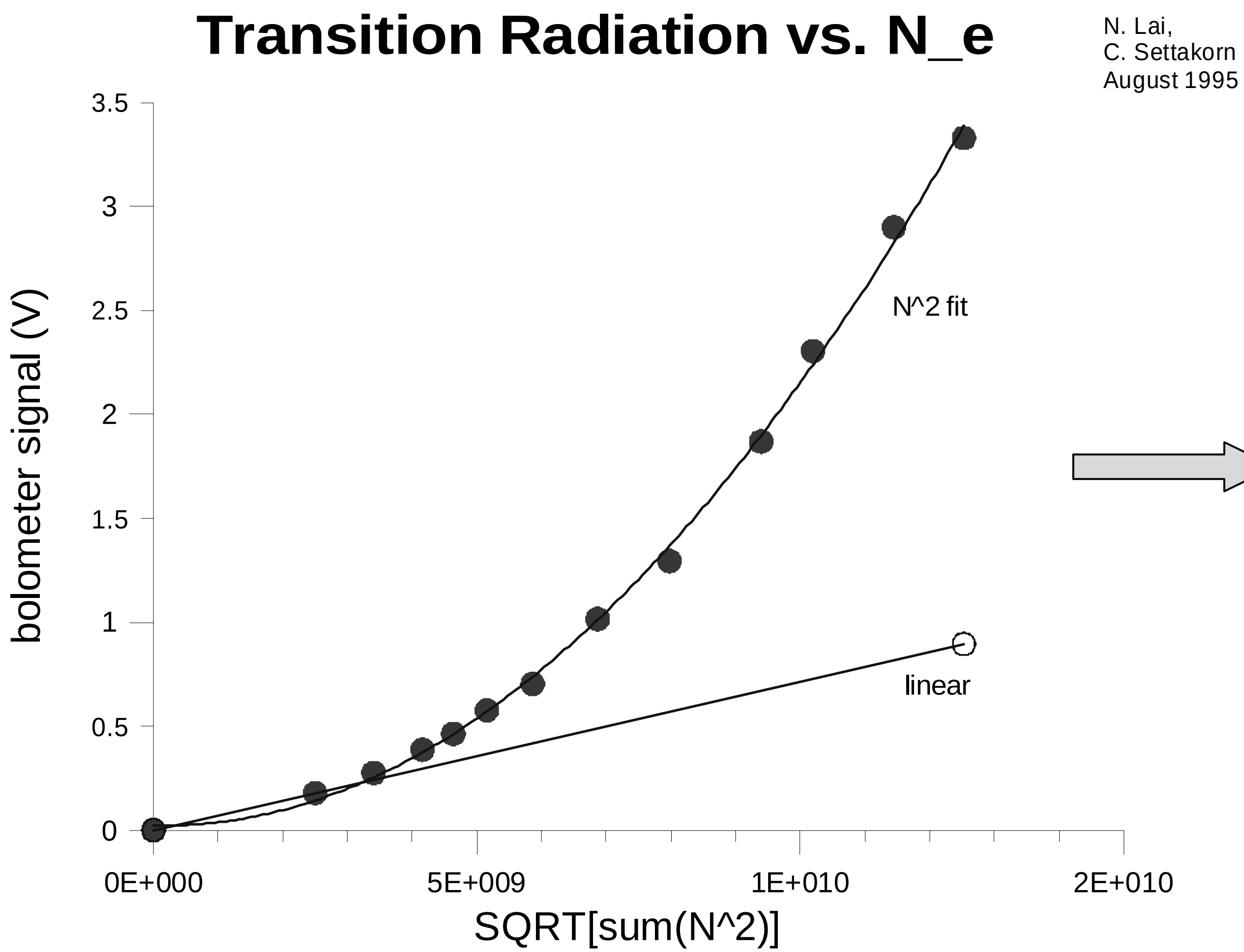
$$P \propto E_0^2 \left(N + \sum_{i \neq j}^{N} e^{\mathrm{i}(\varphi_i - \varphi_j)} \right) = E_0^2 \left[N + N(N-1) \right]$$

radiation power of coherent radiation is proportional to N^2 or proportional to the square of beam intensity

$$\boxed{P \propto N^2}$$

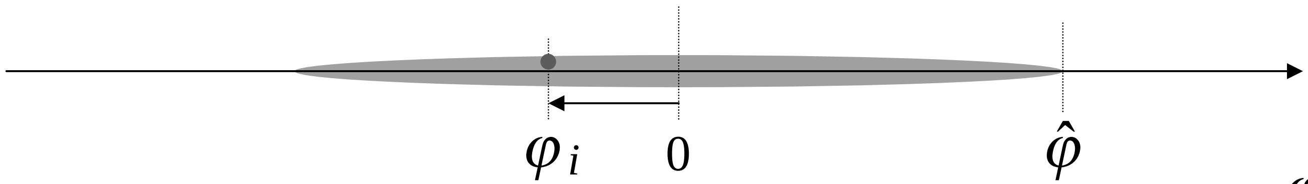
coherent radiation

TR intensity scaling



TR from short electron bunches is coherent

long bunch



$$\varphi = \frac{2\pi}{\lambda} s$$

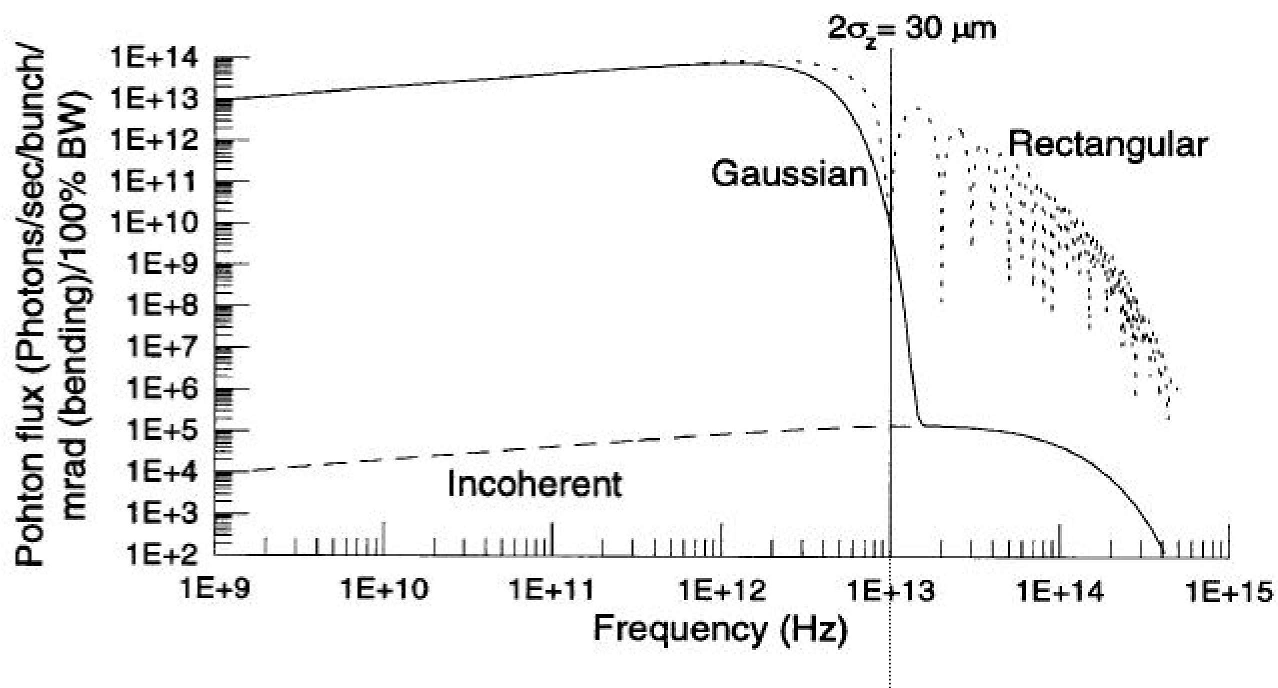
if bunch length $s \gg \lambda$ then $\varphi_i - \varphi_j \gg 2\pi$ for all i,j

$$P \propto E_0^2 \left(N + \underbrace{\sum_{i \neq j}^{N} e^{\mathrm{i}(\varphi_i - \varphi_j)}}_{=0} \right) = E_0^2 N$$

$$\boxed{P \propto N}$$

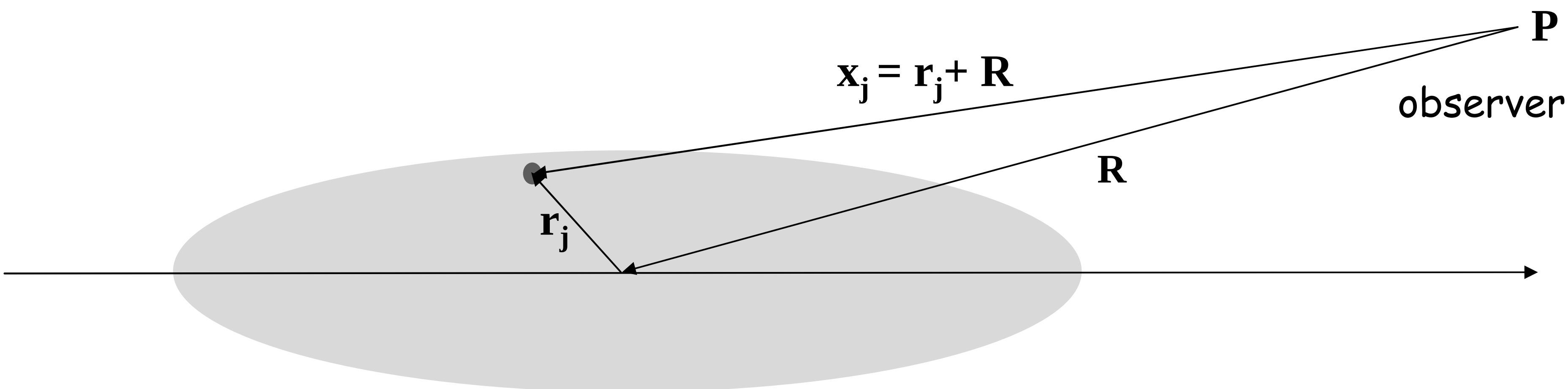
incoherent radiation

every time we have radiation from bunched electron beam,
we have some coherent radiation



λ=30μm

Form Factor - 1



$$E_{\text{tot}}(\omega) = \sum_{j=1}^{N} E_0(\omega)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}_j\mathbf{x}_j} = \sum_{j=1}^{N} E_0(\omega)\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\mathbf{n}_j\mathbf{r}_j}\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\mathbf{n}_j\mathbf{R}} \qquad R \gg |\mathbf{r}|_j$$

intensity
$$I_{\text{total}}(\omega) \propto \left|E_{\text{tot}}(\omega)\right|^2 \approx \left|\sum_{j=1}^{N} E_0(\omega)\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\mathbf{n}_j\mathbf{r}_j}\right|^2 = \sum_{j=1}^{N} E_0(\omega)\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\mathbf{n}_j\mathbf{r}_j} \sum_{k=1}^{N} E_0^*(\omega)\,\mathrm{e}^{\mathrm{i}\frac{\omega}{c}\mathbf{n}_k\mathbf{r}_k}$$

$$= \sum_{j=1}^{N} |E_0(\omega)|^2 + \sum_{\substack{j,k=1 \\ j\neq k}}^{N} |E_0(\omega)|^2\,\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\mathbf{n}_j(\mathbf{r}_j-\mathbf{r}_k)}$$

Form factor - 2

$$I_{\mathrm{total}}(\omega) \propto I_0(\omega)\, N + I_0(\omega) \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\mathbf{n}_j(\mathbf{r}_j-\mathbf{r}_k)}$$

incoherent   coherent radiation

$S(\mathbf{r})$ be the probability to find a particle at $\mathbf{r}$ with $\int S(\mathbf{r})\,\mathrm{d}^3 r = 1$

coherent radiation intensity:

$$\begin{aligned} I_{\mathrm{coh}}(\omega) &\approx I_0(\omega)\, N(N-1) \int \mathrm{d}^3 r \int \mathrm{d}^3 r'\, \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\mathbf{n}(\mathbf{r}-\mathbf{r}')} S(\mathbf{r}) S(\mathbf{r}') \\ &= I_0(\omega)\, N(N-1) \left| \int \mathrm{d}^3 r\, \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\mathbf{n}(\mathbf{r}-\mathbf{r}')} S(\mathbf{r}) \right|^2 \\ &= I_0(\omega)\, N(N-1)\, F(\omega, \mathbf{n}) \end{aligned}$$

$$F(\omega, \mathbf{n}) = \left| \int \mathrm{d}^3 r\, \mathrm{e}^{-\mathrm{i}\frac{\omega}{c}\mathbf{n}\mathbf{r}} S(\mathbf{r}) \right|^2$$ bunch form factor

Form factor - 3

for a Gaussian particle distribution in z : $S(\mathbf{r}) = h(z) = \frac{1}{\sqrt{2\pi}\,\sigma_z} \mathrm{e}^{-\mathrm{i}z^2/2\sigma_z^2}$

and the formfactor is:

$$F(\omega,\mathbf{n}) = \left| \int \mathrm{e}^{-\mathrm{i}\frac{\omega}{c} z\cos\theta} h(\mathrm{z})\,\mathrm{dz} \right|^2 = \mathrm{e}^{-(\omega\sigma_z\cos\theta/c)^2}$$

θ : observation direction with respect to electron beam direction

similarly, for a uniform particle distribution

$$h(\mathrm{z}) = \begin{cases} 1/(2\sigma_z) & \text{for } |\mathrm{z}| \le \sigma_z \\ 0 & \text{otherwise} \end{cases}$$

and

$$F(\omega,\theta) = \left[\frac{2J_1(\omega\sigma_z\,\sin\theta/c)}{\omega\sigma_z\,\sin\theta/c} \frac{\sin(\omega\sigma_z\,\cos\theta/c)}{\omega\sigma_z\cos\theta/c} \right]$$

form factor

Gaussian form factor

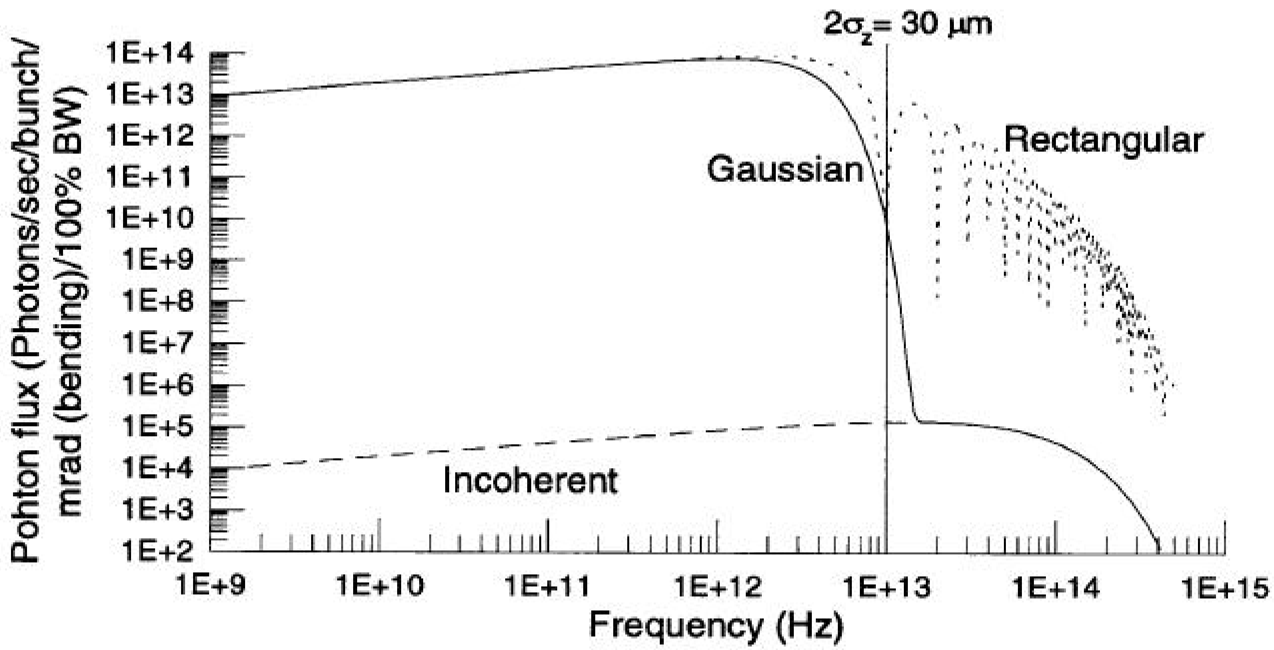
$$F(\omega,\theta) = e^{-(\omega\sigma_z \cos\theta/c)^2}$$

form factor or intensity drops off when $\sigma_z \gtrsim \frac{\lambda}{2\pi}$

or for wavelength $\lambda \lesssim 2\pi\sigma_z$

coherence length

can broad spectrum be coherent ?



yes, with coherence length $\ell_c = \frac{\lambda^2}{\Delta\lambda} \approx \lambda$

Insertion Device Radiation

introduction

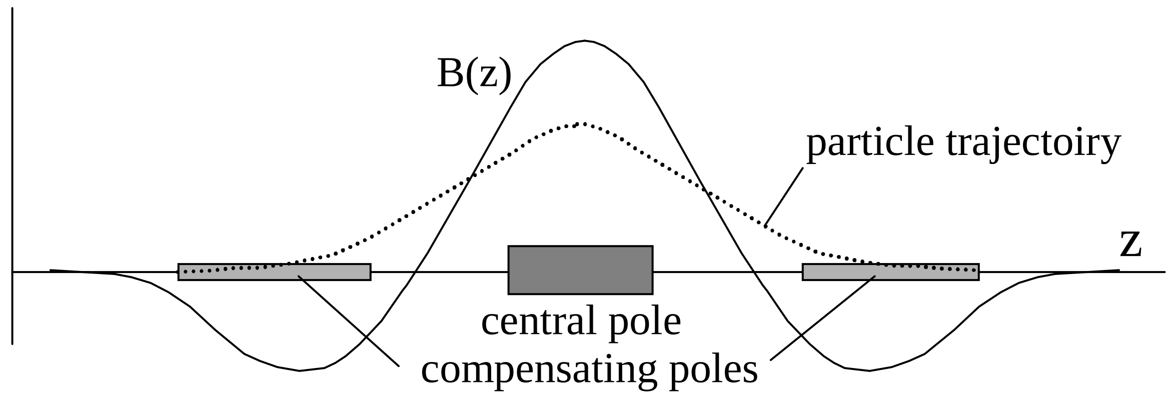
Insertion devices do not change the shape of the storage ring!

$$\int_{-\infty}^{+\infty} B_y(y = 0, z)\mathrm{d}z = 0$$

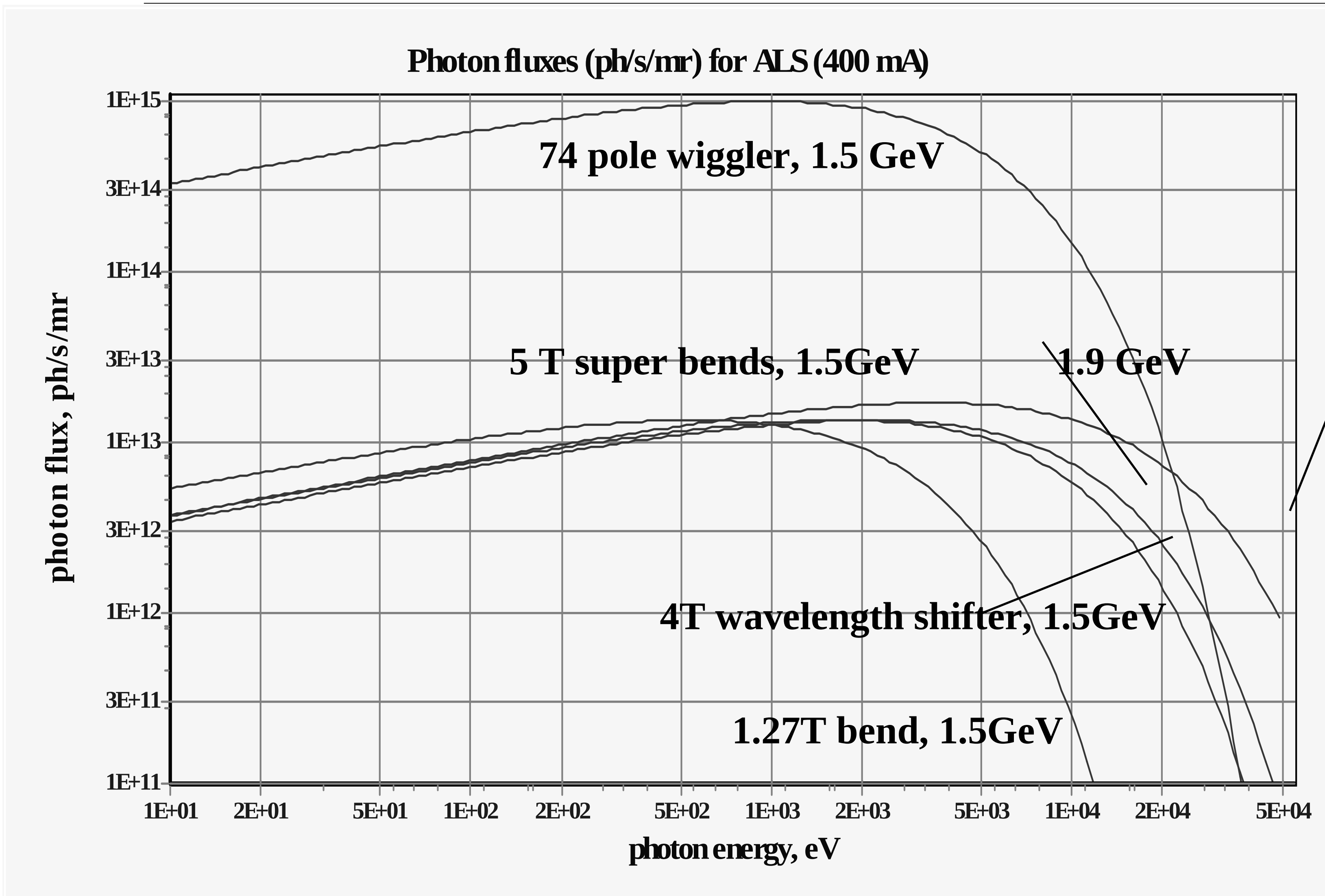
- Wavelength shifter
- Wiggler magnet
- Undulators
- Super bends

Purpose: harden radiation
increase intensity
high brightness monochromatic radiation
elliptically polarized radiation

