Lecture 10

Power Dividers and Couplers

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Power dividers and directional couplers

- **Basic properties of dividers and couplers**

three-port network (T-junction), four-port network (directional coupler), directivity measurement

- **The T-junction power divider**
- **Lossless divider, lossy divider**

- **The Wilkinson power divider**
- **Even-odd mode analysis, unequal power division divider,**
- **N-way Wilkinson divider**

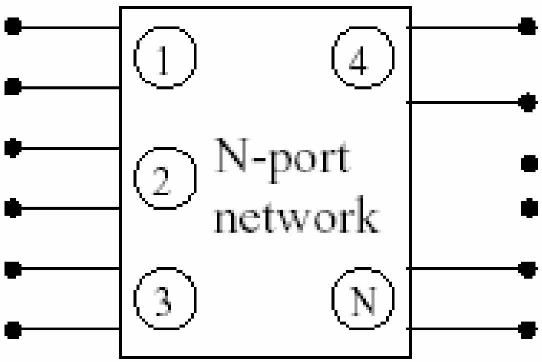
- **The quadrature (90°) hybrid branch-line coupler**

- **Coupled line directional couplers**
- **Even- and odd-mode Z_0, single-section and multisection coupled line couplers**

- **The Lange coupler**
- **The 180° hybrid**
  - rat-race hybrid, tapered coupled line hybrid
- **Other couplers**
  reflectometer

Basic properties of dividers and couplers

- N-port network



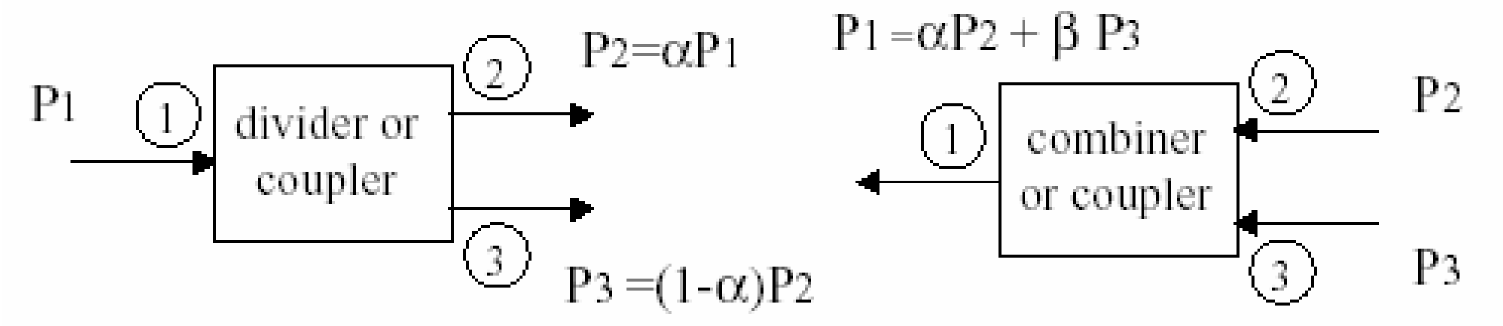
$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & \bullet & S_{1N} \\ S_{21} & S_{22} & S_{23} & \bullet & S_{2N} \\ S_{31} & S_{32} & S_{33} & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ S_{N1} & S_{N2} & \bullet & \bullet & S_{NN} \end{bmatrix} \qquad S_{ij} = \left.\frac{V_i^-}{V_j^+}\right|_{V_k^+=0,\,k\neq j}$$

Discussion

1. Matched ports ⟶ $S_{ij}=0$
2. Reciprocal network ⟶ symmetric property $S_{ij} = S_{ji}$
3. Lossless network ⟶ unitary property

$$\sum_{i=1}^{N} \left|S_{ij}\right|^2 = 1 \;\forall j, \qquad \sum_{i=1}^{N} S_{ik} S_{kj}^* = 0 \quad k \neq j$$

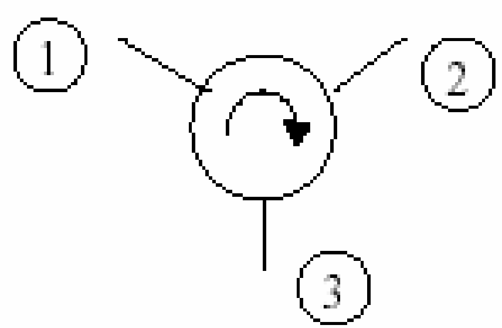
Three-port network (T-junction)



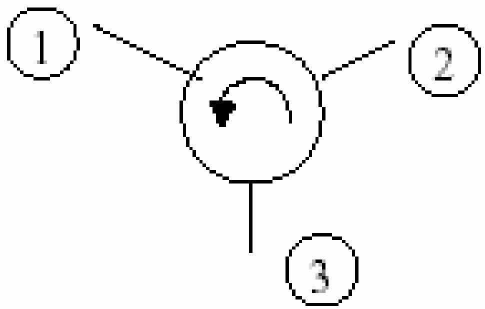
Discussion

1. Three-port network cannot be lossless, reciprocal and matched at all ports.
2. Lossless and matched three-port network is nonreciprocal