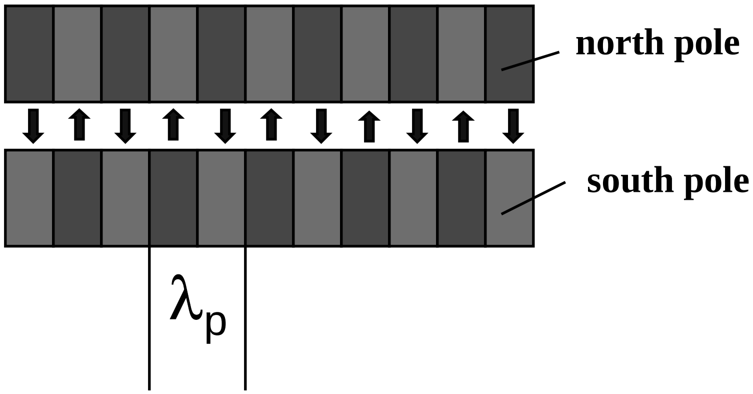
Wave Length Shifter



$$\int_{-\infty}^{+\infty} B_y(y = 0, z)\mathrm{d}z = 0$$


Photon fluxes (ph/s/mr) for ALS (400 mA)
74 pole wiggler, 1.5 GeV
5 T super bends, 1.5GeV
1.9 GeV
4T wavelength shifter, 1.5GeV
1.27T bend, 1.5GeV
photon flux, ph/s/mr
photon energy, eV
1E+15
3E+14
1E+14
3E+13
1E+13
3E+12
1E+12
3E+11
1E+11
1E+01
2E+01
5E+01
1E+02
2E+02
5E+02
1E+03
2E+03
5E+03
1E+04
2E+04
5E+04

Periodically deflecting magnets



magnetic field:

$$B_y(z) = B_0 \cos k_p z$$

Wiggler magnets, strong field
Undulators, weak field

Wiggler magnets produce ordinary, broad band synchrotron radiation;

Intensity increased by factor N_p (# of poles)

K-value

deflection angle per half-pole

$$B_y(z) = B_0 \cos k_p z$$

$$d\theta = \frac{dz}{\rho} = \frac{eB}{cp} dz \qquad \theta = \int \frac{dz}{\rho} = \frac{eB}{cp} \int_0^{\lambda_p/4} \cos k_p z \, dz = \frac{eB_0 \lambda_p}{2\pi cp}$$

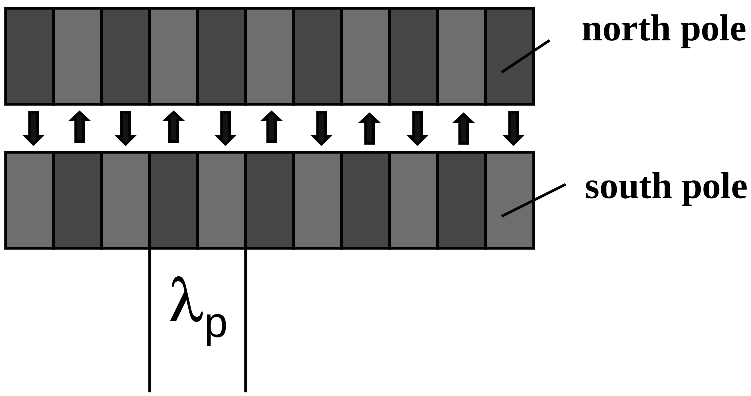
$$\theta = \frac{eB_0 \lambda_p}{2\pi cp} = \frac{K}{\gamma}$$

K: undulator/wiggler strength parameter

$$\boxed{K = \frac{eB_0 \lambda_p}{2\pi mc^2 \beta} = 0.934\, B_0(\mathrm{T})\, \lambda_p(\mathrm{cm})}$$

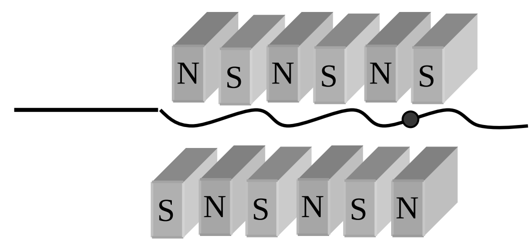
Undulator

assume a magnetic field $$B_y(z) = B_0 \cos k_p z$$



$$k_p = \frac{2\pi}{\lambda}$$

electron performs sinusoidal oscillations



sinusoidal perturbation of field lines

Sinc-function

sinusoidal perturbation of field lines

$\mathbf{N_p}$ undulator periods ⇆ $\mathbf{N_p}$ field oscillations

$$E(t) = \begin{cases} E_0 \sin\omega_0 t & \text{for } -\tfrac{1}{2}N_p T_0 < \omega_0 t < \tfrac{1}{2}N_p T_0 \\ 0 & \text{elsewhere} \end{cases}$$

spectrum

$$E(\omega) = \int E(t)\, e^{-i\omega t} dt = \int_{-\infty}^{\infty} E_0 \sin\omega_0 t\, e^{-i\omega t} dt = E_0$$

$$E_0 \int_{-\frac{1}{2}N_p T_0}^{\frac{1}{2}N_p T_0} e^{-i(\omega_0-\omega)t} dt = E_0 \frac{e^{i(\omega_0-\omega)\frac{1}{2}N_p T_0} - e^{-i(\omega_0-\omega)\frac{1}{2}N_p T_0}}{i(\omega_0-\omega)} = E_0 N_p T_0 \frac{\sin(\omega_0-\omega)\frac{1}{2}N_p T_0}{(\omega_0-\omega)\frac{1}{2}N_p T_0}$$

$$E(\omega) = E_0 N_p T_0 \frac{\sin(\omega_0-\omega)\frac{1}{2}N_p T_0}{(\omega_0-\omega)\frac{1}{2}N_p T_0}$$

ω_0

What is ω_0?

undulator period: λ_p

in electron rest system: $\lambda_p^* = \dfrac{\lambda_p}{\gamma}$

in lab system (Doppler effect): $\omega = \omega_p^* \gamma (1 + \mathbf{n}_z^* \beta)$ or $\lambda = \dfrac{\lambda_p}{\gamma^2 (1+\mathbf{n}_z^* \beta)}$

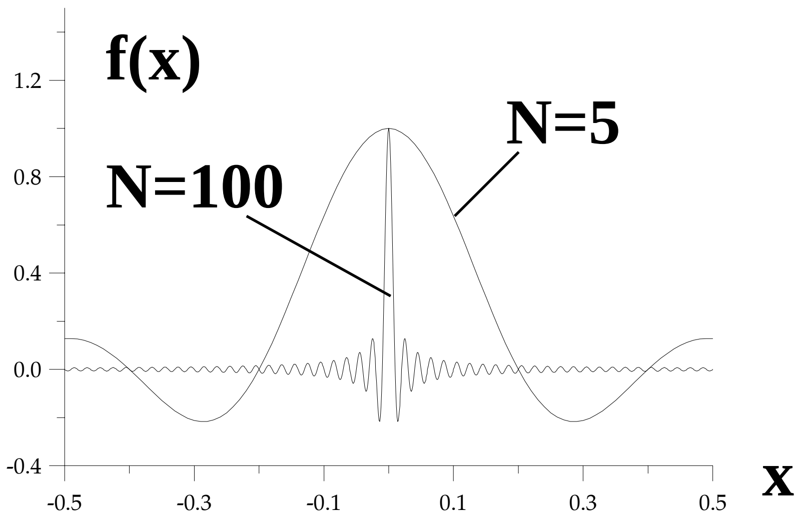
with $$\mathbf{n}_z = \frac{\beta + \mathbf{n}_z^*}{1+\mathbf{n}_z^* \beta} \quad \Rightarrow \quad \lambda = \frac{\lambda_p}{\gamma^2} \frac{\mathbf{n}_z}{\beta + \mathbf{n}_z^*} = \frac{\lambda_p}{\gamma^2} \frac{\cos\theta}{1+\cos\theta^*}$$

$$\sin\theta = \frac{\sin\theta^*}{\gamma(1+\beta\cos\theta^*)} \quad \Rightarrow \quad \theta \approx \frac{\sin\theta^*}{\gamma(1+\beta\cos\theta^*)}$$

or

$$\gamma^2\theta^2 = \frac{\sin^2\theta^*}{(1+\beta\cos\theta^*)^2} = \frac{1-\cos\theta^*}{1+\cos\theta^*} \quad \Rightarrow \quad \cos\theta^* = \frac{1-\gamma^2\theta^2}{1+\gamma^2\theta^2}$$

Sinc-function: $f(x) = \dfrac{\sin \pi N x}{\pi N x}$



f(0) = 1 and

f(y) = 0 for **y = 1/N**

or for $(\omega_0 - \omega)\tfrac{1}{2} N_p T_0 = \pi$

line width: $\dfrac{\delta\omega}{\omega_0} = \pm\dfrac{\omega_0 - \omega}{\omega_0} = \dfrac{2\pi}{T_0}\dfrac{1}{N_p}\dfrac{1}{\omega_0} = \dfrac{1}{N_p}$

$$\boxed{\frac{\delta\omega}{\omega_0} = \pm\frac{1}{N_p}}$$

Fundamental Wavelength

$$\lambda = \frac{\lambda_p}{\gamma^2}\frac{\cos\theta}{1+\cos\theta^*} = \frac{\lambda_p}{\gamma^2}\frac{1}{1+\frac{1-\gamma^2\theta^2}{1+\gamma^2\theta^2}} = \frac{\lambda_p}{2\gamma^2}\left(1+\gamma^2\theta^2\right)$$

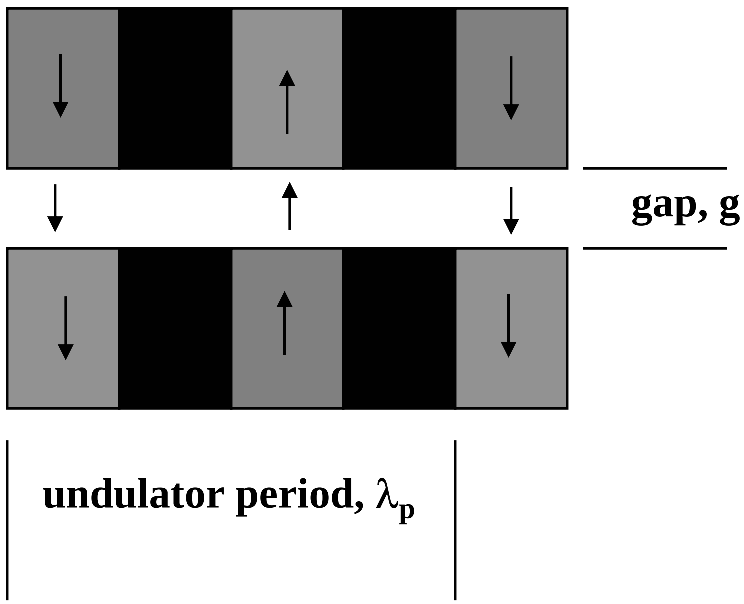
$$\theta^2 = (\theta_{und}+\theta_{obs})^2 = \theta_{und}^2 + 2\theta_{und}\theta_{obs} + \theta_{obs}^2$$

$$\theta_{und} = \frac{K}{\gamma}\cos k_p z \qquad \langle\theta_{und}\rangle = 0 \quad \text{and} \quad \left\langle\theta_{und}^2\right\rangle = \frac{1}{2}\frac{K^2}{\gamma^2}$$

fundamental undulator wavelength:

$$\boxed{\lambda = \frac{\lambda_p}{2\gamma^2}\left(1+\frac{1}{2}K^2+\gamma^2\theta_{obs}^2\right)}$$

permanent magnet undulator



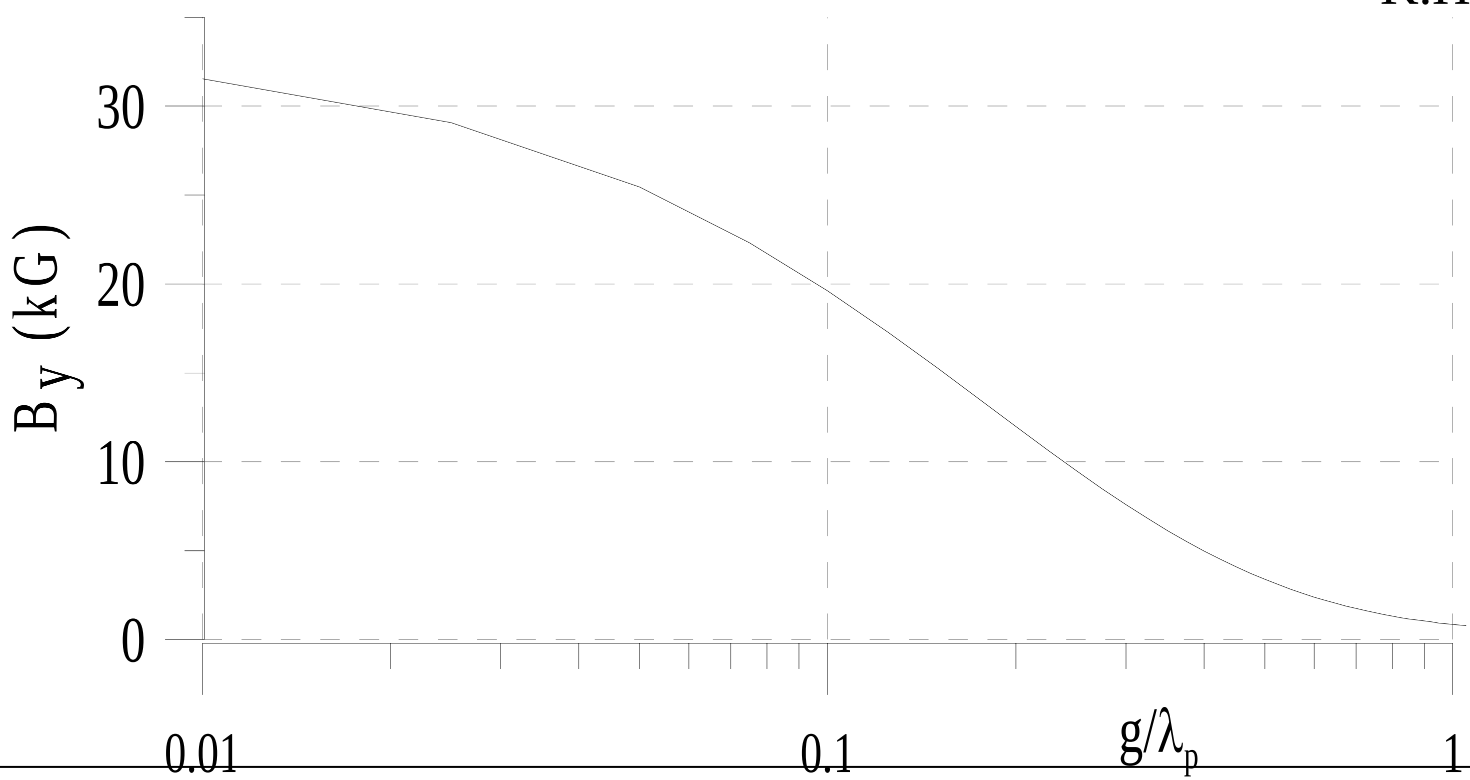
$$\lambda = \frac{\lambda_p}{2\gamma^2}\left(1 + \frac{1}{2}K^2 + \gamma^2\theta_{obs}^2\right)$$

permanent wiggler magnet

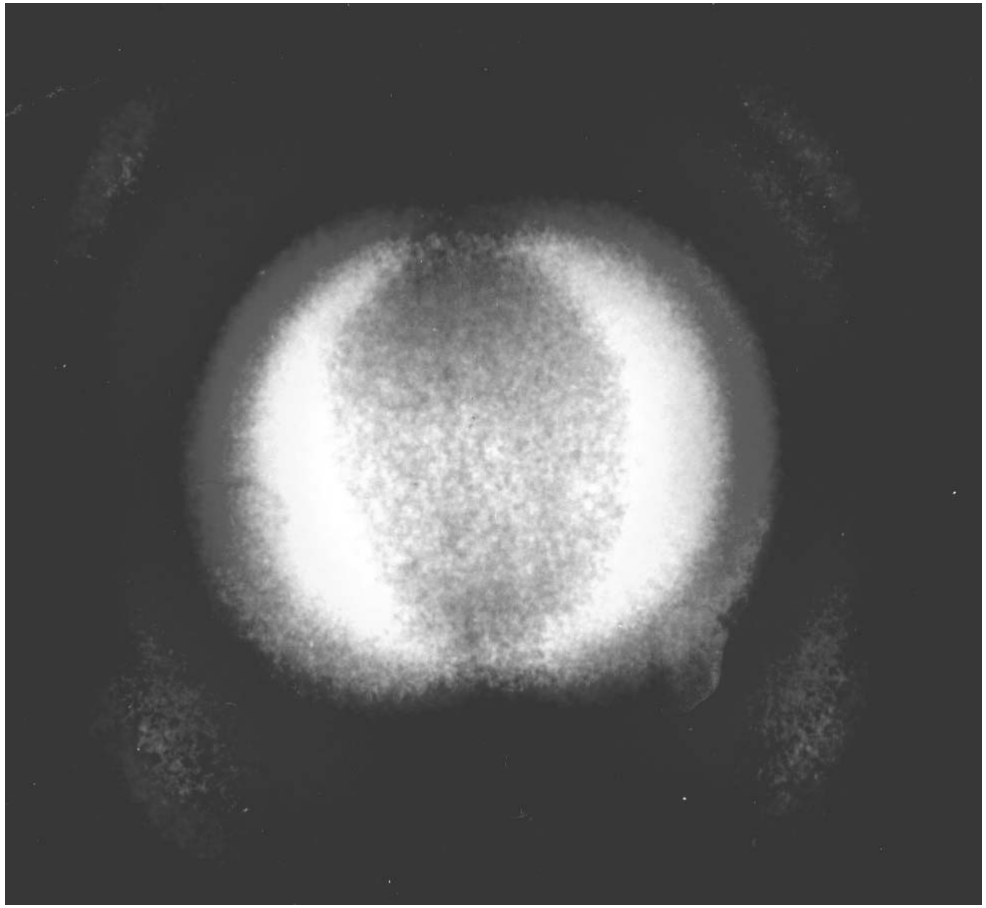
vary field strength by varying gap. For hybrid undulator:

$$B(\mathrm{T}) = 3.3 \exp\left[-\frac{g}{\lambda_p}\left(5.74 - 1.8\frac{g}{\lambda_p}\right)\right]$$

K.Halbach

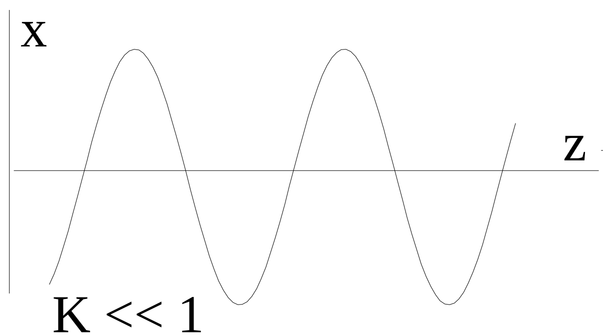


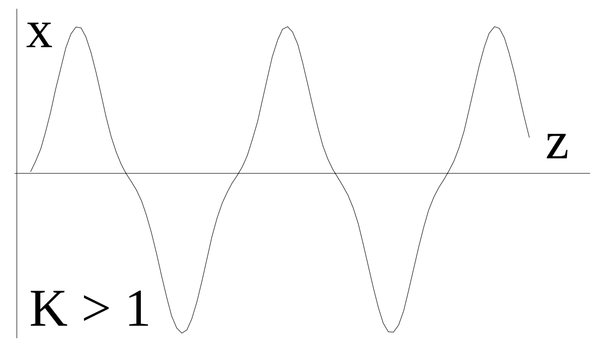
ACO



$$\lambda = \frac{\lambda_p}{2\gamma^2}\left(1 + \frac{1}{2}K^2 + \gamma^2\theta_{obs}^2\right)$$

Stronger undulator field





transverse motion completely non-relativistic

relativistic effect on transverse motion

source of higher harmonics

fundamental wavelength

$$\lambda_i = \frac{\lambda_p}{2i\gamma^2}\left[1 + \frac{1}{2}K^2 + \gamma^2(\theta^2 + \psi^2)\right]$$

i: harmonic number, $i=1,3,5,7\ldots$

$$\lambda_i(\text{Å}) = 1305.6\frac{\lambda_p}{iE^2}\left(1 + \frac{1}{2}K^2\right)$$

$$\varepsilon_i(\text{eV}) = 9.4963 \bullet \frac{iE^2}{\lambda_p\left(1+\frac{1}{2}K^2\right)}$$

$$\sigma_\theta = \frac{1}{\gamma}\sqrt{\frac{1+\frac{1}{2}K^2}{2iN_p}}$$

$$B_y(z) = B_0 \cos k_p z$$ this is what we want

Maxwell tells us what we can get! $B_y(y,z) = B_0\, b(y) \cos k_p z$

$$\nabla \times \mathbf{B} = \mathbf{0} \quad \frac{\partial B_z}{\partial y} = \frac{\partial B_y}{\partial z} = -B_0\, b(y)\, k_p \sin k_p z$$

and $B_y = -B_0\, b(y)\,(1 - \cos k_p z)$

$$\nabla \cdot \mathbf{B} = \mathbf{0} \qquad \mathbf{B} \neq \mathbf{B}(\mathbf{x}) \quad \frac{\partial B_z}{\partial z} = -B_0 \frac{\partial b(y)}{\partial y} \cos k_p z$$

form $\dfrac{\partial^2 B_z}{\partial y \partial z} \Rightarrow \dfrac{\partial^2 b(y)}{\partial^2 y} = k_p^2\, b(y)$ $\quad b(y) = a_1 \cosh k_p y + a_2 \sinh k_p y$

$$\begin{aligned} B_x &= 0 \\ B_y &= B_0 \cosh k_p y \cos k_p z \\ B_z &= -B_0 \sinh k_p y \sin k_p z \end{aligned}$$

trajectory

beam dynamics

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{\mathbf{n}}{\rho} = \frac{e}{mc^2\gamma}\left[\frac{\mathbf{v}}{v}\times\mathbf{B}\right]$$

$$\frac{d^2x}{dt^2} = -\frac{eB_0}{mc\gamma}\frac{dz}{dt}\cos k_p z$$

$$\frac{d^2z}{dt^2} = +\frac{eB_0}{mc\gamma}\frac{dx}{dt}\cos k_p z$$

$$\frac{dx}{dt} = -c\beta\frac{K}{\gamma}\sin k_p z$$

$$\frac{dz}{dt} = +c\beta\left(1 - \frac{K^2}{2\gamma^2}\sin^2 k_p z\right)$$

drift velocity

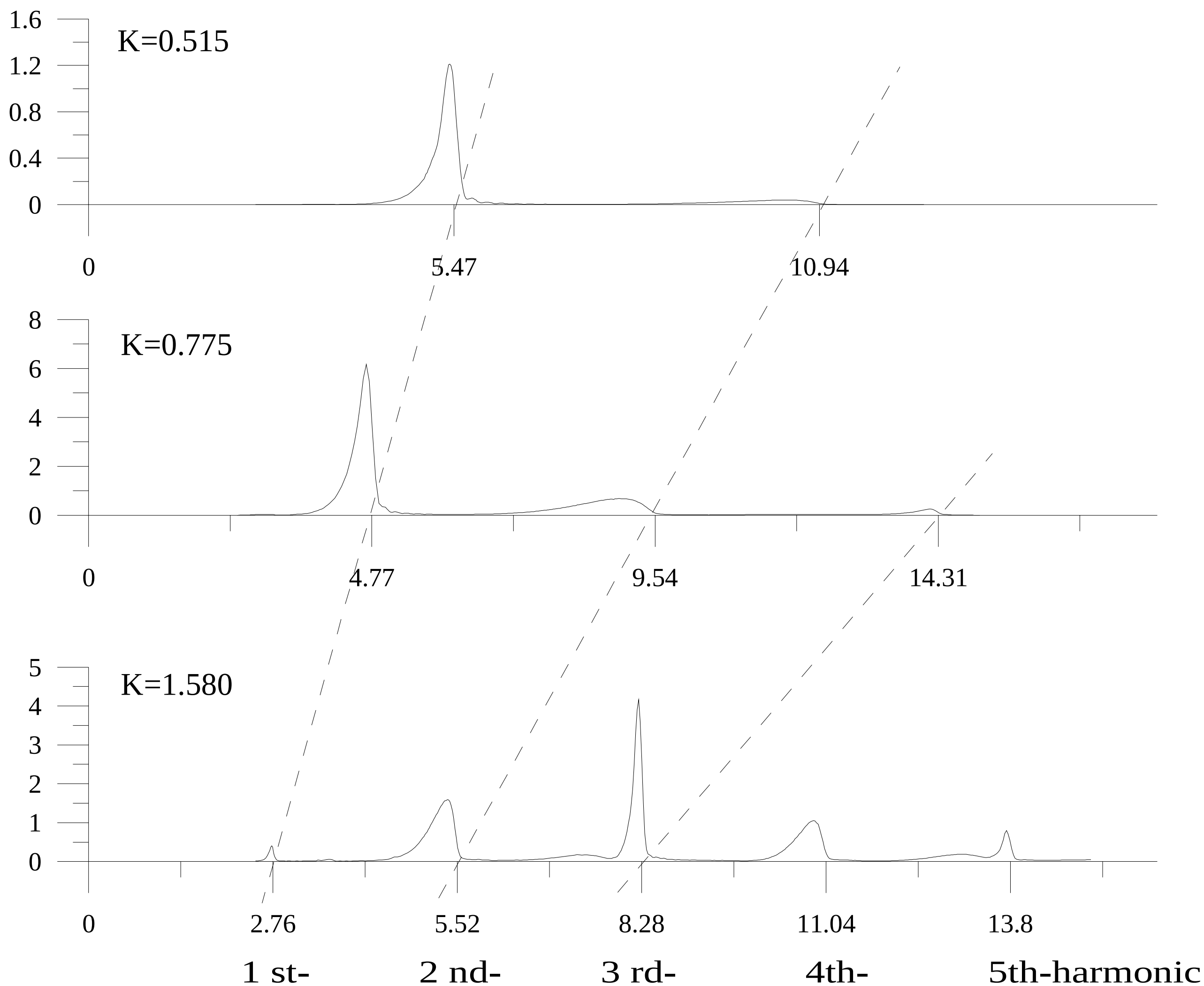
$$\bar{\beta} = \beta\left(1 - \frac{K^2}{4\gamma^2}\right)$$

$$x(t) = a\cos(k_p c\bar{\beta}t)$$

$$z(t) = c\bar{\beta}t + \tfrac{1}{8}k_p a^2\sin(2k_p c\bar{\beta}t)$$

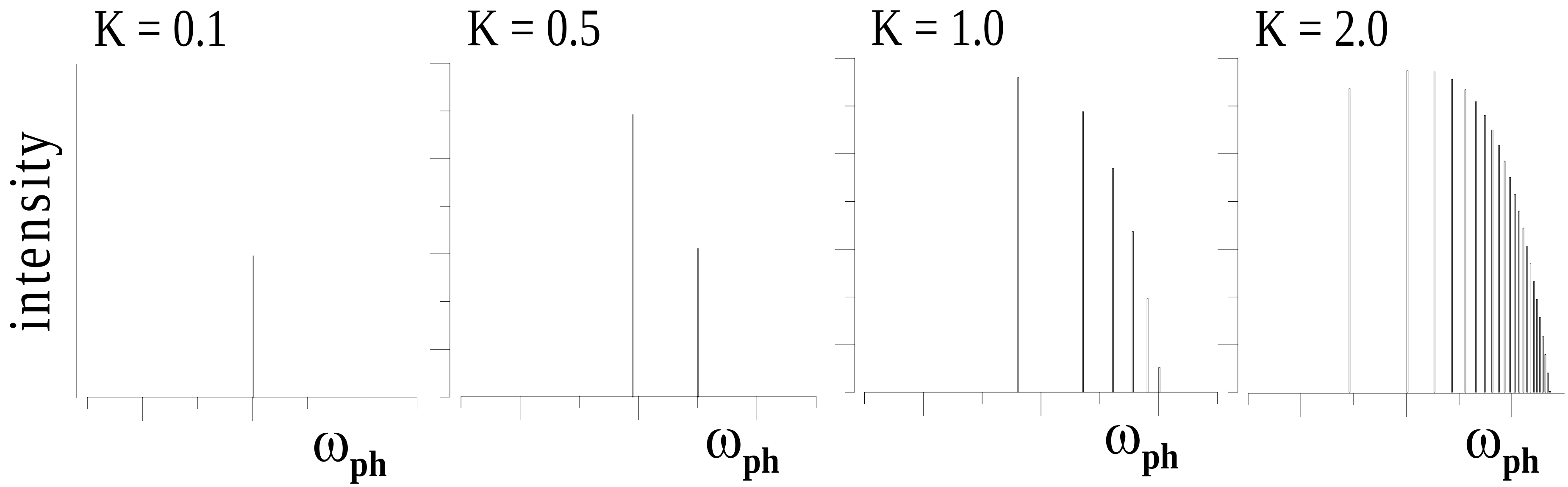
$$a = \frac{K}{\gamma k_p}$$

PEP-Undulator, 77 mm, 27 periods, 7.1 GeV



Undulator-wiggler

transition from undulator to wiggler radiation



critical photon energy from wiggler magnet at angle y with axis

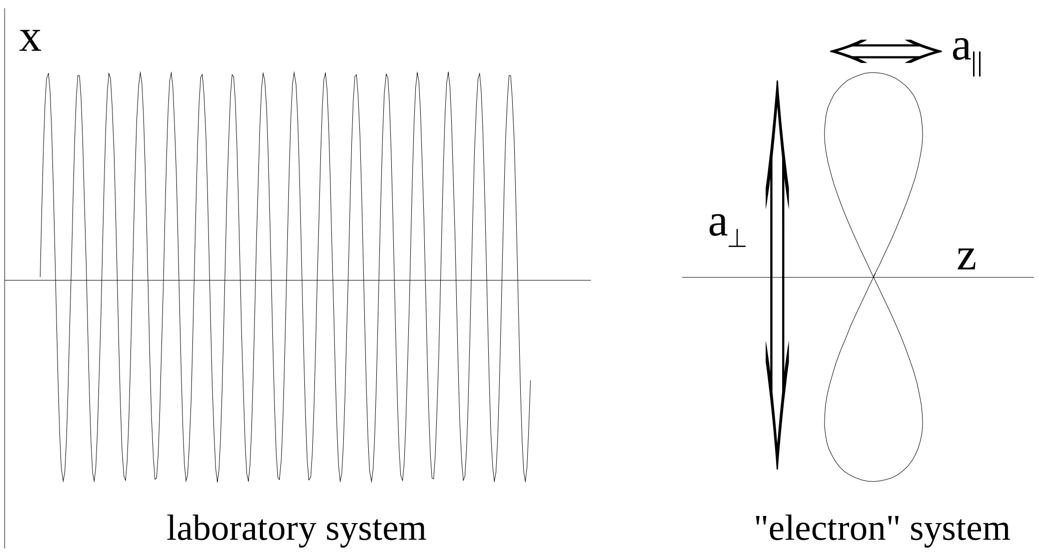
$$\varepsilon_c(\psi) = \varepsilon_c(0)\sqrt{1-\left(\frac{\gamma\psi}{K}\right)^2}$$

homework !

increase undulator strength K

$$x(t) = a\cos(k_p c\bar{\beta} t)$$

$$z(t) = c\bar{\beta} t + \tfrac{1}{8} k_p a^2 \sin(2k_p c\bar{\beta} t)$$



longitudinal oscillation generates even harmonics: $i = 2,4,6,8\ldots$

energy loss per undulator/wiggler pass

$$\Delta E_{\mathrm{rad}} = \frac{1}{3} r_{\mathrm{c}} mc^2 \gamma^2 K^2 k_{\mathrm{p}}^2 L_{\mathrm{u}}$$

$$\Delta E_{\mathrm{rad}}(\mathrm{eV}) = 0.07257 \frac{\mathrm{E}^2 \mathrm{K}^2}{\lambda_{\mathrm{p}}^2} L_{\mathrm{u}}$$

tot. radiation power

$$P(\mathrm{W}) = 0.07257 \frac{\mathrm{E}^2 \mathrm{K}^2 \mathrm{N} \mathrm{I}}{\lambda_{\mathrm{p}}}$$

undulator photon flux

$$\frac{d\dot{N}_{ph}(\omega)}{d\Omega} = \alpha\gamma^2 N_p^2 \frac{\Delta\omega}{\omega}\frac{I}{e} \times \sum_{i=1}^{\infty} i^2\,\mathrm{Sinc}\,(F_\sigma^2 + F_\pi^2)$$

$$\mathrm{Sinc} = \left(\frac{\sin\pi N_p \Delta\omega_i/\omega_1}{\pi N_p \Delta\omega_i/\omega_1}\right)^2$$

$$F_\sigma = \frac{2\gamma\theta\Sigma_1\cos\varphi - K\Sigma_2}{1+\frac{1}{2}K^2+\gamma^2\theta^2} \qquad F_\pi = \frac{2\gamma\theta\Sigma_1\sin\varphi}{1+\frac{1}{2}K^2+\gamma^2\theta^2}$$

$$\Sigma_{1,i} = \sum_{m=-\infty}^{\infty} J_{-m}(u) J_{i-2m}(v)$$

$$\Sigma_{2,i} = \sum_{m=-\infty}^{\infty} J_{-m}(u)\left[J_{i-2m-1}(v) + J_{i-2m+1}(v)\right]$$

$$u = \frac{\omega}{\omega_1}\frac{\bar{\beta}K^2}{4(1+\frac{1}{2}K^2+\gamma^2\theta^2)} \qquad v = \frac{\omega}{\omega_1}\frac{2\bar{\beta}K^2\gamma\theta\cos\varphi}{1+\frac{1}{2}K^2+\gamma^2\theta^2}$$

pin hole radiation

$$\left.\frac{d\dot{N}_{ph}(\omega)}{d\Omega}\right|_i = \alpha\gamma^2 N_p^2 \frac{\Delta\omega}{\omega}\frac{I}{e}\frac{i^2 K^2 [JJ]^2}{\left(1+\frac{1}{2}K^2\right)^2}$$

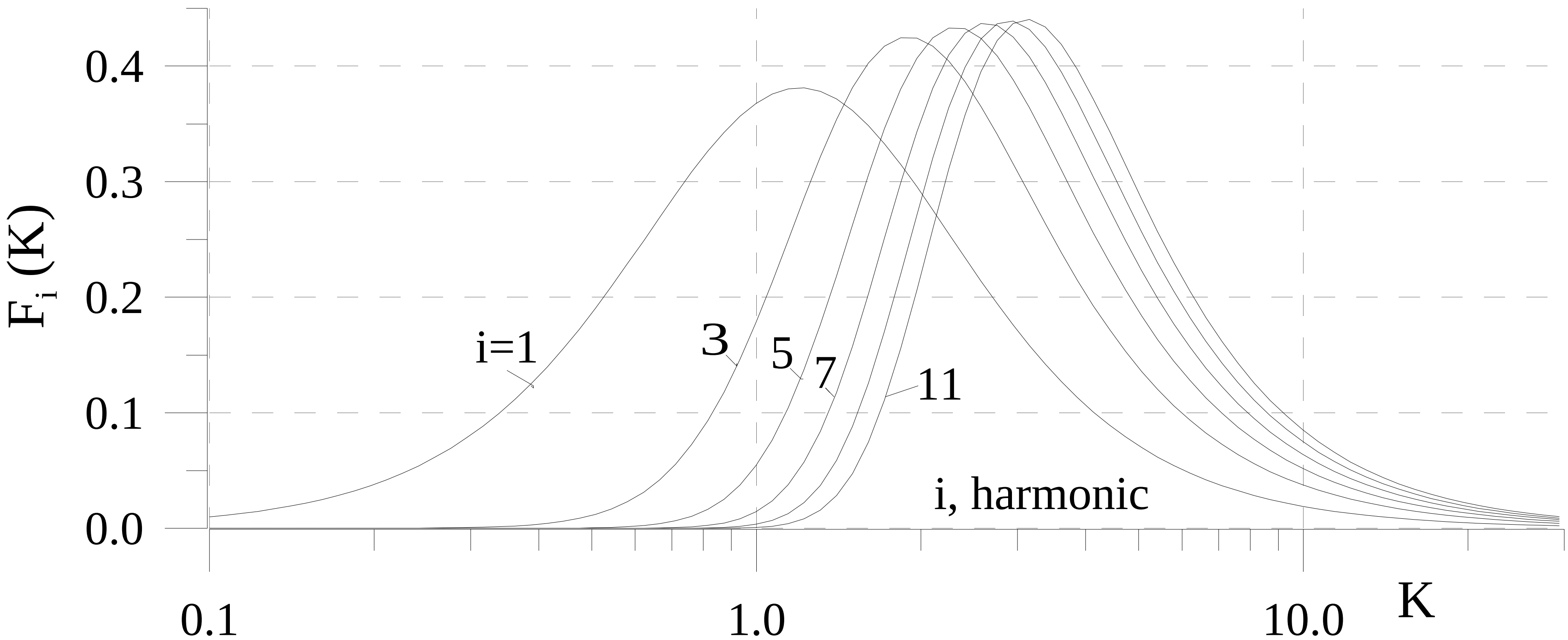
$$= 1.7466 \cdot 10^{23} E^2(\mathrm{GeV}^2) I(\mathrm{A}) N_p^2 \frac{\Delta\omega}{\omega} f_i(K),$$

$$f_i(K) = \frac{i^2 K^2 [JJ]^2}{\left(1+\frac{1}{2}K^2\right)^2}$$

$$[JJ] = \left[J_{\frac{i-1}{2}}(x) - J_{\frac{i+1}{2}}(x)\right]$$

$$x = \frac{iK^2}{4+2K^2}$$

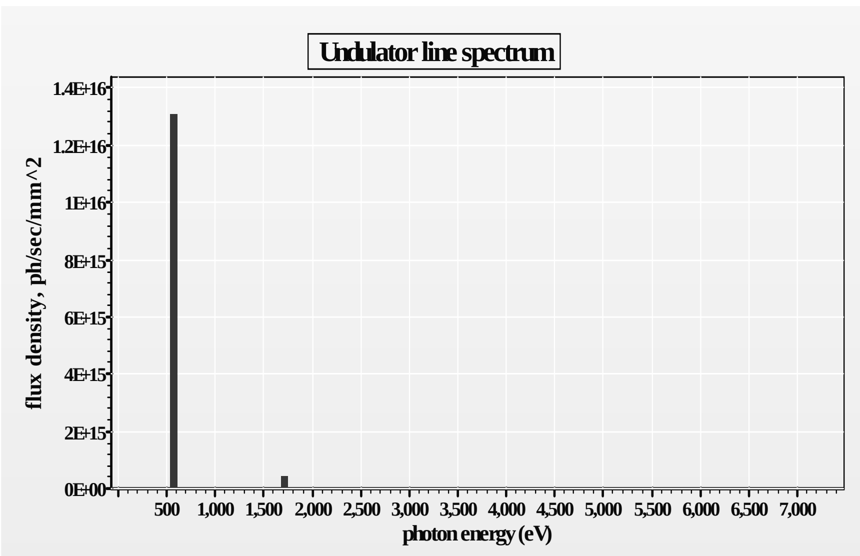
F_{ik}

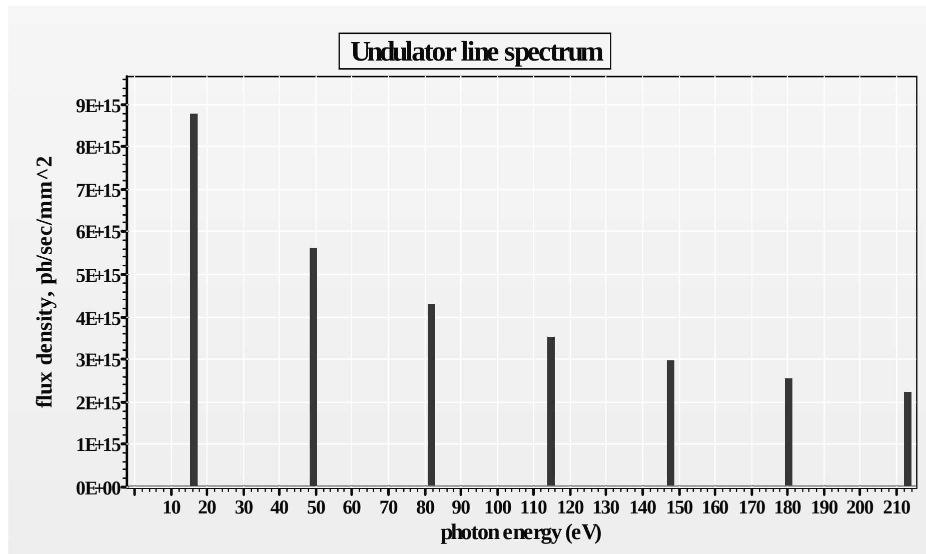


Tuning range

tuning range: $$\lambda = \frac{\lambda_p}{2\gamma^2}\left(1 + \frac{1}{2}K^2 + \gamma^2\theta_{obs}^2\right)$$

compare two undulators in SIAM: λ_p= 20 mm and = 50 mm





λ_p= 20 mm, 100 periods
0.32 < K < 0.63

λ_p= 50 mm, 40 periods
1.1 < K < 5.6

$$\lambda = \frac{\lambda_p}{2\gamma^2}\left(1 + \frac{1}{2}K^2 + \gamma^2\theta_{obs}^2\right)$$

make λ_p very short $\Rightarrow$ to get x-rays !?

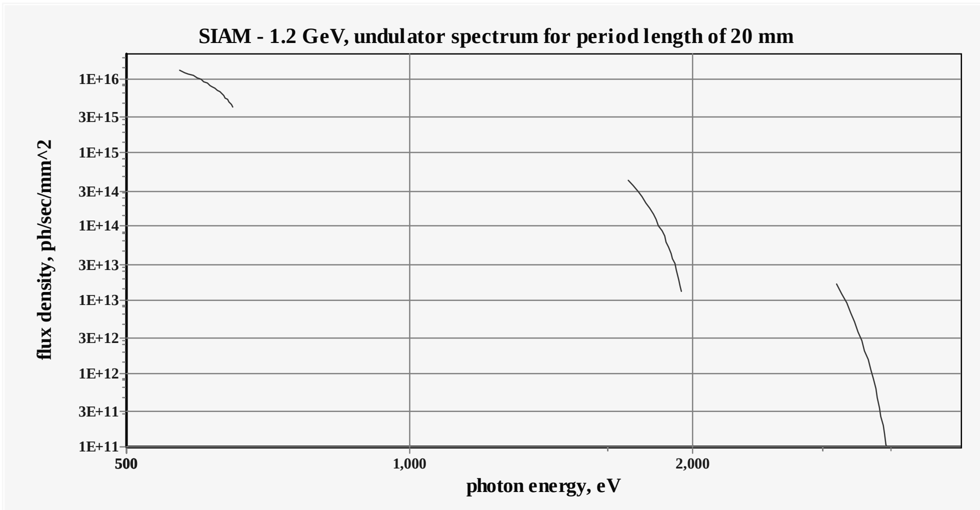
does not work well !

$$K = 0.934\,B(\mathrm{T})\,\lambda_p(\mathrm{cm})$$

short λ_p leads generally to small value of K !

intensity is low
tuning range is narrow

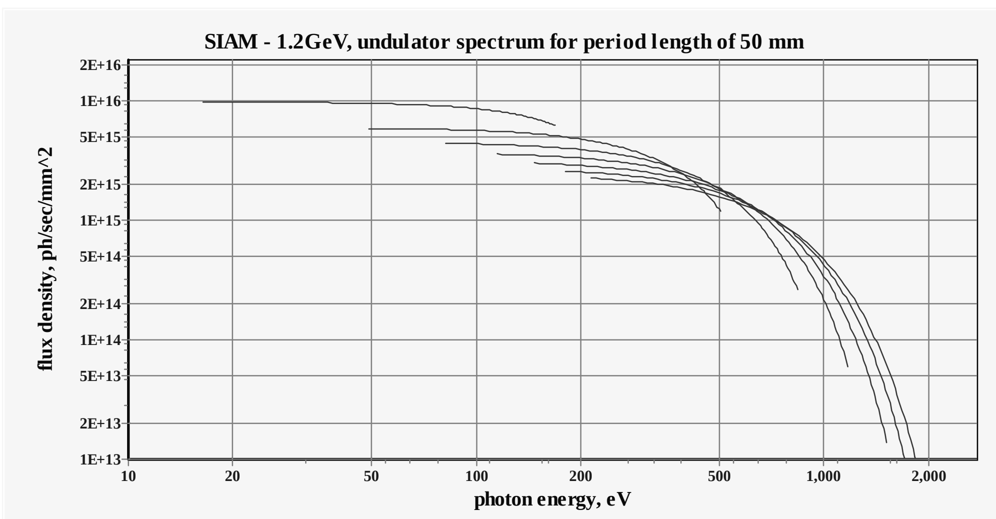
Spectral-tuning



λ_p= 20 mm, 100 periods

0.33 < K < 0.63

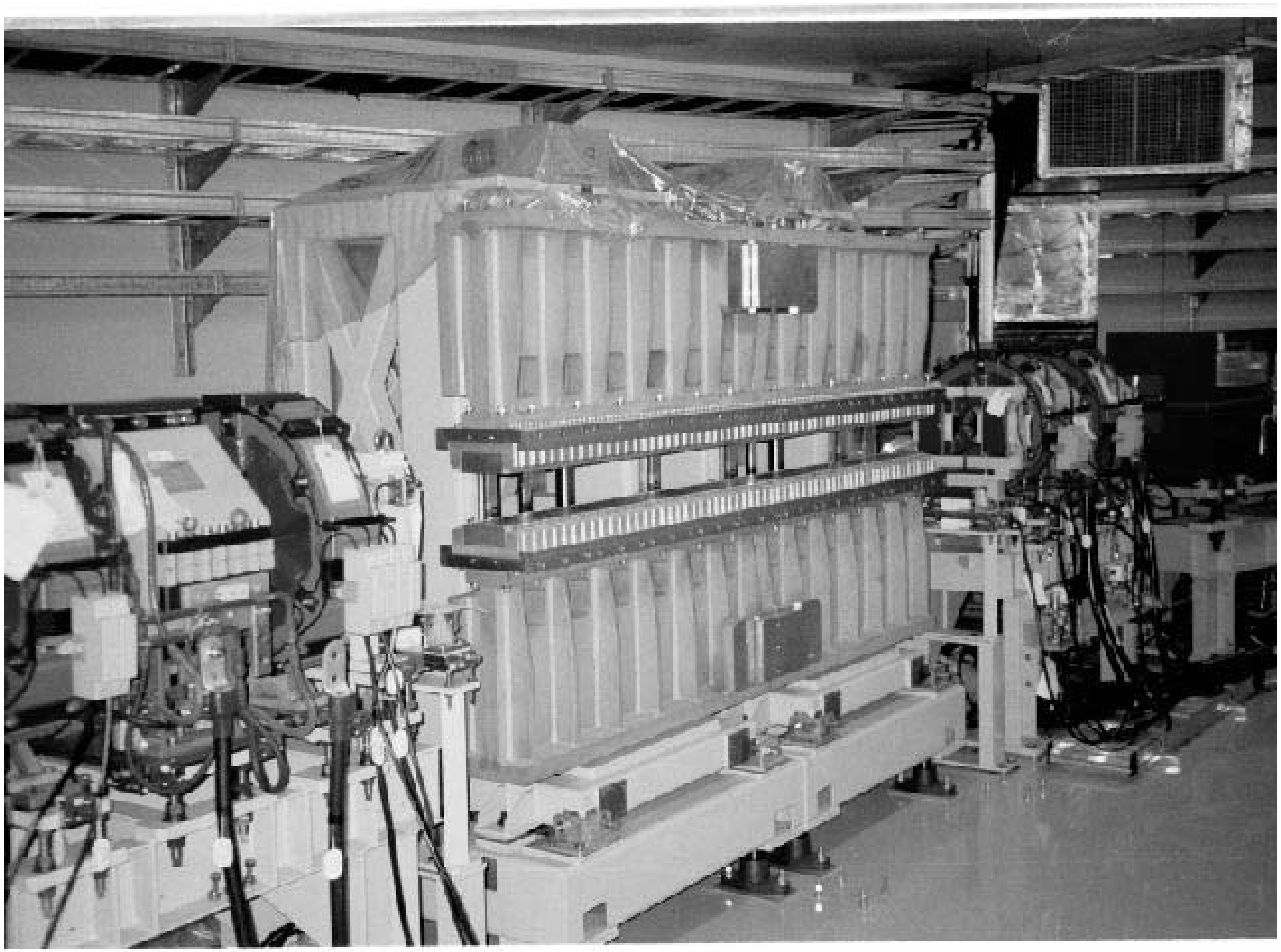
10 mm < gap < 30 mm



λ_p= 50 mm, 40 periods

1.1 < K < 5.6

10 mm < gap < 30 mm



SUBARU : 2.3 m undulator, λ_p = 7.6 cm, 30 periods

Subaru-Und

Subaru_10m

SUBARU : 10.8 m undulator, λ_p = 5.4 cm, 200 periods to electron: undulator field looks like EM-wave

so does a laser field !

how about colliding an electron beam with a laser beam ?