$\longrightarrow$ circulator



$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

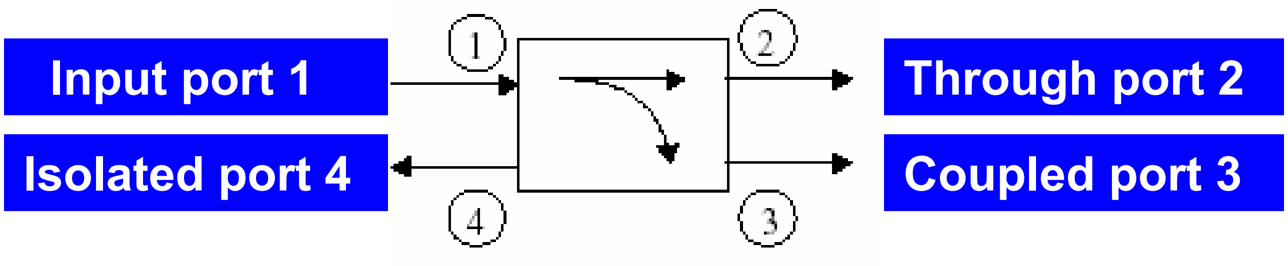


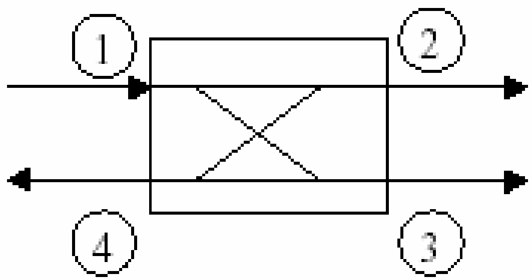
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

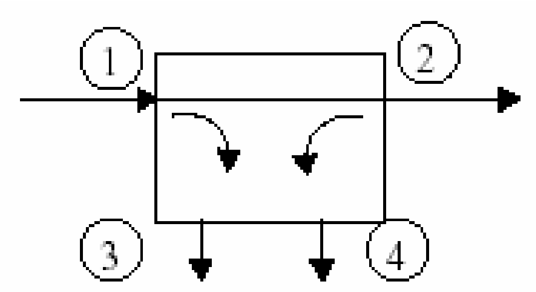
3. Matched and reciprocal three-port network is lossy
→ resistive divider

4. Lossless and perfect isolation three-port network cannot be matched at all ports.

Four-port network (directional coupler)







Coupling: $C(dB) \equiv 10\log\frac{P_1}{P_3}$

Directivity: $D(dB) \equiv 10\log\frac{P_3}{P_4}$

Isolation: $I(dB) \equiv 10\log\frac{P_1}{P_4} = C + D$

Discussion

1. Matched, reciprocal and lossless four-network $\longrightarrow$ symmetrical (90°) directional coupler or antisymmetrical (180°) directional coupler.

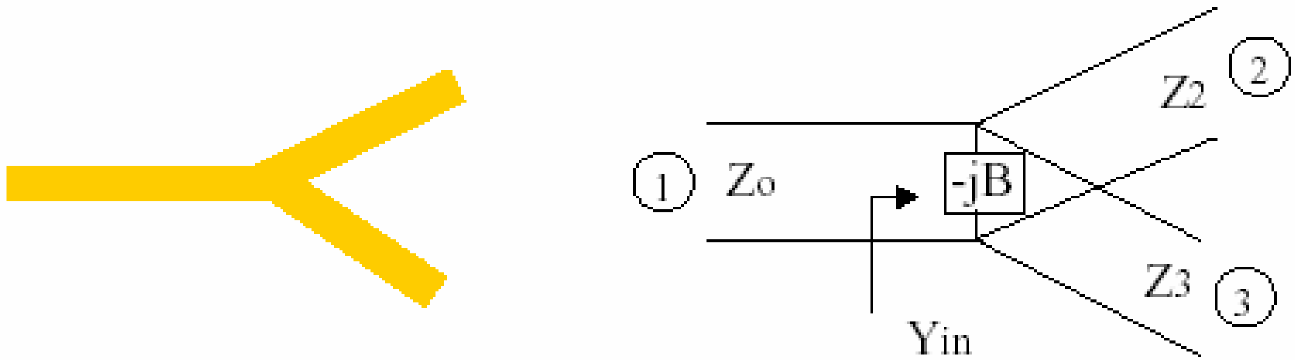
$$\begin{bmatrix} 0 & \alpha & j\beta & 0 \\ \alpha & 0 & 0 & j\beta \\ j\beta & 0 & 0 & \alpha \\ 0 & j\beta & \alpha & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & -\beta \\ \beta & 0 & 0 & \alpha \\ 0 & -\beta & \alpha & 0 \end{bmatrix}$$

2. C = 3dB $\longrightarrow$ 90° hybrid (quadrature hybrid, symmetrical coupler), 180° hybrid (magic-T hybrid, rate-race hybrid)

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & j & 0 \\ 1 & 0 & 0 & j \\ j & 0 & 0 & 1 \\ 0 & j & 1 & 0 \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