Laser backscattering

$$\lambda_\gamma = \frac{\lambda_p}{4\gamma^2}\left(1 + \frac{1}{2}\gamma^2\vartheta^2\right)$$

K is very small !

Photon flux

$$N_{sc} = \sigma_{Th}\,\mathcal{L}$$

Scattering cross section

$$\sigma_{Th} = \frac{8\pi}{3} r^2 = 6.65 \times 10^{-25}\,\mathrm{cm}^2$$

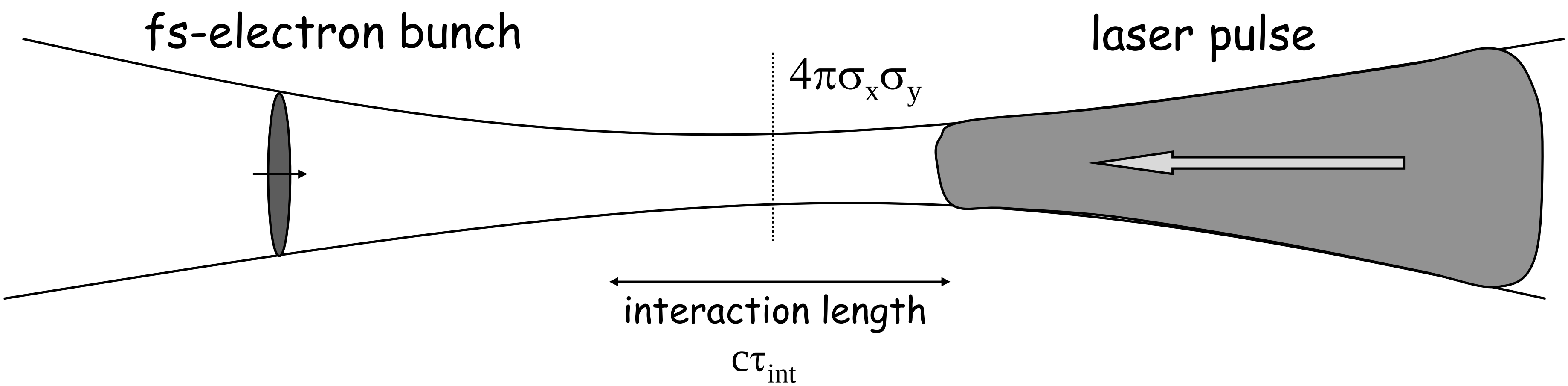
Luminosity

$$\mathcal{L} = \frac{N_e \dot{N}_{ph}}{4\pi\,\sigma_x \sigma_y}$$

Effective laser photon flux

$$N_{ph} = \frac{P_{Laser}}{\epsilon_{ph}}\,c\tau_{int}$$

Thompson scattering



example:

$$\left.\begin{array}{l} P_{laser} = 100\ \text{kW} \\ \varepsilon_{ph} = 1.24\ \text{eV} \\ c\tau = 1\ \text{m} \\ N_e = 10^9 \\ 4\pi\sigma_x\sigma_y = 10^{-2}\ \text{cm}^2 \end{array}\right\} \quad N_{sc} = 3.4\ 10^{10}\ \text{photons/fs-pulse}$$

Reaction Rate

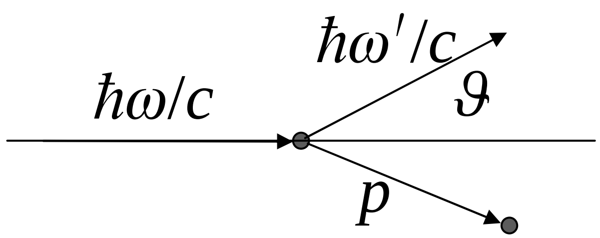
Undulator radiation as Compton effect – 1

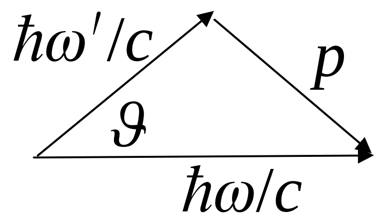
for electron undulator field looks just like EM field or photon

we treat the collision of an electron with a photon in the electron rest system

energy conservation: $$\hbar\omega + mc^2 = \hbar\omega' + \sqrt{(cp)^2 + (mc^2)}$$

momentum conservation:





$$p^2 = \left(\frac{\hbar\omega}{c}\right)^2 + \left(\frac{\hbar\omega'}{c}\right)^2 - 2\frac{\hbar^2}{c^2}\omega\omega'\cos\vartheta$$

or $$p^2 = \frac{\hbar^2}{c^2}\left[(\omega-\omega')^2 + 2\omega\omega'(1-\cos\vartheta)\right]$$

Undulator radiation as Compton effect – 2

combine with energy conservation and eliminate electron momentum

$$\frac{\hbar}{mc^2}(1-\cos\vartheta) = \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{2\pi c}(\lambda' - \lambda) \approx 0$$

with $\frac{2\pi\hbar c}{mc^2} = 2.4 \cdot 10^{-12}\mathrm{m}$

in electron system no change in photon energy !

undulator period in electron system: $\lambda = \frac{\lambda_\mathrm{p}}{\gamma}$

observed radiation from moving source:

$$\lambda_\mathrm{lab} = \frac{\lambda_\mathrm{p}}{2\gamma^2}\left(1 + \tfrac{1}{2}K^2\right)$$

Photon Beam Brightness

Brightness is:

photon density in 6-dimensional phase space

$$\mathcal{B} = \frac{\dot{\mathcal{N}}}{4\pi \sigma_x \sigma_{x'} \sigma_y \sigma_{y'} \frac{\Delta\omega}{\omega}}$$

diffraction limited brightness $\sigma_{x,y}\sigma_{x',y'} = \frac{1}{2}\sigma_r \sigma_{r'} = \frac{\lambda}{4\pi}$

$$\mathcal{B}_{\mathrm{diff}} = \frac{4\dot{\mathcal{N}}}{\lambda^2 \frac{\Delta\omega}{\omega}}$$

we have not only one electron, but many

finite beam size $\sigma_{e,x,y}$

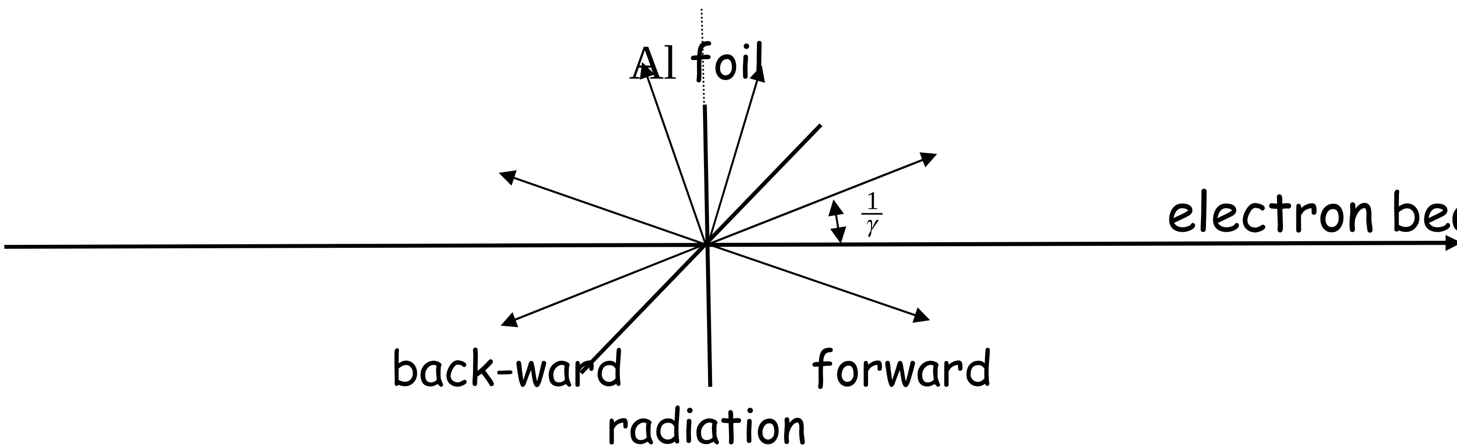
finite beam divergence $\sigma_{e,x',y'}$

$$\boxed{\mathcal{B} = \frac{\dot{\mathcal{N}}}{4\pi\sigma_{t,x}\sigma_{t,x'}\sigma_{t,y}\sigma_{t,y'}\frac{\Delta\omega}{\omega}}}$$

$$\sigma_{t,x} = \sqrt{\tfrac{1}{2}\sigma_r^2 + \sigma_{e,x}^2}\,,\ \text{etc.}$$

Transition Radiation

generate Transition Radiation

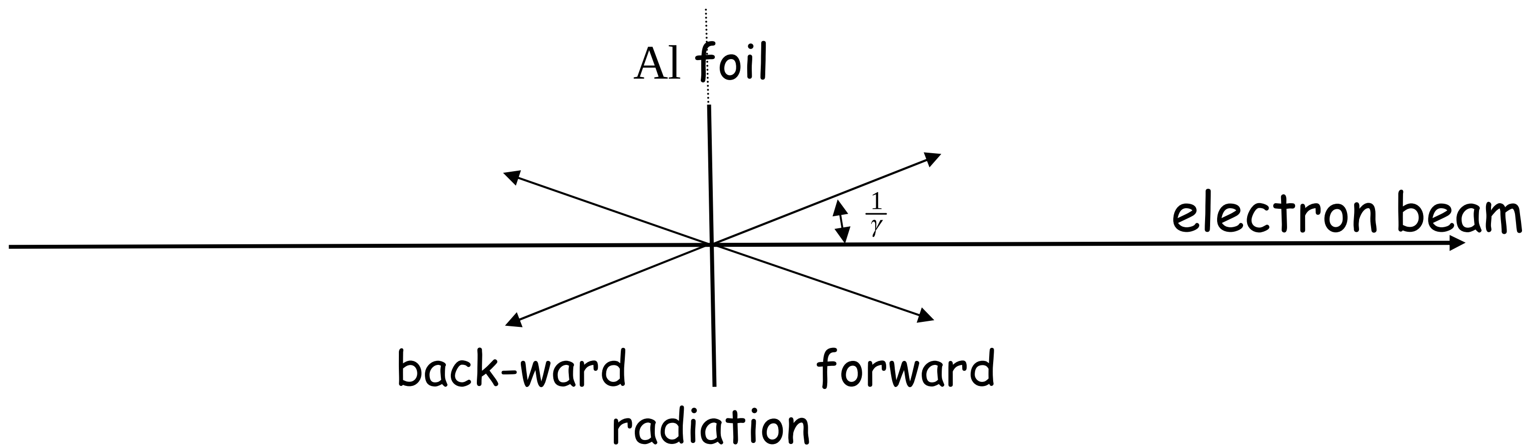


tilt radiator by 45^0

inconvenient to separate electron beam from TR

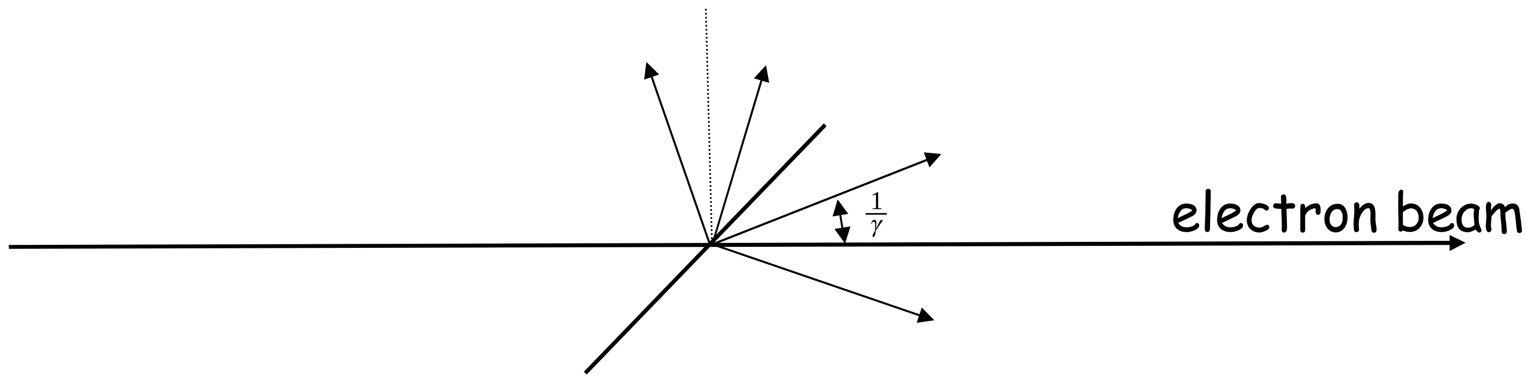
now, TR can be extracted normal to electron beam through window

TR



inconvenient to separate electron beam from TR

90 degree TR



tilt radiator by 45^0

now, TR can be extracted normal to electron beam through window

TR theory - 1

Poynting vector $\quad \mathbf{S}_{\mathrm{r}} = \frac{c}{4\pi}\left[\frac{4\pi}{\mu_0 c}\right] B_{\mathrm{ret}}^2 \mathbf{n}$

radiation power $\quad \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \mathbf{S}_{\mathrm{r}} \mathbf{n} R_{\mathrm{ret}}^2 \Delta\Omega = \frac{c}{4\pi}\left[\frac{4\pi}{\mu_0 c}\right] R_{\mathrm{ret}}^2 B_{\mathrm{ret}}^2 \Delta\Omega$

need some tools:

Fourier transforms

$$B(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega$$

$$B(\omega) = \int_{-\infty}^{\infty} B(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t$$

Parceval's theorem

$$\int_{-\infty}^{\infty} B^2(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} B^2(\omega)\,\mathrm{d}\omega$$

TR theory - 2

$$\frac{d\varepsilon}{dt} = \mathbf{S}_{\mathrm{r}}\mathbf{n}R_{\mathrm{ret}}^2\,\Delta\Omega = \frac{c}{4\pi}\left[\frac{4\pi}{\mu_0 c}\right]R_{\mathrm{ret}}^2 B_{\mathrm{ret}}^2\Delta\Omega$$

$$d\varepsilon(t) = \frac{c}{4\pi}\left[\frac{4\pi}{\mu_0 c}\right]R_{\mathrm{ret}}^2\Delta\Omega\, B_{\mathrm{ret}}^2(t)\,dt$$

$$d\varepsilon(\omega) = \frac{c}{4\pi}\left[\frac{4\pi}{\mu_0 c}\right]R_{\mathrm{ret}}^2\Delta\Omega\,\frac{1}{2\pi}B_{\mathrm{ret}}^2(\omega)2\,d\omega$$

use only $\omega>0$!

$$B(\omega)\ ?$$

spectrum: $0 < \omega < \omega_{\mathrm{plasma}}$

our interest is in $\omega \ll \omega_{\mathrm{plasma}}$

$B(t) \neq 0$ during time τ only

$$e^{i\omega t} \approx 1$$

$$\boxed{B_{\mathrm{ret}}(\omega) = \int_{-\infty}^{\infty} B_{\mathrm{ret}}(t)\,e^{i\omega t}\,dt \approx \int_{-\tau/2}^{\tau/2} B_{\mathrm{ret}}(t)\,dt}$$

for $\omega \ll \omega_{\mathrm{plasma}}$

TR theory – 3

$$B_{\text{ret}}(t)\ ?\qquad B_{\text{ret}}(t) = \nabla\times\mathbf{A}_{\text{ret}} = \left\{\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z};\ldots\right\}_{t_{\text{r}}=t-R/c}$$

$$\frac{\partial A_z}{\partial y} = \frac{\partial A_z}{\partial t_{\text{r}}}\frac{\partial t_{\text{r}}}{\partial y};\ldots \qquad R^2 = \left(x-x_{\text{r}}\right)^2 + \left(y-y_{\text{r}}\right)^2 + \left(z-z_{\text{r}}\right)^2$$

$$\frac{\partial t_{\text{r}}}{\partial y} = -\frac{1}{c}\frac{y-y_{\text{r}}}{R} = \frac{n_y}{c};\ldots$$

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{1}{c}\frac{\partial A_z}{\partial t_{\text{r}}}n_y - \frac{1}{c}\frac{\partial A_y}{\partial t_{\text{r}}}n_z$$

n is unit vector from observer to electron

$$B_{\text{ret}}(t) = \nabla\times\mathbf{A}_{\text{ret}} = \frac{1}{c}\left(\mathbf{n}\times\frac{\partial\mathbf{A}}{\partial t_{\text{r}}}\right)_{\text{r}} = \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t_{\text{r}}}\left(\mathbf{n}\times\mathbf{A}\right)_{\text{r}}$$

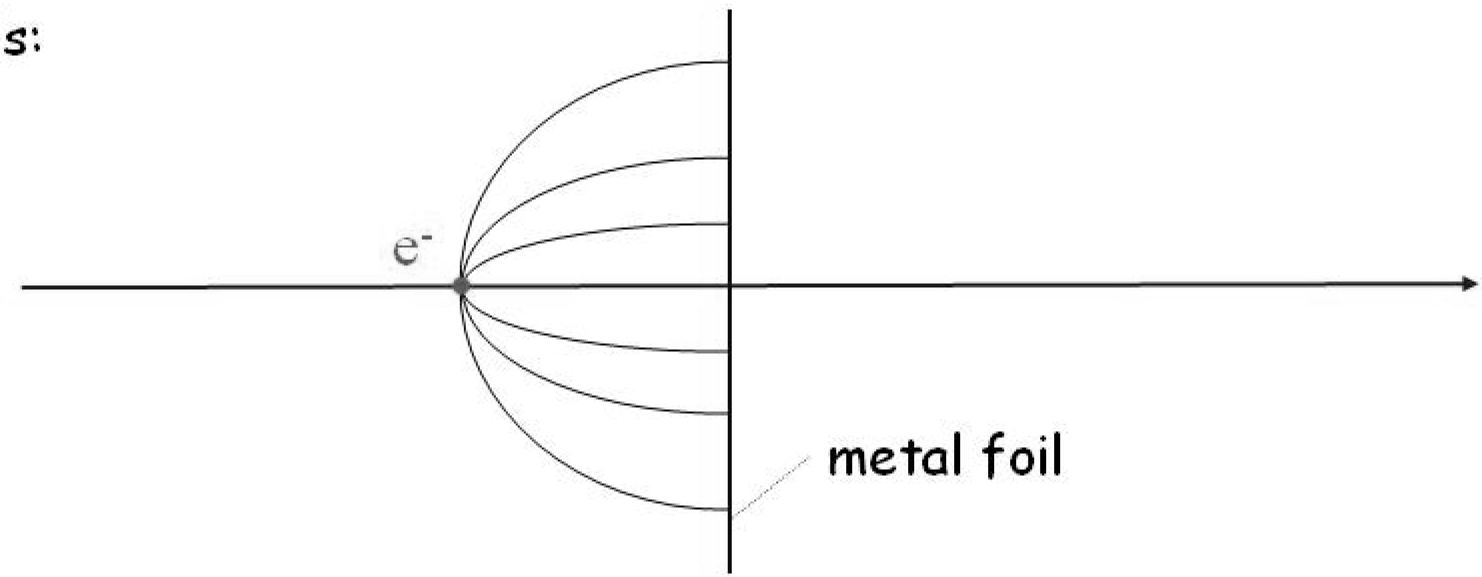
$$\boxed{B_{\text{ret}}(\omega) \approx \int_{-\tau/2}^{\tau/2} B_{\text{ret}}(t)\,\mathrm{d}t = \frac{1}{c}\left(\mathbf{n}\times\mathbf{A}\right)_{\text{r}}\Big|_{\text{initial}}^{\text{final}}}$$

TR theory – 4

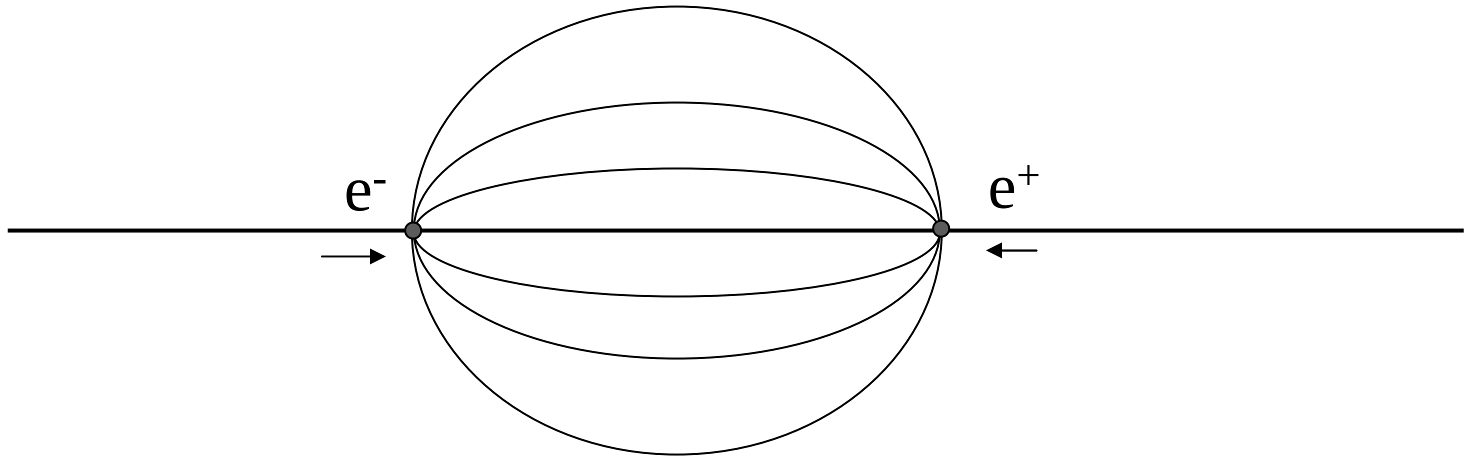
$$(\mathbf{n}\times\mathbf{A})_{\mathrm{r}}\Big|_{\mathrm{initial}}^{\mathrm{final}} \;?$$

Lienard - Wiechert Potentials $\mathbf{A} = \left.\frac{e\boldsymbol{\beta}}{R(1+\boldsymbol{\beta}\mathbf{n})}\right|_{\mathrm{ret}}$

fields:



field and metal foil can be replaced by head-on moving electron and positron



field

$$\mathbf{A} = \underbrace{\left.\frac{e\boldsymbol{\beta}}{R(1+\boldsymbol{\beta}\mathbf{n})}\right|_{\mathrm{ret}}}_{\mathrm{e}^{+}} + \underbrace{\left.\frac{(-e)(-\boldsymbol{\beta})}{R(1-\boldsymbol{\beta}\mathbf{n})}\right|_{\mathrm{ret}}}_{\mathrm{e}^{-}}$$

$$\mathrm{d}\varepsilon(\omega) = \frac{c}{4\pi}\left[\frac{4\pi}{\mu_0 c}\right] R_{\mathrm{ret}}^2 \Delta\Omega \, \frac{1}{2\pi} B_{\mathrm{ret}}^2(\omega)\,\mathrm{d}\omega$$

$$B_{\mathrm{ret}}(\omega) \approx \int_{-\tau/2}^{\tau/2} B_{\mathrm{ret}}(t)\,\mathrm{d}t = \frac{1}{c}(\mathbf{n}\times\mathbf{A})_{\mathrm{r}}\Big|_{\mathrm{initial}}^{\mathrm{final}}$$

TR theory - 6

$$\frac{d\varepsilon}{d\omega d\Omega} = \frac{\cancel{c}}{4\pi}\left[\frac{4\pi}{\mu_0 c}\right]\cancel{R}^2_{\text{ret}}\ \frac{\cancel{2}}{\cancel{2}\pi}\ \frac{e^2}{c^{\cancel{2}}}\left[\frac{\mathbf{n}\times\boldsymbol{\beta}}{\cancel{R}(1+\boldsymbol{\beta}\mathbf{n})} + \frac{\mathbf{n}\times\boldsymbol{\beta}}{\cancel{R}(1-\boldsymbol{\beta}\mathbf{n})}\right]^2_{\text{ret}}$$

with $\boldsymbol{\beta} = \beta\mathbf{z}$

$$\frac{d\varepsilon}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c}\left[\frac{4\pi}{\mu_0 c}\right]\underbrace{(\mathbf{n}\times\mathbf{z})^2}_{\sin^2\theta}\left[\frac{2\beta}{1-\beta^2\underbrace{(\mathbf{nz})^2}_{\cos^2\theta}}\right]^2_{\text{ret}}$$

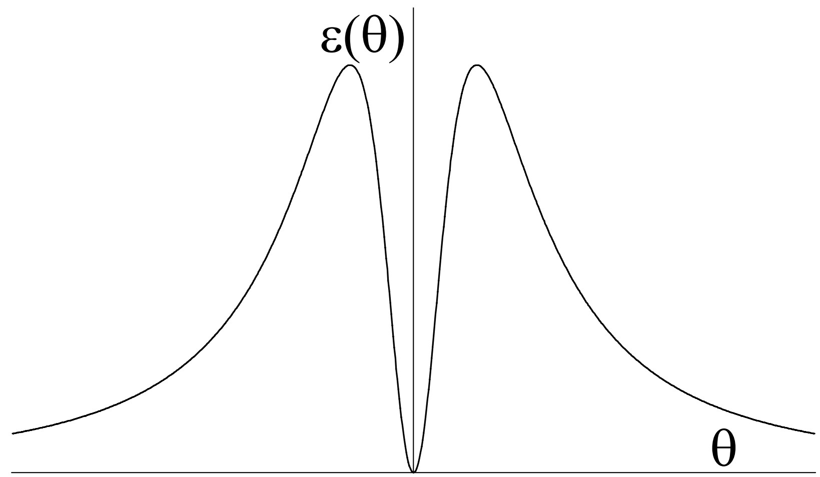
θ observation angle with respect to z

spectral and spatial radiation power distribution of transition radiation

$$\boxed{\frac{d\varepsilon}{d\omega d\Omega} = \frac{r_{\text{c}} mc^2}{\pi^2 c}\ \frac{\beta^2\sin^2\theta}{(1-\beta^2\cos^2\theta)^2}}$$

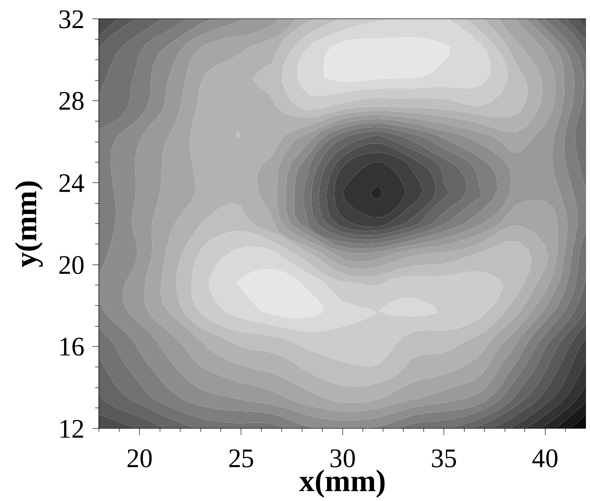
TR radiation distribution

theoretical
radiation distribution



measured
spatial distribution

C. Settakorn, 1998



TR total radiation power

with $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\Phi$

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega} = \frac{2r_{\mathrm{c}}mc^2}{\pi c}\int_0^{\pi/2}\frac{\beta^2\sin^3\theta}{\left(1-\beta^2\cos^2\theta\right)^2}\mathrm{d}\theta$$

$$= \frac{r_{\mathrm{c}}mc^2}{\pi c}\int_{-1}^{1}\frac{\beta^2(1-x^2)}{(1-\beta^2x^2)^2}\mathrm{d}x$$

$$= \frac{r_{\mathrm{c}}mc^2}{\pi c}\left(-1+\frac{(1+\beta^2)\arctan\beta}{\beta}\right)$$

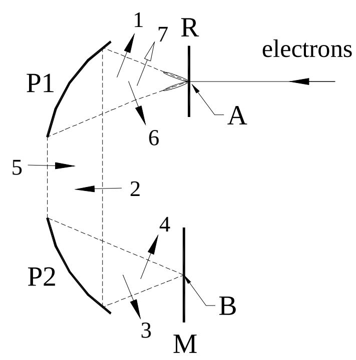
$$= \frac{r_{\mathrm{c}}mc^2}{\pi c}(-1+2\ln 2+2\ln\gamma) \approx \frac{2r_{\mathrm{c}}mc^2}{\pi c}\ln\gamma$$

spectral TR disribution

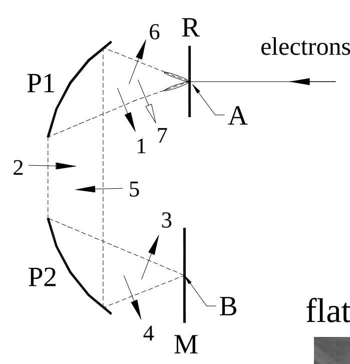
$$\boxed{\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega} = \frac{2r_{\mathrm{c}}mc^2}{\pi c}\ln\gamma}$$

for $\omega \ll \omega_{\mathrm{plasma}}$

Stimulated Transition Radiation

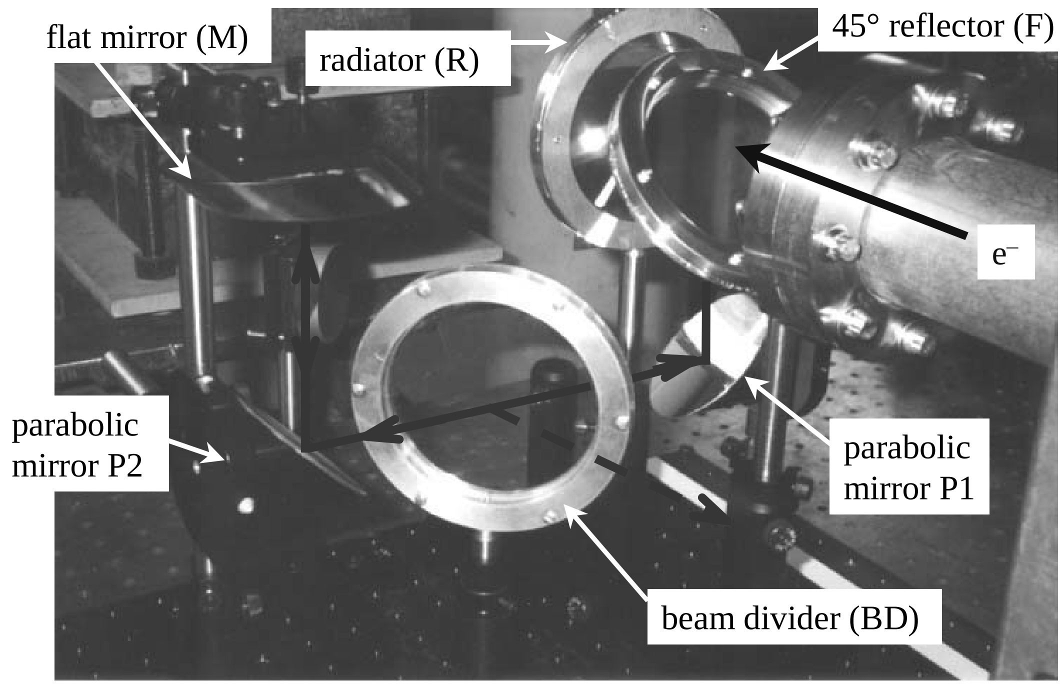


(a)

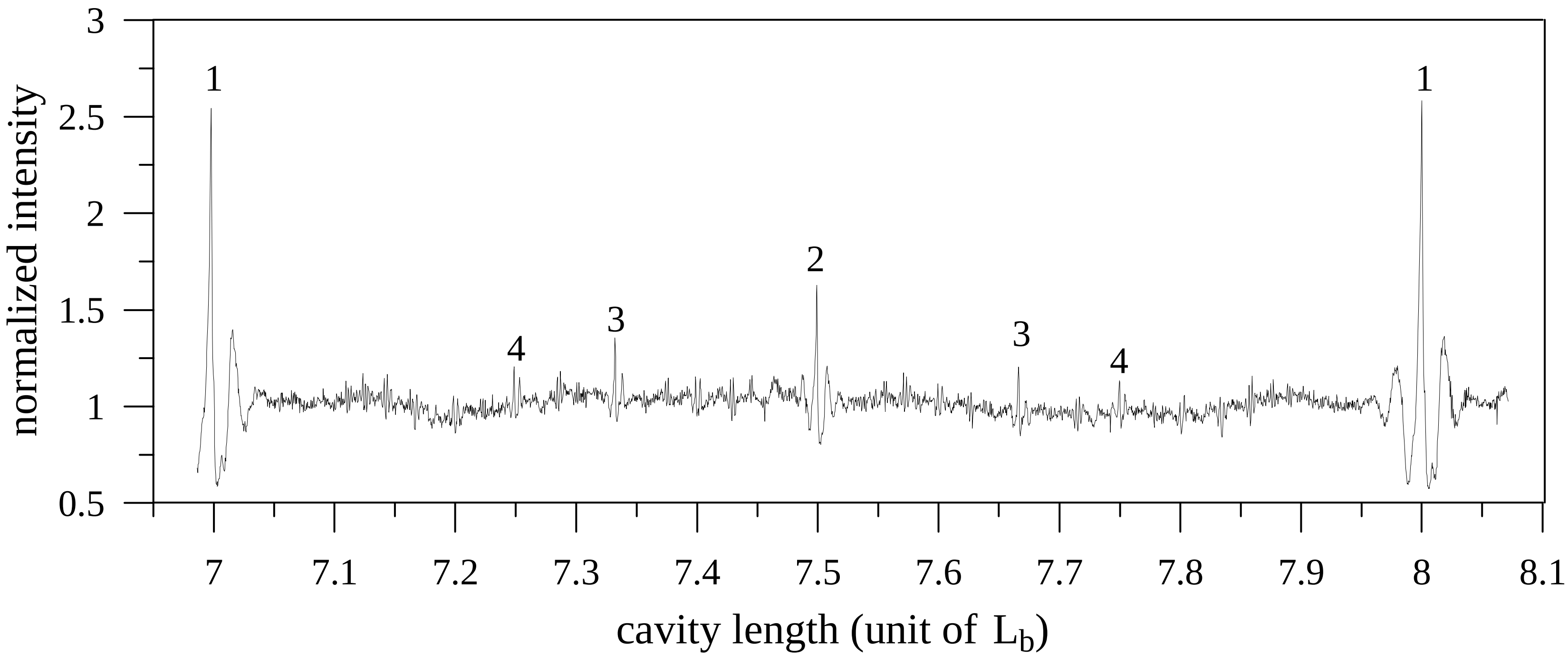


(b)

C. Settakorn

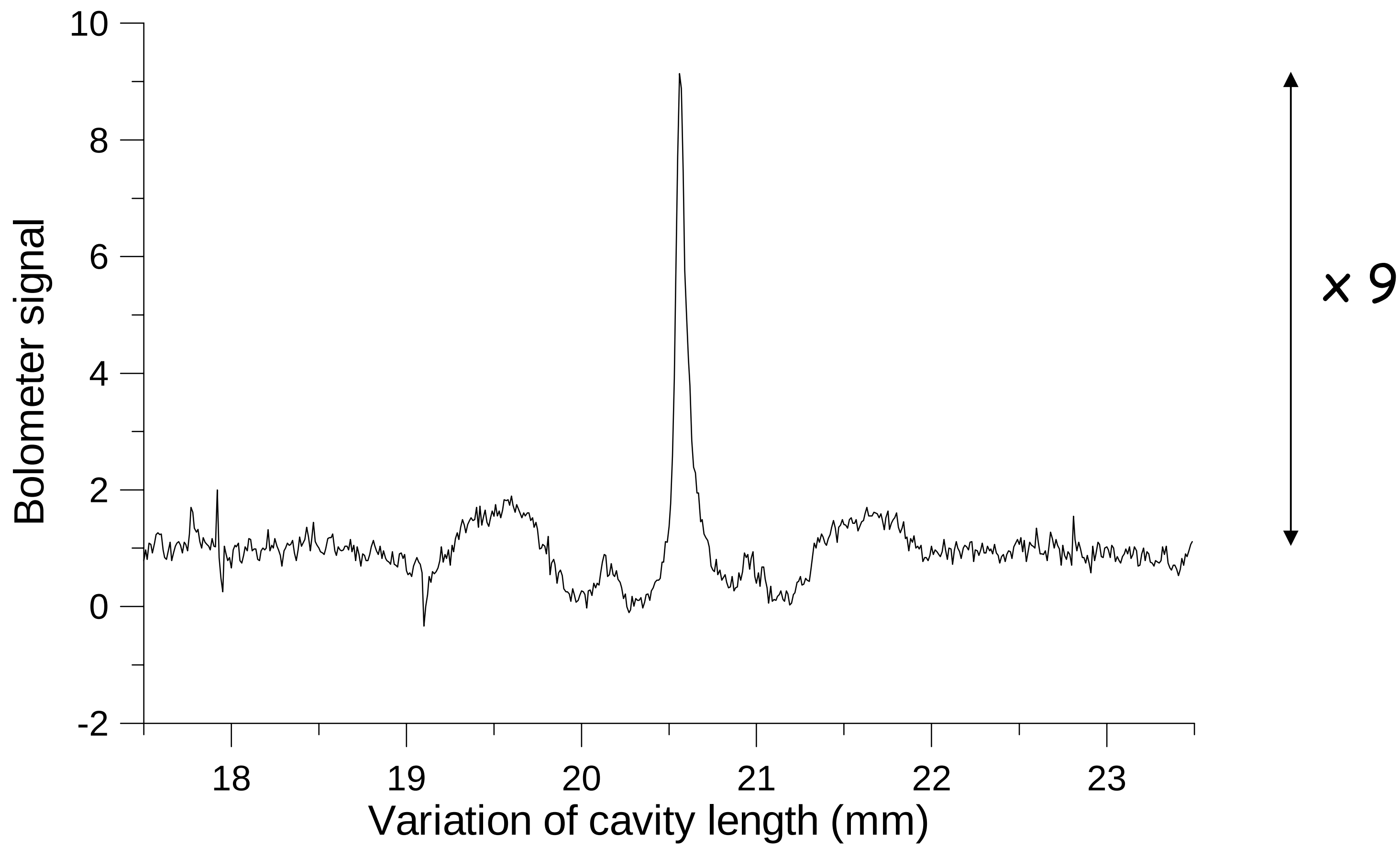


scan

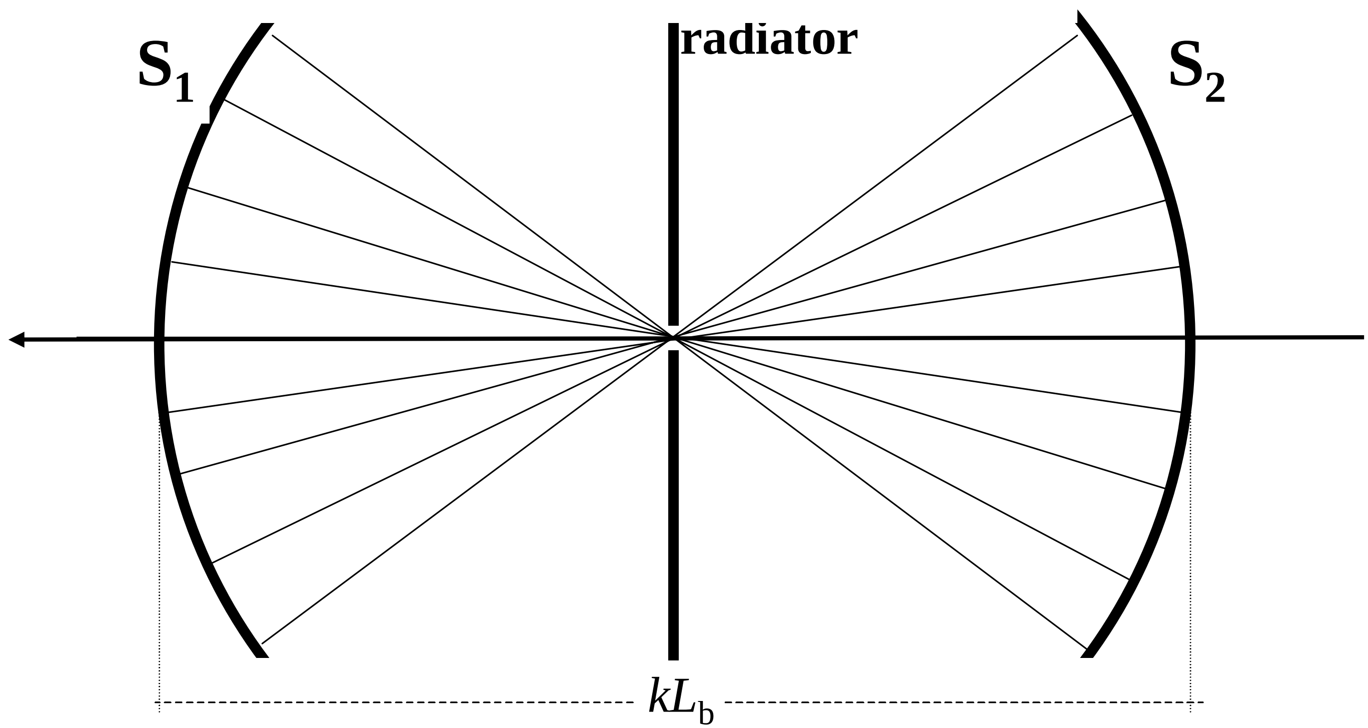


Recorded radiation intensity as a function of optical cavity length (C. Settakorn)

maximum enhancement



Max. enhancement of STR achieved so far (C.Settakorn) try this one ?