The T-junction power divider

- **Lossless divider**

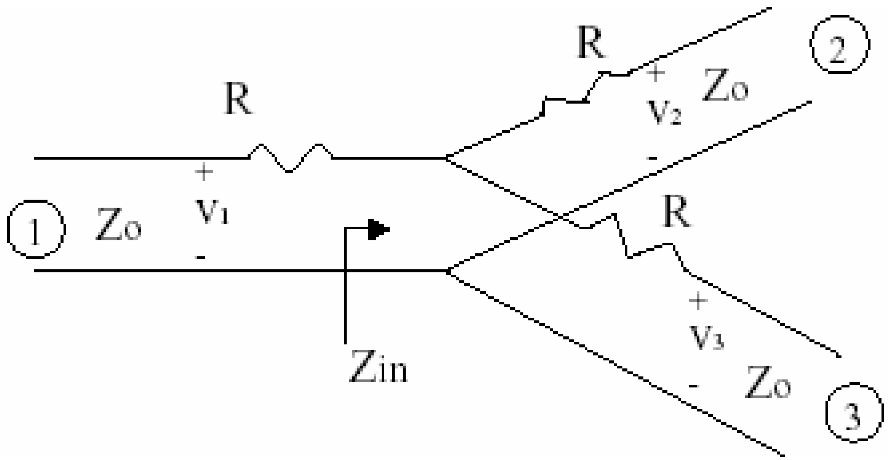


$$Y_{in} = jB + \frac{1}{Z_2} + \frac{1}{Z_3} = \frac{1}{Z_o} \rightarrow B = 0$$

"not practical"

Lossless divider has mismatched ports

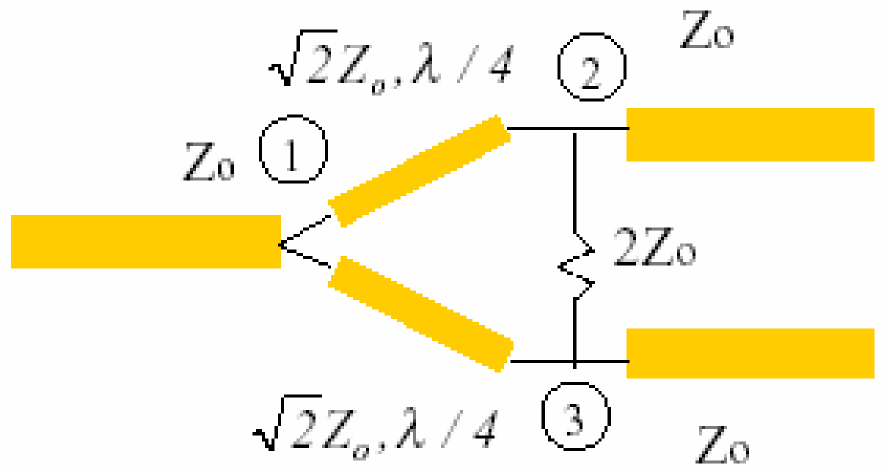
Resistive (lossy) divider



matched ports $\Rightarrow (R+Z_o)//(R+Z_o)+R=Z_o \quad \rightarrow \quad R=\dfrac{Z_o}{3}$

Wilkinson power divider

- **Basic concept**



$$\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -j & -j \\ -j & 0 & 0 \\ -j & 0 & 0 \end{bmatrix}$$

Input port 1 matched, port 2 and port 3 have equal potential

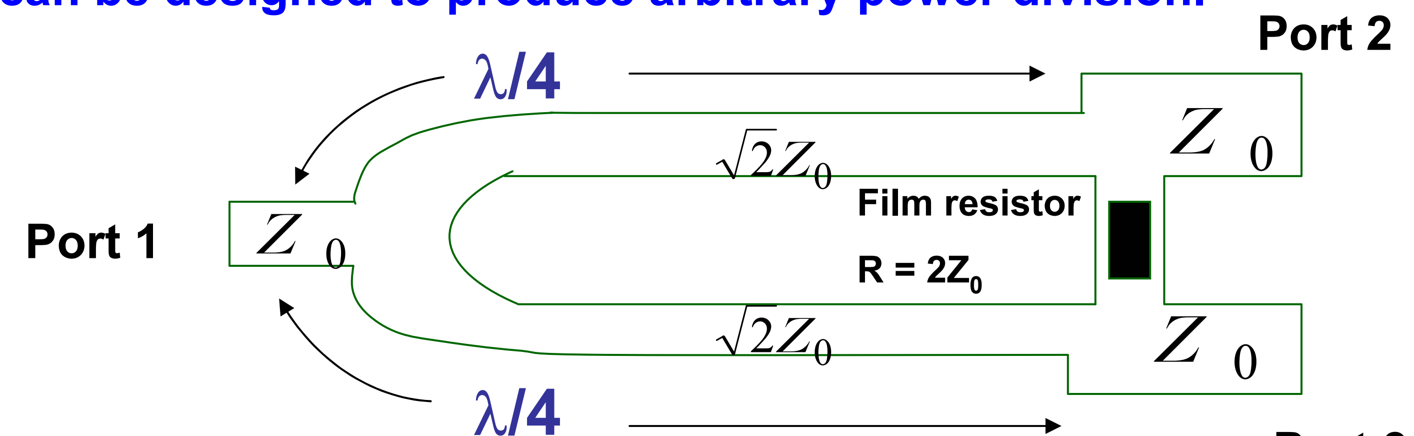
$$\Rightarrow \sqrt{2}Z_0, \lambda/4$$

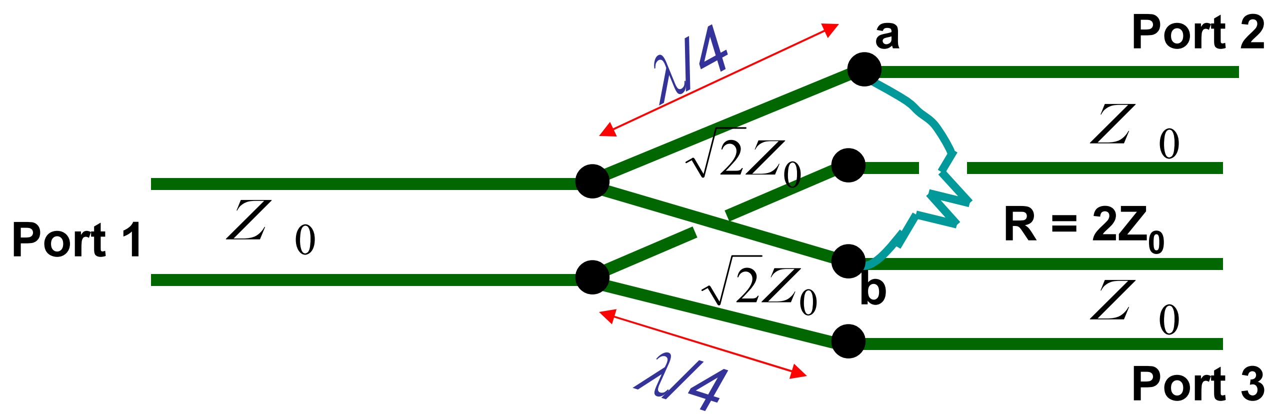
Input port 2, port 1 and port 3 have perfect isolation

⇒ **lossy, matched and good isolation (equal phase) three-port divider**

Wilkinson power divider

- The Wilkinson power divider has these advantages:
  1. It is lossless when output ports are matched.
  2. Output ports are isolated.
  3. It can be designed to produce arbitrary power division.





Wilkinson power divider

- If we inject a TEM mode signal at port 1, equal in-phase signals reach points a and b. Thus, no current flows through the resistor, and equal signals emerge from port 2 and port 3. The device is thus a 3dB power divider. Port 1 will be matched if the λ/4 sections have a characteristic impedance $\sqrt{2}Z_0$.

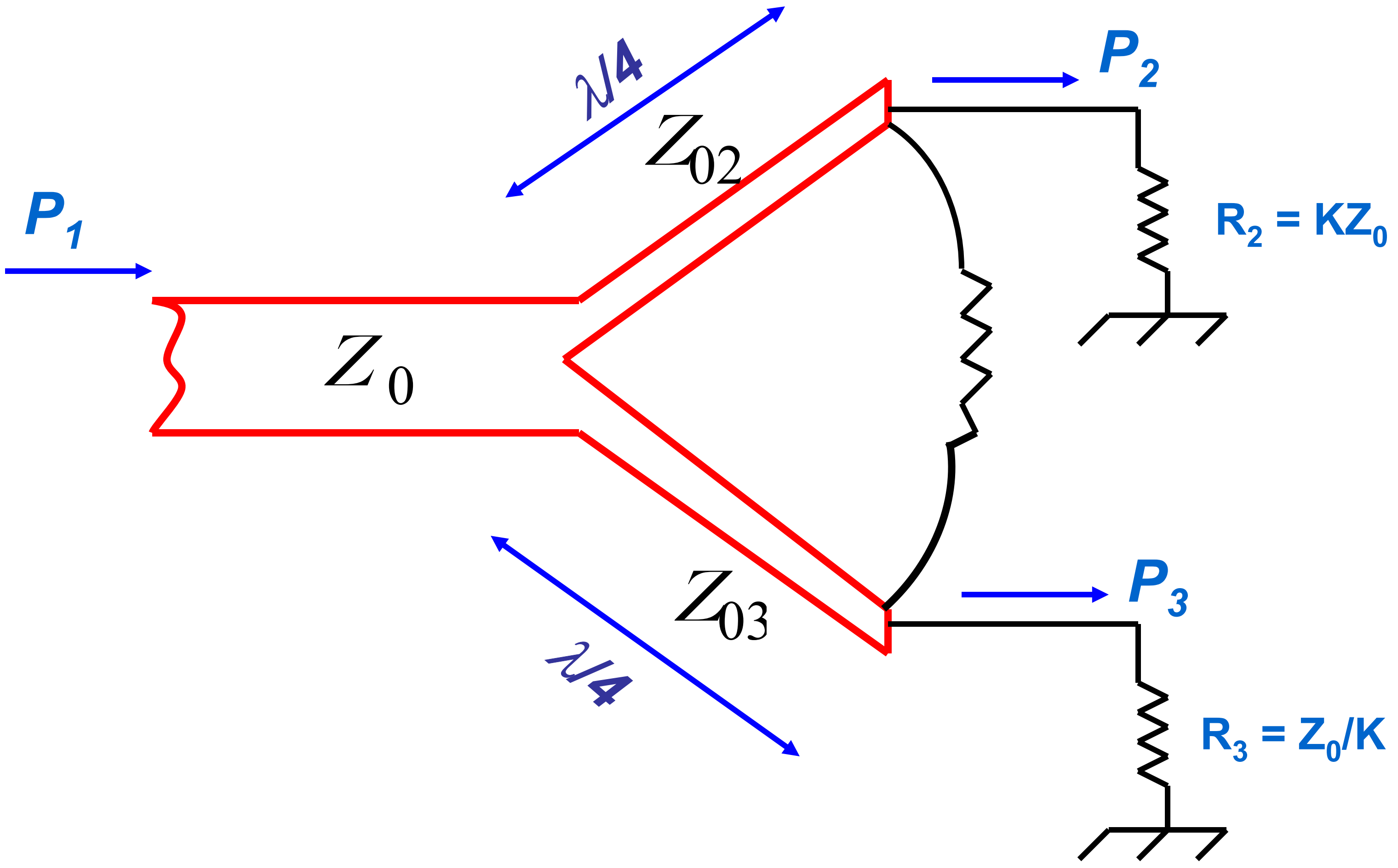
- If we now inject a TEM mode signal at port 2, with matched loads placed on port 1 and on port 3, the resistor is effectively grounded at point b. Equal signals flow toward port 1, and down into the resistor, with port 2 seeing a match. Half the incident power emerges from port 1 and half is dissipated in the resistor film.