Free Electron Laser, Optical Klystron and SASE

principle

stimulate the emission of EM radiation from relativistic electron beam by the interaction with an external EM field.

make electrons move against the EM field
to loose energy into EM wave

how ?

how?

energy gain/loss of electron from/to EM field

$$\Delta W = -e\int \overrightarrow{E_{\mathrm{L}}}\,\mathrm{d}\vec{s} = -e\int \vec{v}\,\overrightarrow{E_{\mathrm{L}}}\,\mathrm{d}t = 0$$

because $\vec{v} \perp \overrightarrow{E_{\mathrm{L}}}$

how do we get better coupling ?

need particle motion in the direction of electric field from EM wave

undulator

trajectory

$$\frac{d^2x}{dt^2} = -\frac{eB_0}{mc\gamma}\frac{dz}{dt}\cos k_p z$$
$$\frac{d^2z}{dt^2} = +\frac{eB_0}{mc\gamma}\frac{dx}{dt}\cos k_p z$$

$$\frac{dx}{dt} = -c\beta\frac{K}{\gamma}\sin k_p z$$
$$\frac{dz}{dt} = +c\beta\left(1 - \frac{K^2}{2\gamma^2}\sin^2 k_p z\right)$$

drift velocity

$$\bar{\beta} = \beta\left(1 - \frac{K^2}{4\gamma^2}\right)$$

$$x(t) = a\cos(k_p c\bar{\beta}t)$$

$$a = \frac{K}{\gamma k_p}$$

$$z(t) = c\bar{\beta}t + \frac{1}{8}k_p a^2 \sin(2k_p c\bar{\beta}t)$$

synchronicity condition

$$\Delta W = -e\int v_x E_{x\mathrm{L}}\,\mathrm{d}t = -e\int \left[c\frac{K}{\gamma}\sin\left(k_\mathrm{u}s\right)\right]\left[E_{x\mathrm{L},0}\cos\left(k_\mathrm{L}s-\omega_\mathrm{L}t+\varphi_0\right)\right]\mathrm{d}t$$

$$= -\frac{ecKE_{x\mathrm{L},0}}{\gamma}\int\Big\{\sin\left[\left(k_\mathrm{L}+k_\mathrm{u}\right)s-\omega_\mathrm{L}t+\varphi_0\right]$$

$$-\cancel{\sin\left[\left(k_\mathrm{L}-k_\mathrm{u}\right)s-\omega_\mathrm{L}t+\varphi_0\right]}\Big\}\mathrm{d}t$$

get continuous energy transfer if $\Psi_\pm = \left(k_\mathrm{L}\pm k_\mathrm{u}\right)\overline{s}-\omega_\mathrm{L}t+\varphi_0 \approx \text{const.}$

$$\frac{\mathrm{d}\Psi_\pm}{\mathrm{d}t} = \left(k_\mathrm{L}+k_\mathrm{u}\right)\frac{\mathrm{d}\bar{s}}{\mathrm{d}t}-\omega_\mathrm{L} \approx 0 = \left(k_\mathrm{L}+k_\mathrm{u}\right)\beta\left(1-\frac{K^2}{4\gamma^2}\right)-k_\mathrm{L}$$

condition for continuous energy transfer

$$k_\mathrm{u} = \frac{k_\mathrm{L}}{2\gamma^2}\left(1+\tfrac{1}{2}K^2\right) \quad \text{or} \quad \lambda_\mathrm{L} = \frac{\lambda_\mathrm{u}}{2\gamma^2}\left(1+\tfrac{1}{2}K^2\right)$$

energy gain/loss per unit path length

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{\mathrm{d}W}{c\mathrm{d}t}\frac{1}{mc^2} = -\frac{ecKE_{x\mathrm{L},0}}{2\gamma mc^2}\sin\left[\left(k_\mathrm{L}+k_\mathrm{u}\right)s-\omega_\mathrm{L}t+\varphi_0\right]$$

where $s = ct\bar{\beta} + \frac{K^2}{8\gamma^2 k_\mathrm{u}}\sin\left(2k_\mathrm{u}ct\right)$

define $\eta = \frac{k_\mathrm{L}K^2}{8\gamma^2 k_\mathrm{u}}$ and $K_\mathrm{L} = \frac{eE_{x\mathrm{L},0}}{k_\mathrm{u}mc^2}$

and the energy gain becomes

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = -\frac{k_\mathrm{u}K_\mathrm{L}K}{2\gamma}\left[J_0(\eta)-J_1(\eta)\right]\sin\left[\left(k_\mathrm{L}+k_\mathrm{u}\right)\bar{s}-\omega_\mathrm{L}t+\varphi_0\right]$$

the phase varies slowly for
particles off the resonance energy $\gamma_\mathrm{r}^2 = \frac{k_\mathrm{L}}{2k_\mathrm{u}}\left(1+\frac{1}{2}K^2\right)$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}s} = k_\mathrm{u}\left(1-\frac{\gamma_\mathrm{r}^2}{\gamma^2}\right) = 2\frac{k_\mathrm{u}}{\gamma_\mathrm{r}}\Delta\gamma$$

where $\Delta\gamma = \gamma - \gamma_\mathrm{r}$

$$\frac{d\Delta\gamma}{ds} = \frac{d\gamma}{ds} - \frac{d\gamma_r}{ds} = -\frac{k_u K_L K}{2\gamma}\left[J_0(\eta) - J_1(\eta)\right]\sin\Psi$$

$$\frac{d^2\Psi}{ds^2} = 2\frac{k_u}{\gamma_r}\frac{d\Delta\gamma}{ds} = -\frac{k_u^2 K_L K}{\gamma_r\gamma}\left[J_0(\eta) - J_1(\eta)\right]\sin\Psi$$

Pendulum equation $$\boxed{\frac{d^2\Psi}{ds^2} + \Omega_L^2 \sin\Psi = 0}$$

with $$\Omega_L^2 = \frac{k_u^2 K_L K}{\gamma_r\gamma}\left[J_0(\eta) - J_1(\eta)\right]$$

Gain

gain of laser field: $\Delta W_{\mathrm{L}} = -mc^2 \Delta\gamma$

stored energy in laser field $W_{\mathrm{L}} = \frac{1}{2}\epsilon_0 E_{\mathrm{L},0}^2 V$

gain of laser field per electron

$$G_1 = \frac{\Delta W_{\mathrm{L}}}{W_{\mathrm{L}}} = -\frac{2mc^2}{\epsilon_0 E_{\mathrm{L},0}^2 V}\Delta\gamma = -\frac{mc^2\gamma_{\mathrm{r}}}{\epsilon_0 E_{\mathrm{L},0}^2 V k_{\mathrm{u}}}\Delta\Psi'$$

or for all electrons

$$G = -\frac{e^2 k_{\mathrm{u}} K^2}{\epsilon_0 mc^2}\frac{n_{\mathrm{b}}}{\gamma_{\mathrm{r}}^3}\left[J_0(\eta) - J_1(\eta)\right]^2\frac{\langle\Delta\Psi'\rangle}{\Omega_{\mathrm{L}}^4}$$

where n_{b} is the electron density

the average variation of $\langle\Delta\Psi'\rangle$ can be calculated from the phase equation

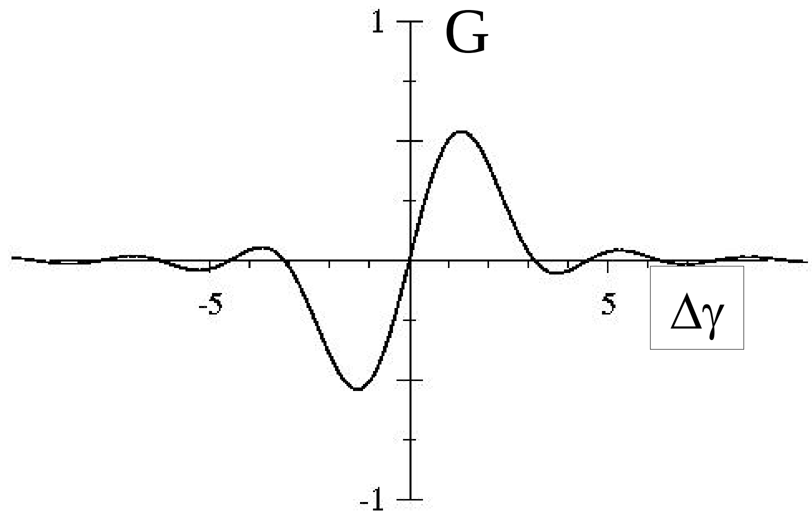
FEL gain per pass $G = -\frac{\pi e^2 K^2 N_u \lambda_u^2}{4\epsilon_0 mc^2}\frac{n_b}{\gamma_r^3}[J_0(\eta) - J_1(\eta)]^2 \frac{d}{dw}(\frac{\sin w}{w})^2$

with $w = \frac{2\pi N_u}{\gamma_r}(\gamma_0 - \gamma_r)$ and $\eta = \frac{k_L K^2}{8\gamma^2 k_u}$

gain curve $\frac{d}{dw}(\frac{\sin w}{w})^2$

for finite gain: $\Delta\gamma \neq 0$

adjust beam energy slightly higher than resonance energy

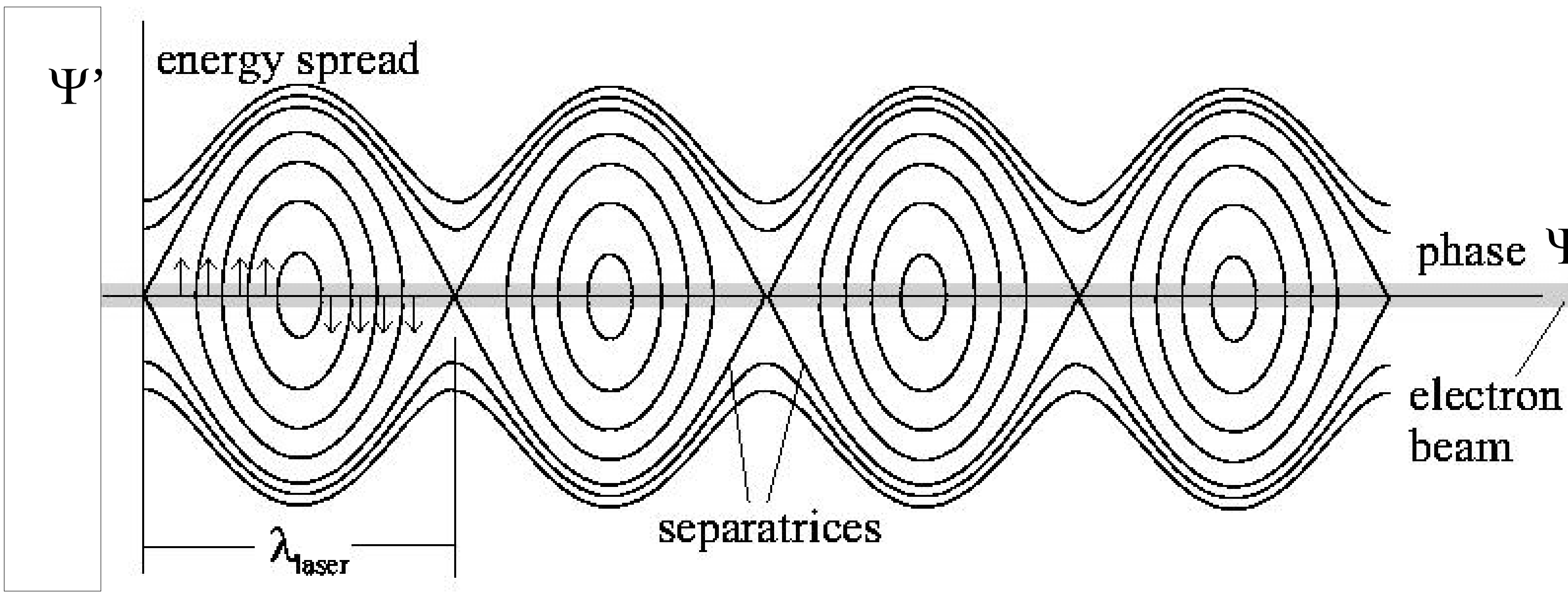


laser energy $W_L = W_{L,0} e^{Gn}$ n number of passes

phase space motion

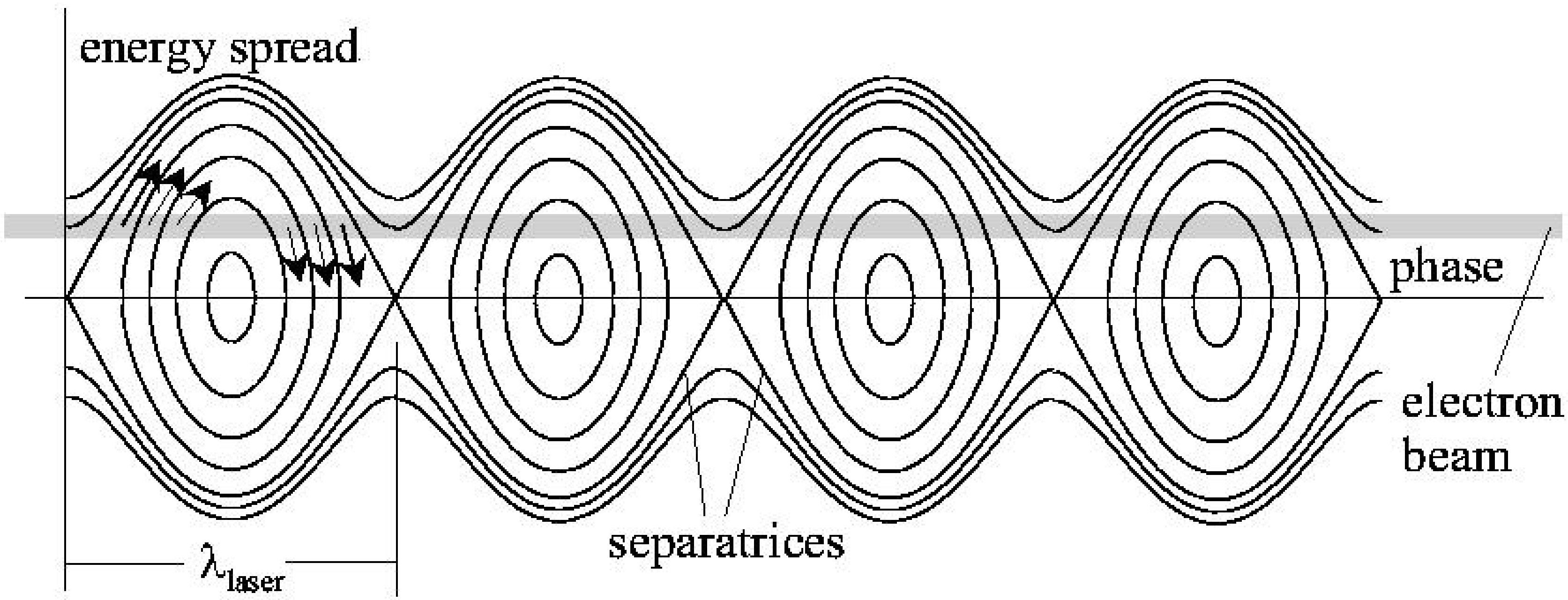
Pendulum equation $\frac{d^2\Psi}{ds^2} + \Omega_L^2 \sin\Psi = 0$ | $\cdot\Psi'$ and integrate

$$\frac{1}{2}\Psi'^2 - \Omega_L^2 \cos\Psi = \text{const}$$



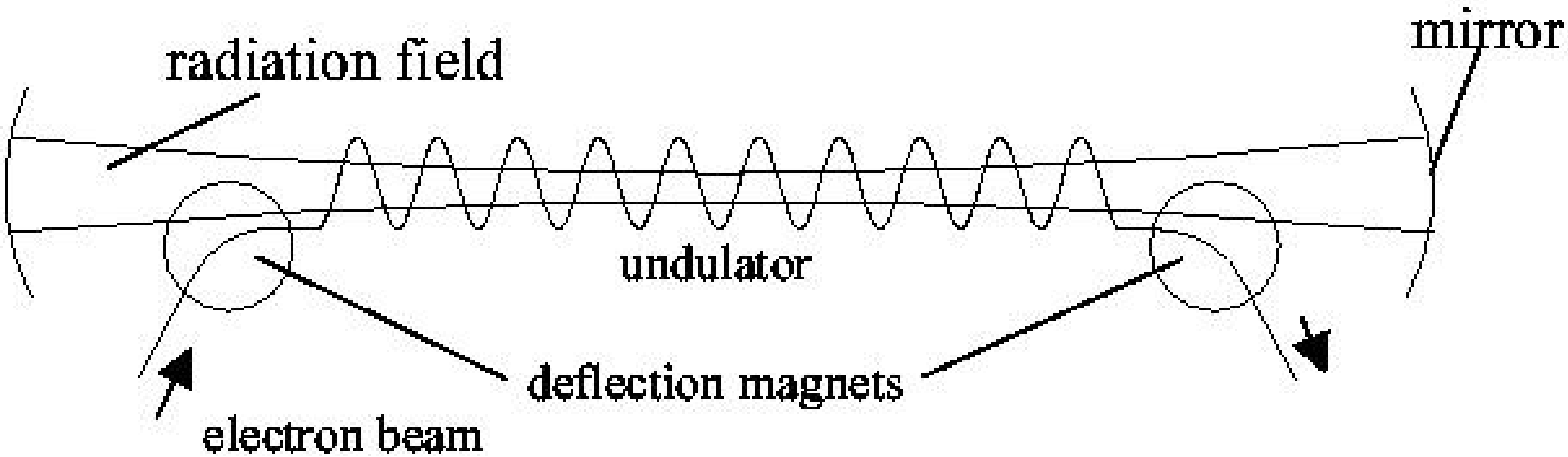
no net energy transfer !

phase space motion



for $\gamma_0 > \gamma_r$ → energy transfer to laser field !

FEL schematic



velocity of wave: c

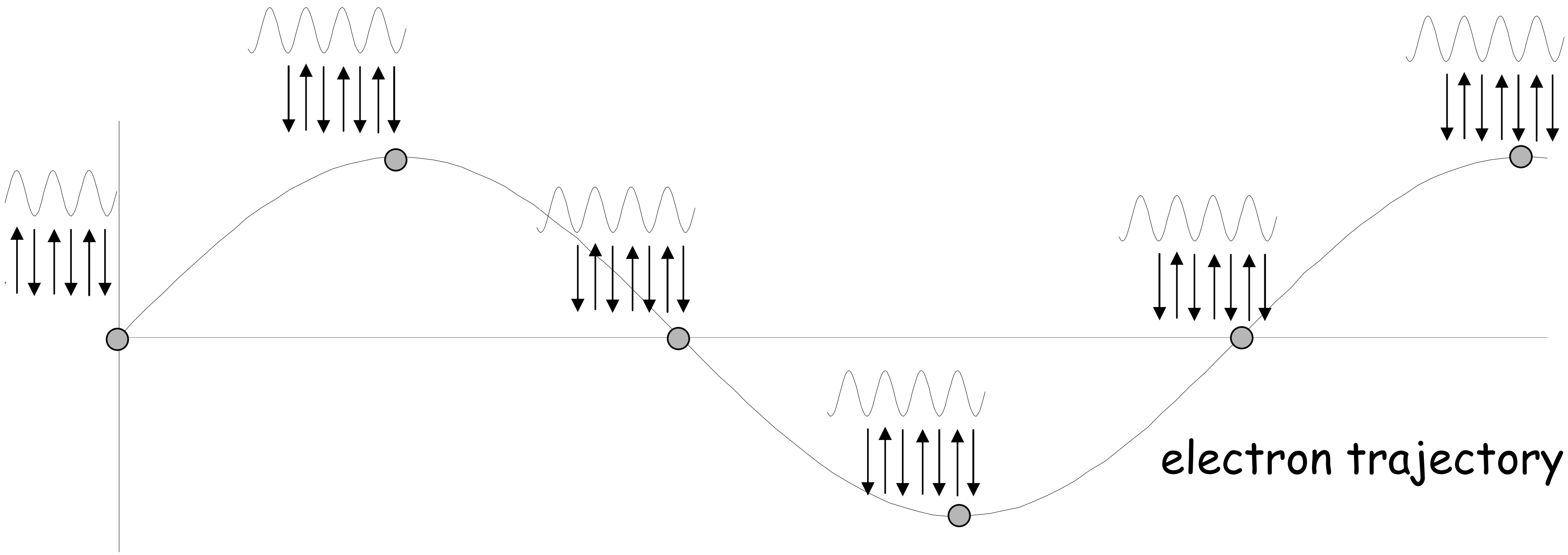
average drift velocity of electron: $\bar{\beta} = \beta\left(1 - \frac{K^2}{4\gamma^2}\right)$

time for electron to travel one period: $\tau = \frac{\lambda_u}{c\bar{\beta}} = \frac{\lambda_u}{c\beta\left(1-\frac{K^2}{4\gamma^2}\right)}$

distance wave propagates in time τ : $s_\gamma = \frac{\lambda_u c}{c\beta\left(1-\frac{K^2}{4\gamma^2}\right)}$

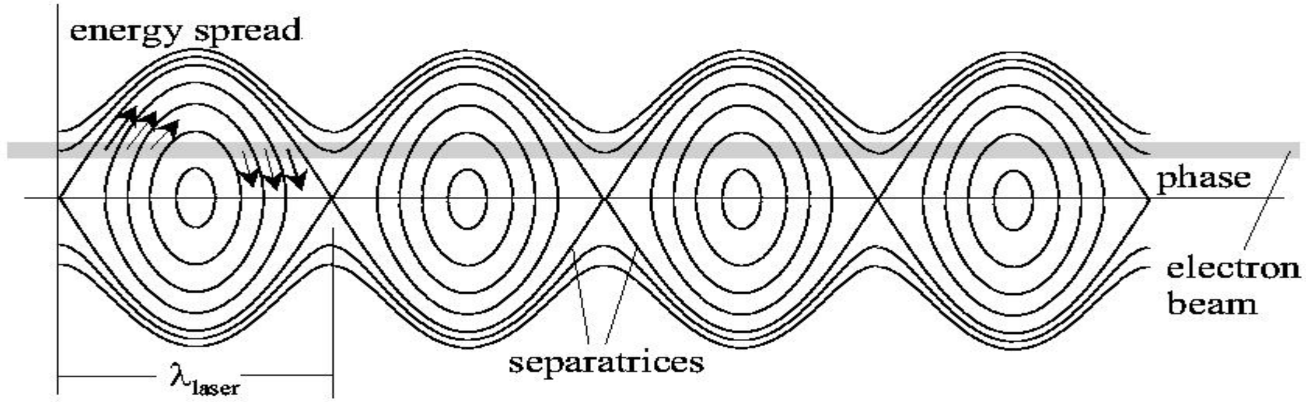
$$\delta s = \frac{\lambda_u}{\beta\left(1-\frac{K^2}{4\gamma^2}\right)} - \lambda_u \approx \lambda_u\left[\frac{1}{\beta}\left(1 + \frac{K^2}{4\gamma^2}\right) - 1\right] \approx \frac{\lambda_u}{2\gamma^2}\left(1 + \tfrac{1}{2}K^2\right)$$

or $\delta s = \lambda_\gamma$ EM wave propagates one wavelength ahead of electron per period

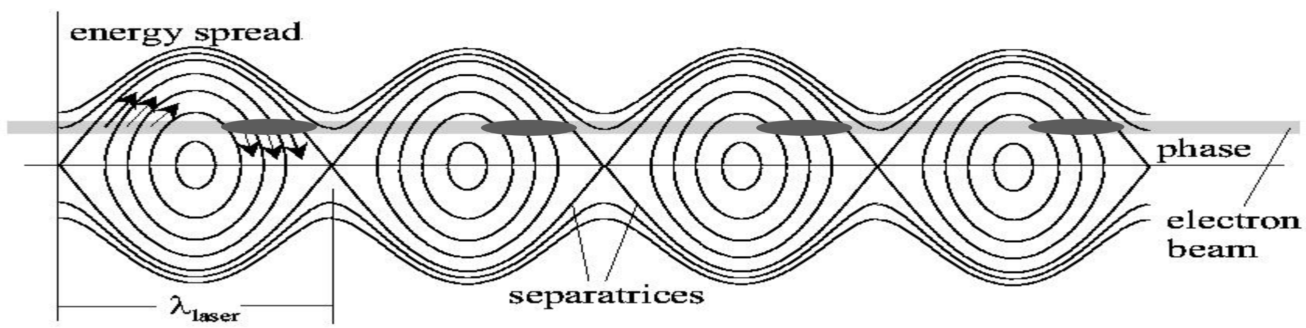


electron move constantly against external field

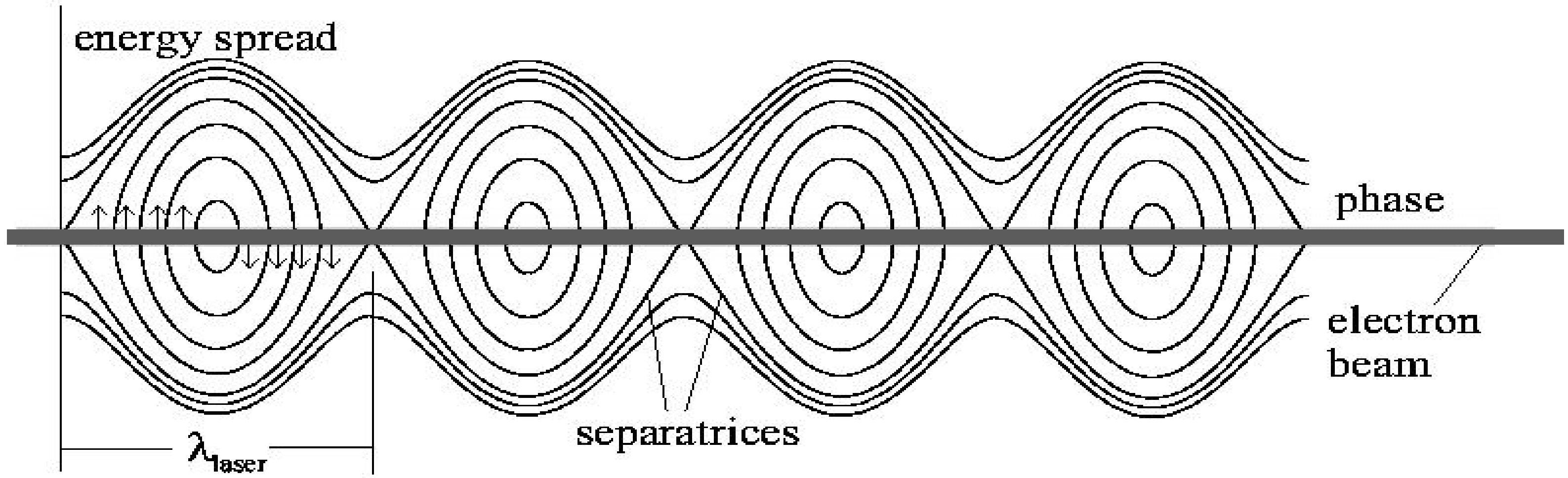
Optical Klystron



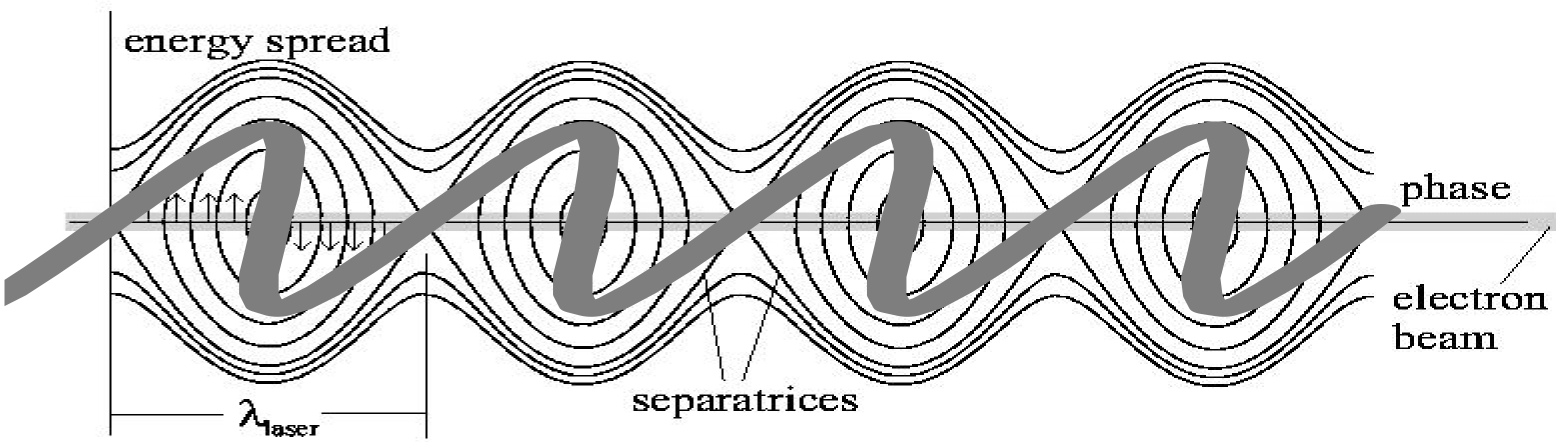
this works, but is not very efficient
bunched beam would be better



beam bunching

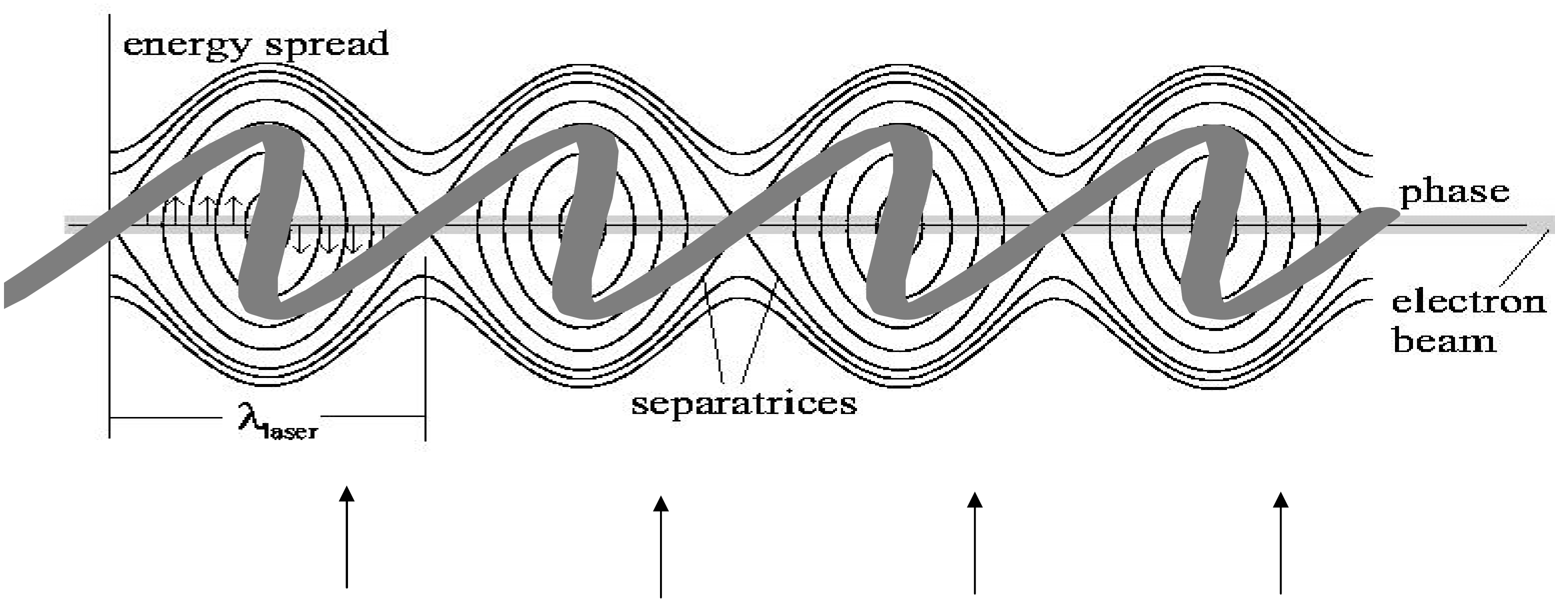


at undulator exit:



FEL action results in a bunched beam

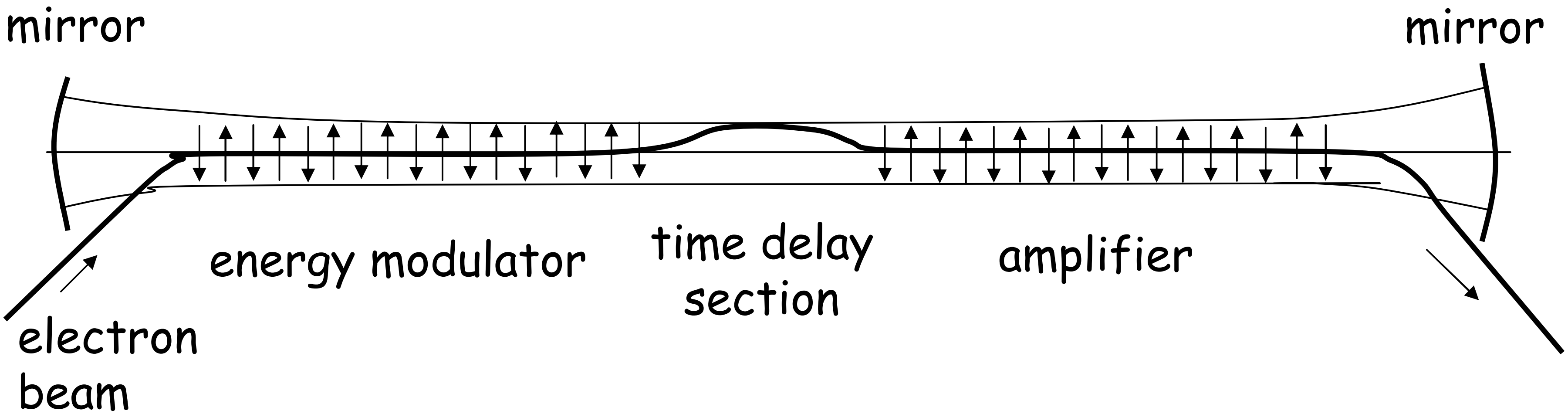
but the bunches are not at the right point



we need bunches here

need time delay section

Principle of optical klystron



SASE

FEL works only for wavelength where mirrors exist

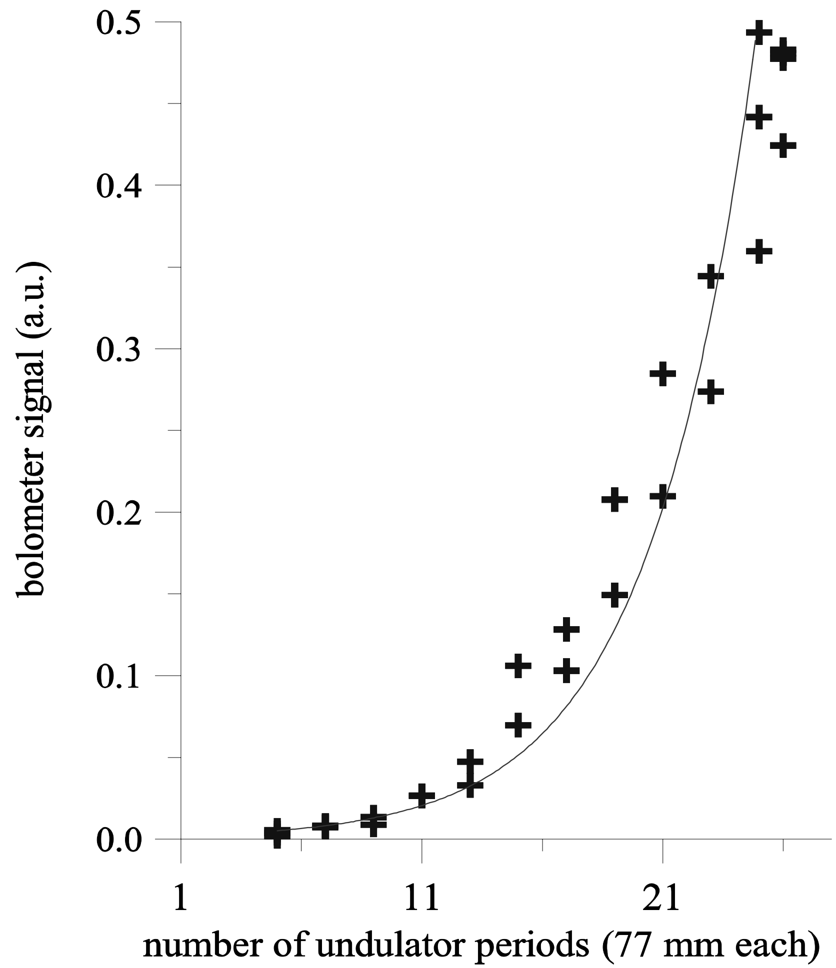
mostly visible, IR, FIR and microwaves

how about an x-ray free electron laser ?

amplification can occur only in one pass !

it can work !

53 um SASE



0.5
0.4
0.3
0.2
0.1
0.0
bolometer signal (a.u.)
1
11
21
number of undulator periods (77 mm each)

how does that work?

consider bunch

there is always a density fluctuation

fluctuation acts like a bunch, emitting coherent radiation

coherent radiation propagates faster than electrons

field acts back on bunch generating periodic energy variation

energy variation transforms into bunching at desired wavelength

generating even more radiation growing exponentially

need long undulator: ~ 100 m (SLAC)

for 1A radiation: need electron energy about 15 GeV

need high quality, high intensity, low emittance beam

X-rays from Low Energy Electron Beams

types of radiation

- Thomson/Compton Scattering
- Channeling Radiation
- Parametric x-rays
- Smith-Purcell Radiation
- Crystalline Undulator
- Resonant Transition Radiation
- Stimulated Transition Radiation
- ?????

Compton scattering

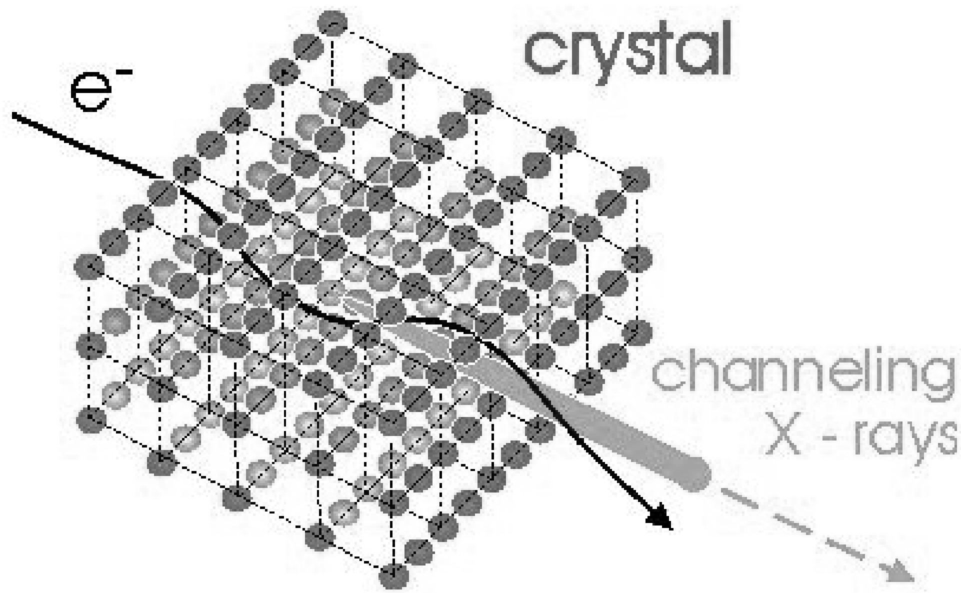
Thomson backscattering

$$\varepsilon_{\mathrm{ph}}(eV) = 4.959 \frac{\gamma^2}{\lambda_L(\mu m)}$$

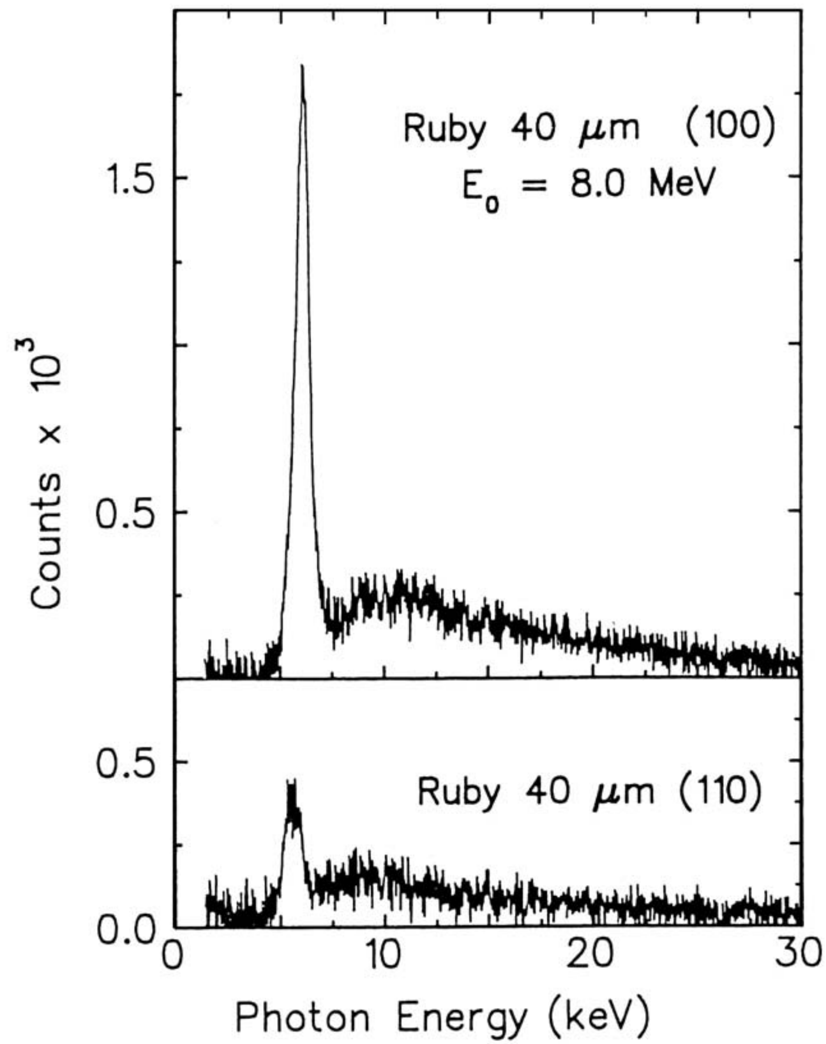
For 25 MeV electrons:

incoming radiation	**backscattered radiation**
coherent FIR, 100 - 1000 μm	12-120 eV
CO_2 Laser, 10μm	1200 eV
Yag Laser, 1μm	12.0 keV

Channeling radiation

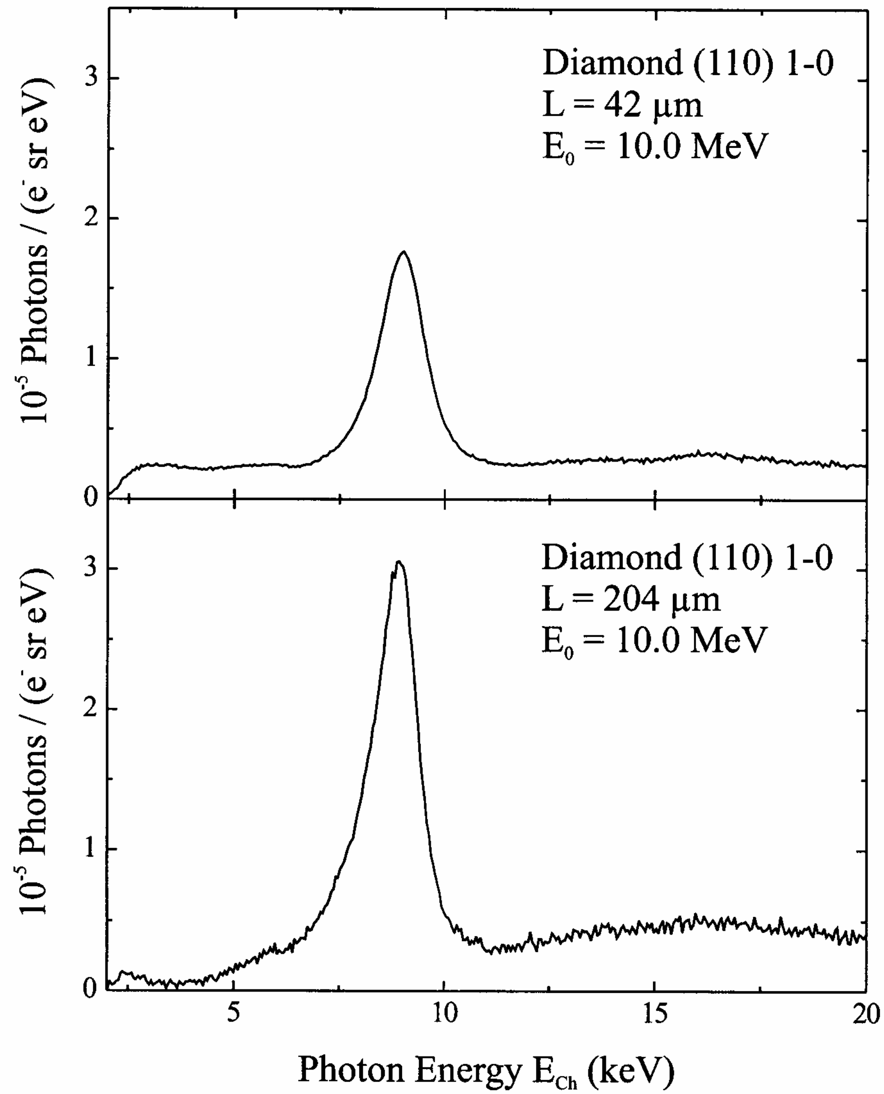


Channeling-radiation spectra from 8.0-MeV electrons along the (100) and (110) planes of ruby, obtained at the superconducting linac at Darmstadt [Freudenberger *et al*. NIM B**119** (1996) 123].