- Similar performance occurs when port 1 and port 2 are terminated in matched loads, and a TEM mode signal is injected at port 3.

- If we choose the terminal planes at 1.0 wavelengths from the three Tee junctions, the scattering matrix is

$$[S]=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -j & -j \\ -j & 0 & 0 \\ -j & 0 & 0 \end{bmatrix}$$

Wilkinson power divider for unequal power splits



Wilkinson power divider

Design a Wilkinson power divider with a power division ration of 3 dB and a source impedance of 50 Ω

Solution:

$$\frac{P_3}{P_2} = 0.5(3dB)$$

$$K^2 = \frac{P_3}{P_2} = \frac{1}{2} \Rightarrow K = 0.707$$

$$Z_{03} = Z_0\sqrt{\frac{1+K^2}{K^3}} = 50\sqrt{\frac{1+0.5}{(.5)(.707)}} = 103.0\,\Omega$$

$$Z_{02} = K^2 Z_{03} = (.5)(103\Omega) = 51.5\Omega$$

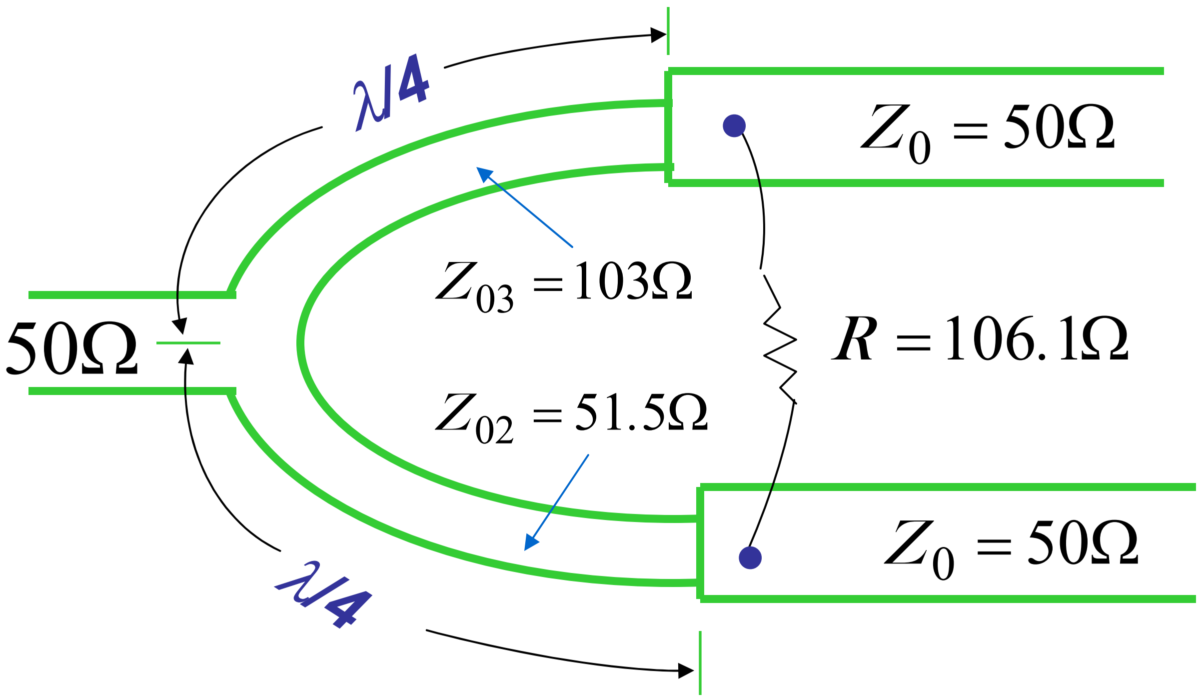
$$R = Z_0\left(K + \frac{1}{K}\right) = 50\left(0.707 + \frac{1}{0.707}\right) = 106.1\Omega$$

Wilkinson power divider

The output impedances are

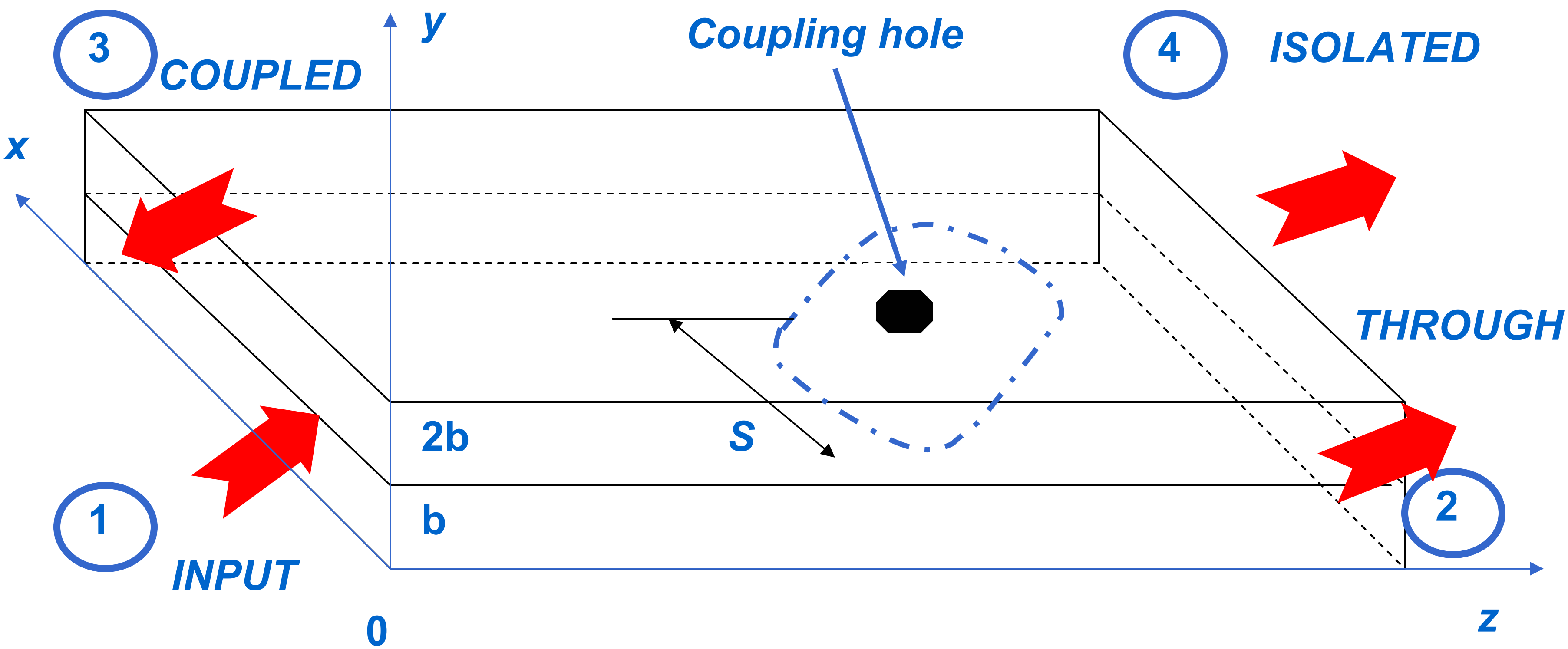
$$R_2 = Z_0 K = 50(0.707) = 35.35\Omega$$

$$R_3 = Z_0 / K = 50 / 0.707 = 70.72\Omega$$



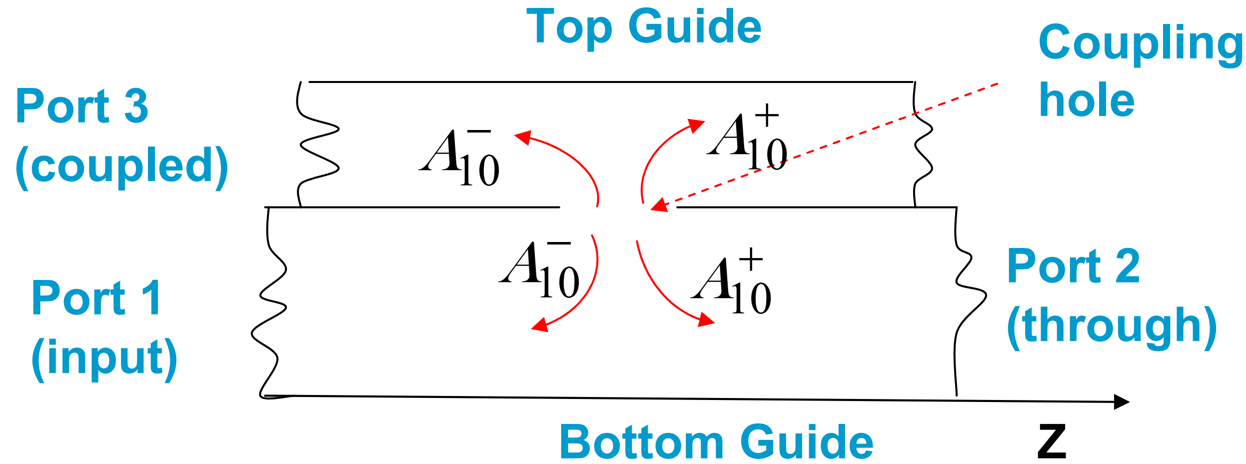
Bethe-Hole Directional Coupler

This the simplest form of a waveguide directional coupler. A small hole in the common broad wall between two rectangular guides provides 2 wave components that add in phase at the coupler port, and are cancelled at the isolation port.