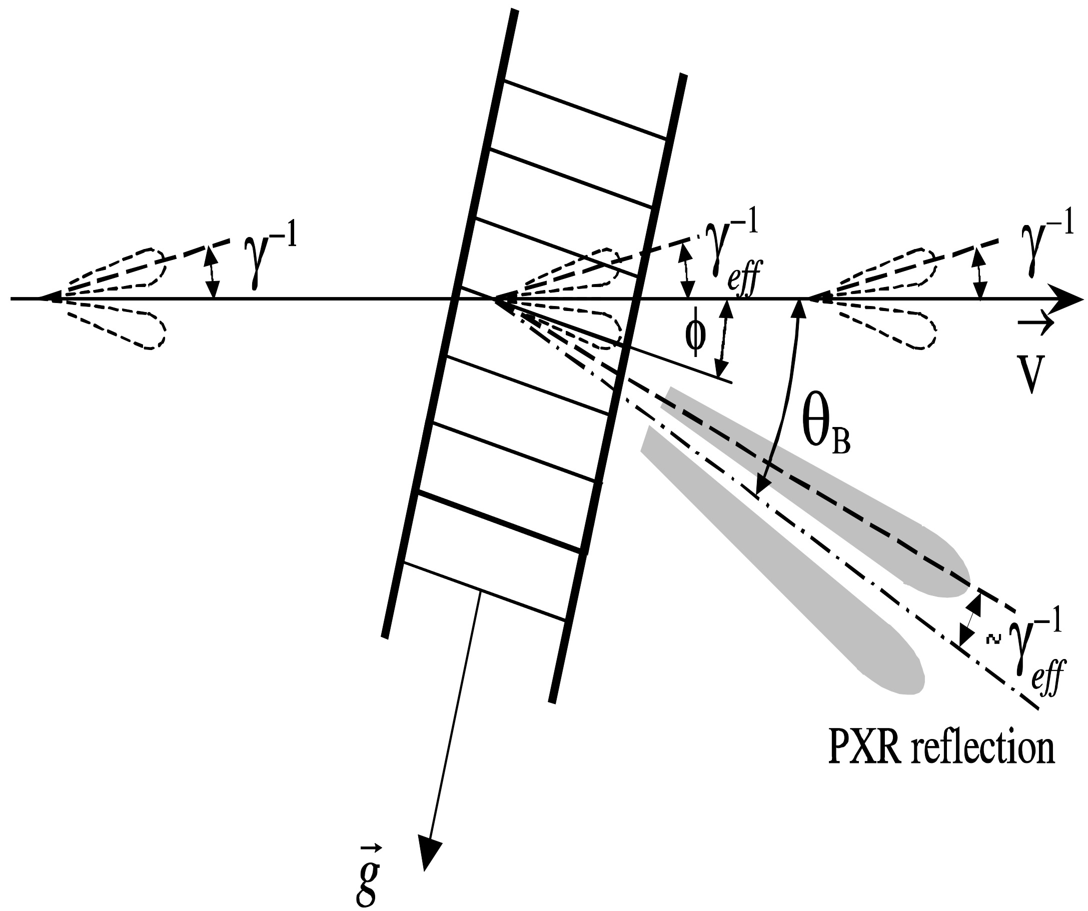


Channeling - 2

Channeling radiation spectra obtained from diamond crystals after substraction of bremsstrahlung background (H. Genz)

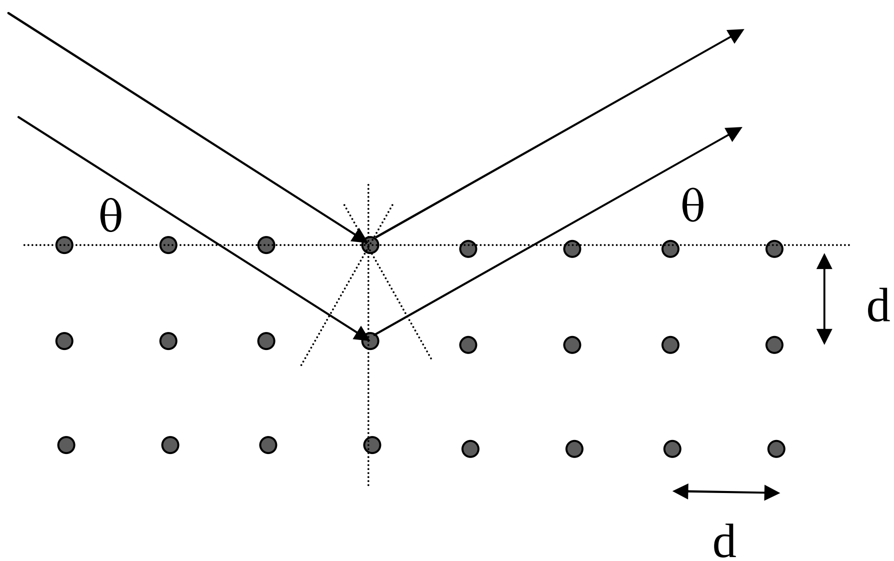


PXR as diffraction of virtual photons associated with relativistic charged particles (A. Shchagin)



Photon energy of PXR

PXR is emitted under the Bragg reflection rule



Bragg reflection

$$2d\sin\theta = n\lambda$$

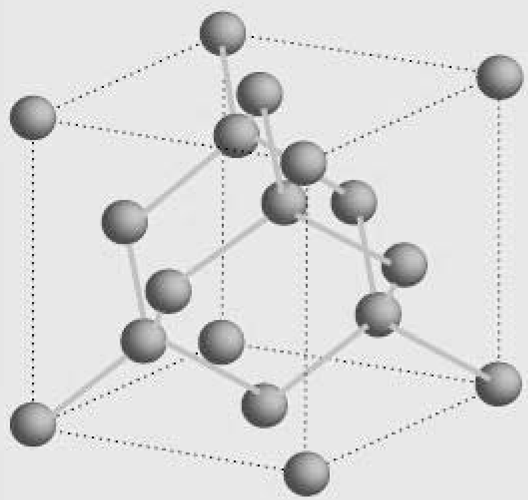
example

for Si d = 1.8 A
θ = 22.5 deg

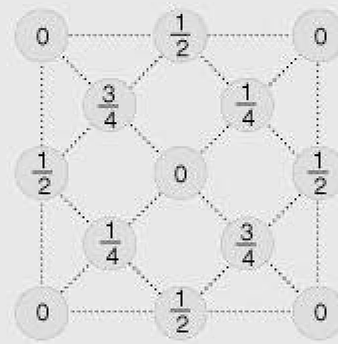
λ = 1.4 A or 8.8 keV

Si crystal

Silicon Crystal Structure



after Kittel



The above illustration shows the arrangement of the silicon atoms in a unit cell, with the numbers indicating the height of the atom above the base of the cube as a fraction of the cell dimension.

Silicon crystallizes in the same pattern as diamond, in a structure which Ashcroft and Mermin call "two interpenetrating face-centered cubic" primitive lattices. The lines between silicon atoms in the lattice illustration indicate nearest-neighbor bonds. The cube side for silicon is 0.543 nm. Germanium has the same diamond structure with a cell dimension of .566 nm.

PXR

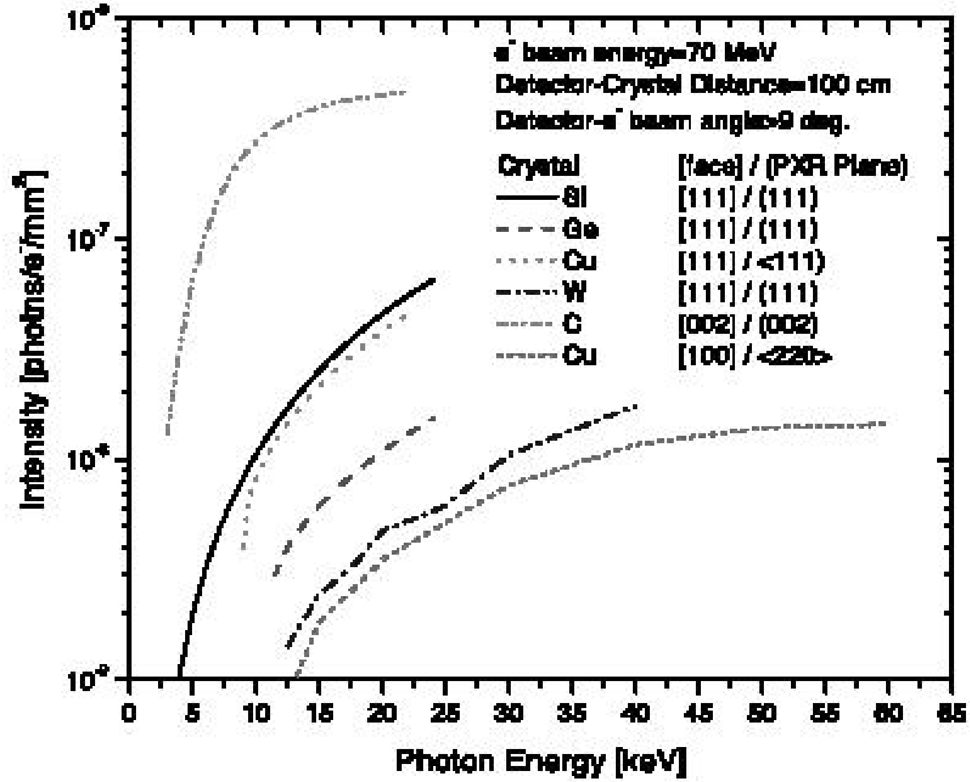
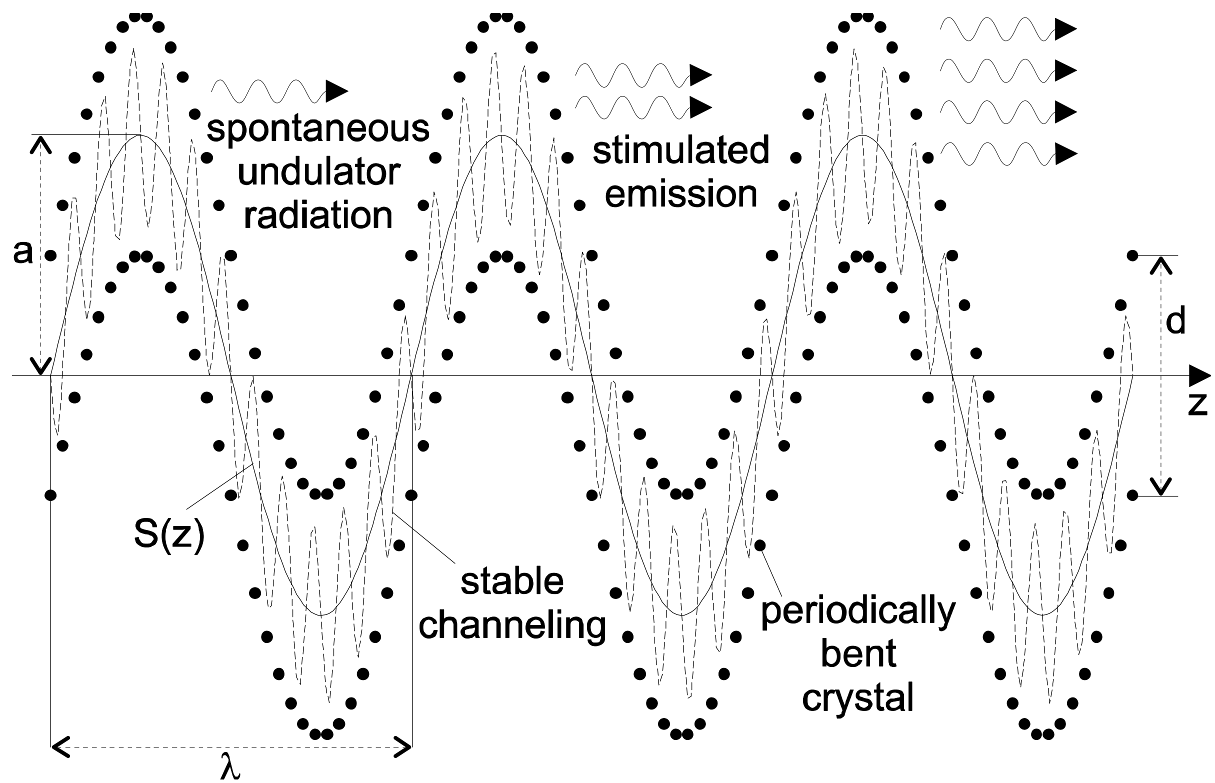


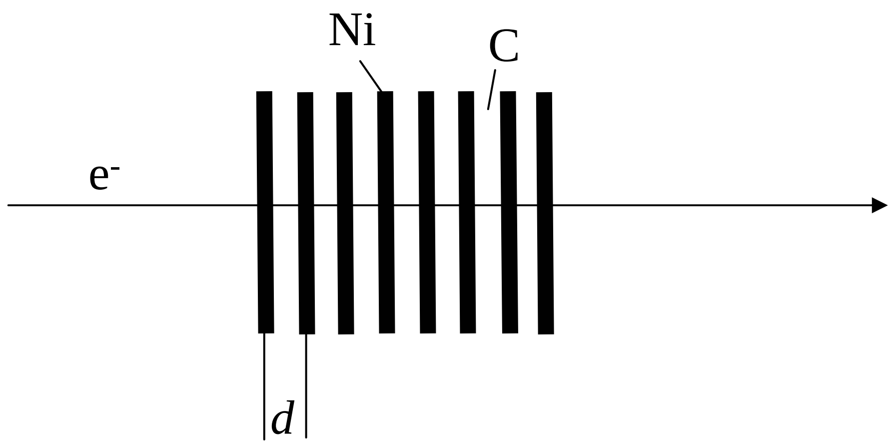
Fig. 1. Calculated X-ray intensities for several 500 μm thick crystal targets for several reflection planes and crystal face planes. The upper energy limit of each case was limited by a requirement of an angle of more than 9 deg between the detector and the electron beam axis.



Crystalline undulator (schematic). Vertical scale magnified by 10^4. (Krause et.al.)

Resonant TR-theory

use stack of layered low-Z/high-Z material



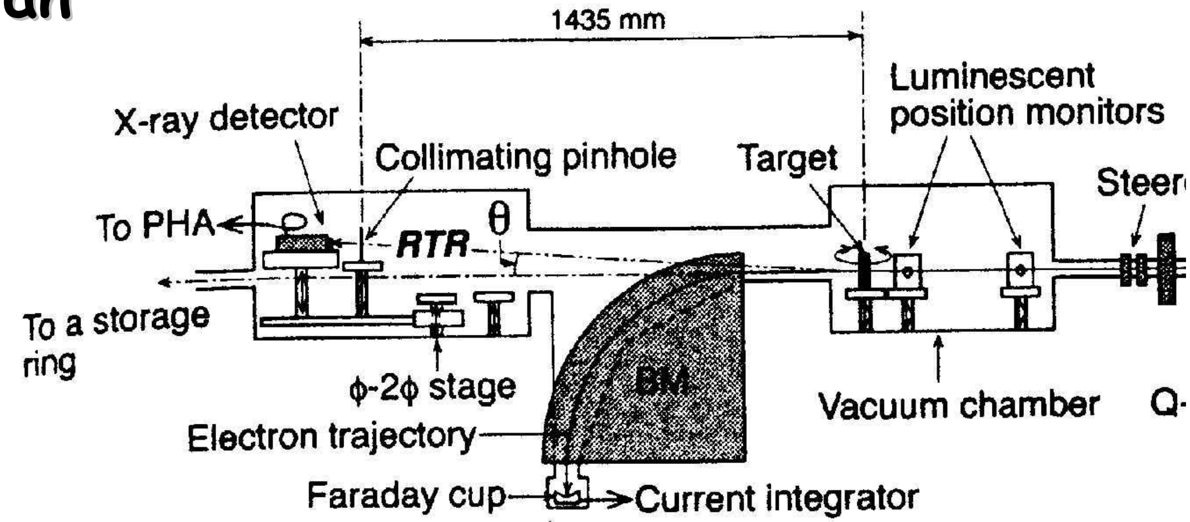
resonance condition in electron frame: $d^* = \frac{d}{\gamma} = n\lambda^*$

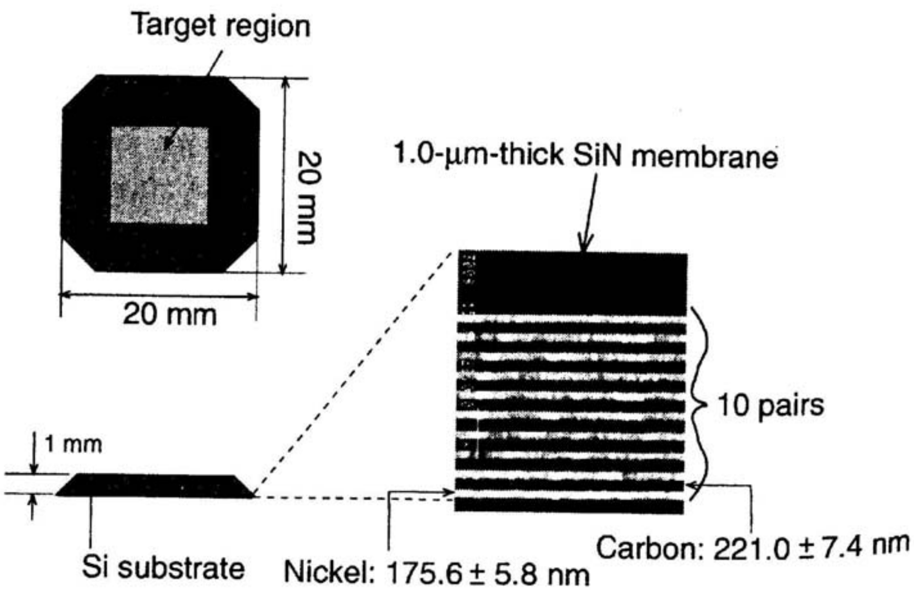
radiation observed in lab frame: $\lambda = \frac{d}{2\gamma^2}(1+\gamma^2\theta^2)$

for λ = 1A and E = 20 MeV, we need d = 320 nm

Resonant TR-Ispirian

RTR experimental Setup at the Yerevan Physics Institute

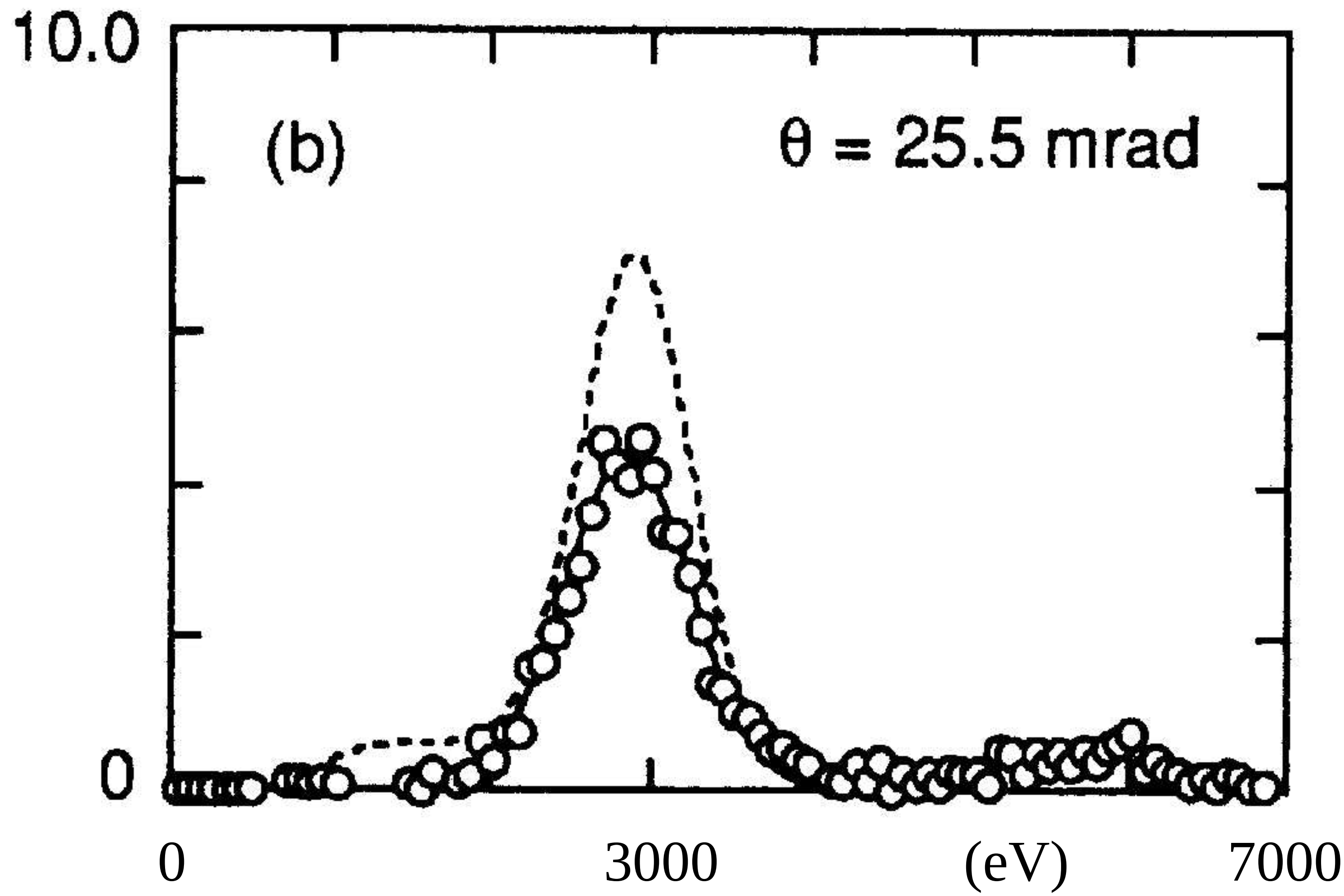




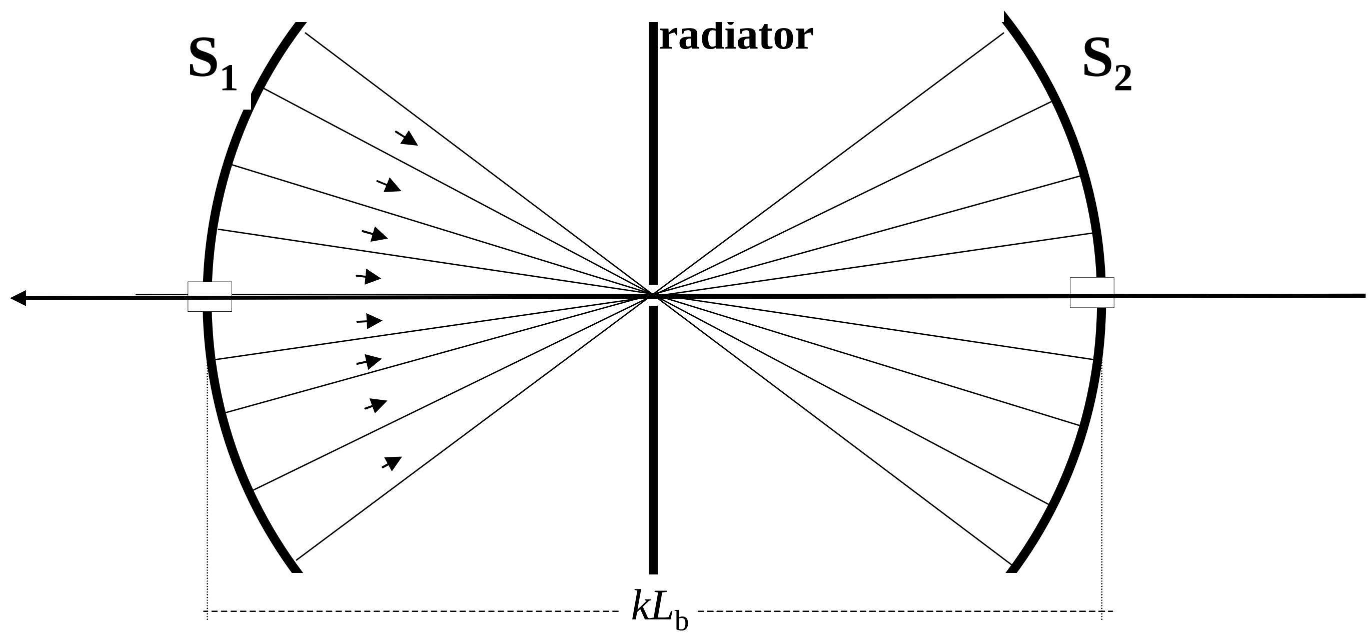
Multi layer radiator

K.A. Ispirian

Resonant TR-Ispirian



STR



FIR radiation: $50 < \lambda < 1000\ \mu m$

VUV-radiation: $150\ eV < \varepsilon_{ph} < 7\ eV$