Bethe-Hole Directional Coupler

Let the incident wave at Port 1 be the dominant TE_{10} mode:



$$E_y = A \sin\frac{\pi x}{a} e^{-j\beta z}$$

$$H_x = -\frac{A}{Z_{10}} \sin\frac{\pi x}{a} e^{-j\beta z}$$

$$H_z = \frac{j\pi A}{\beta a Z_{10}} \cos\frac{\pi x}{a} e^{-j\beta z}$$

A **= amplitude of electric field (V m^{-1})**

$$Z_0 = \frac{\eta_0}{\sqrt{1-(\lambda_0/2a)^2}}$$ **= wave impedance, dominate mode,Ω**

$$\beta = \kappa_0\sqrt{1-(\lambda_0/2a)^2}$$ **= phase constant *rad/m***

$$\kappa_0 = 2\pi/\lambda_0$$

Bethe-Hole Directional Coupler

In the bottom guide the amplitude of the forward scattered wave is given by

$$A_{10}^{+} = -\frac{j\omega A}{P_{10}}\left[\varepsilon_0 \alpha_e \sin^2\frac{\pi s}{a} - \frac{\mu_0 \alpha_m}{Z_{10}^2}\left(\sin^2\frac{\pi s}{a} + \frac{\pi^2}{\beta^2 a^2}\cos^2\frac{\pi s}{a}\right)\right]$$

while the amplitude of the reversed scattered wave is given by

$$A_{10}^{-} = -\frac{\omega A}{P_{10}}\left[\varepsilon_0 \alpha_e \sin^2\frac{\pi s}{a} + \frac{\mu_0 \alpha_m}{Z_{10}^2}\left(\sin^2\frac{\pi s}{a} - \frac{\pi^2}{\beta^2 a^2}\cos^2\frac{\pi s}{a}\right)\right]$$

where

$$P_{10} = \frac{ab}{Z_{10}}$$

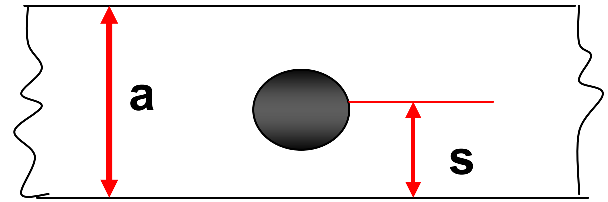
For round coupling hole or radius r_0, we have

$$\alpha_e = \frac{2}{3} r_0^2$$ electric polarizability

$$\alpha_m = \frac{4}{3} r_0^2$$ magnetic polarizability

Bethe-Hole Directional Coupler

Let ***s*** = offset distance to hole



We can then show that

$$\sin\frac{\pi s}{a} = \frac{\lambda_0}{\sqrt{2\left(\lambda_0^2 - a^2\right)}}$$

The coupling factor for a single-hole Bethe Coupler is

$$C = 20\log\left|\frac{A}{A_{10}^{-}}\right| \quad (dB)$$

and its directivity is

$$D = 20\log\left|\frac{A_{10}^{-}}{A_{10}^{+}}\right| \quad (dB)$$

Bethe-Hole Directional Coupler

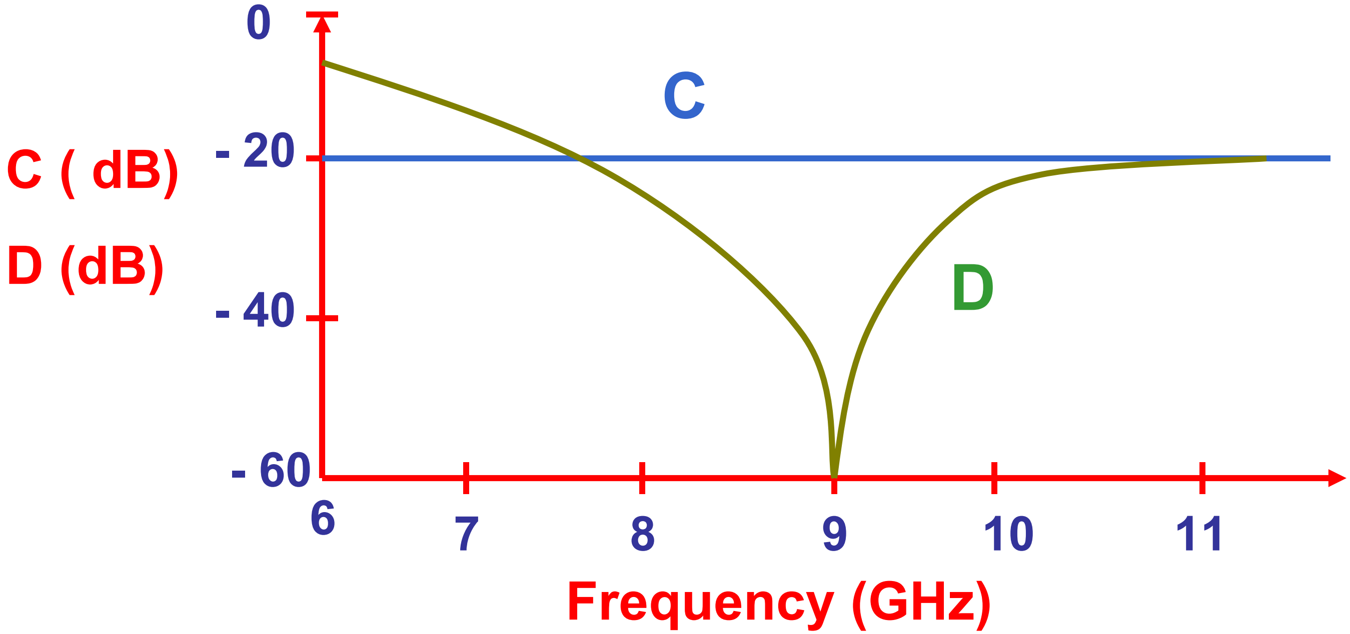
Design procedure:

1. Use $$\sin\frac{\pi s}{a} = \frac{\lambda_0}{\sqrt{2\left(\lambda_0^2 - a^2\right)}}$$ **to find position of hole.**

2. Use $$C = 20\log\left|\frac{A}{A_{10}^{-}}\right| \ (dB)$$ **to determine the hole radius r_0 to give the required coupling factor.**

Bethe-Hole Directional Coupler

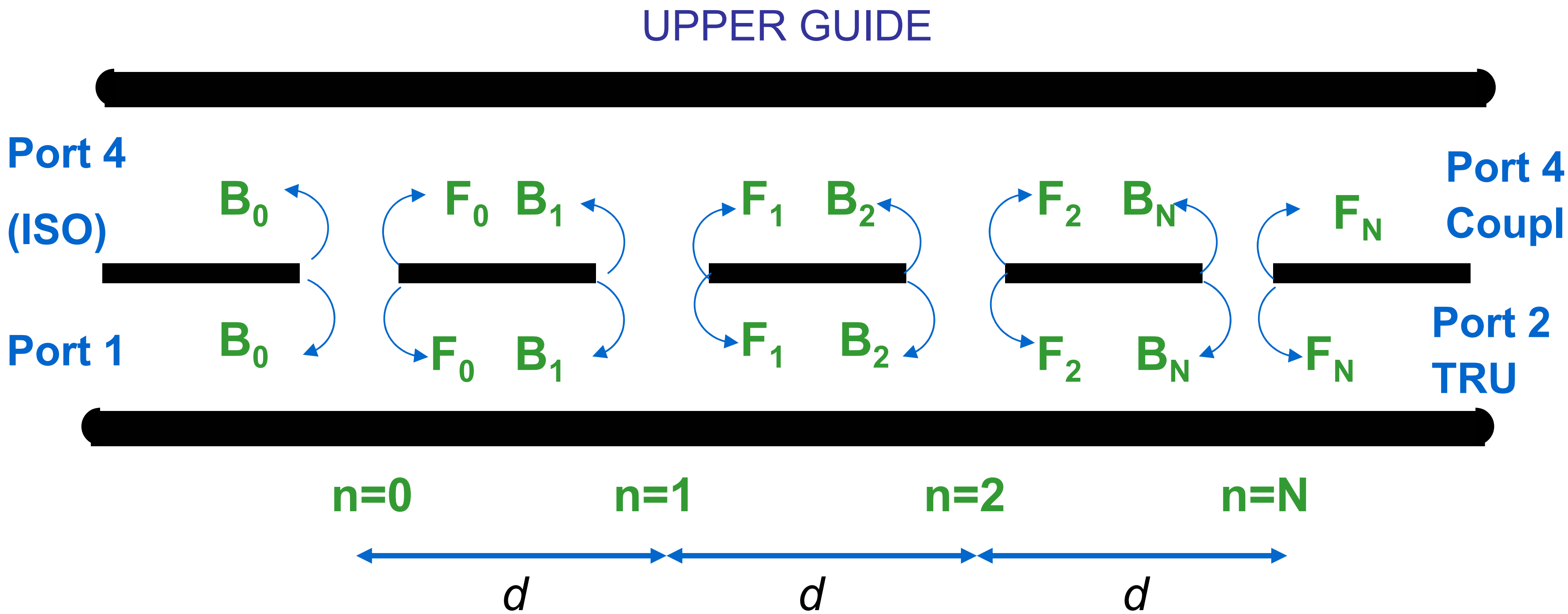
Typical x-Band -20 dB coupler



Note: Coupling very broad band, directivity is very narrow band (for single-hole coupler)

We can achieve improved directivity bandwidth by using an array of equispaced holes.

Bethe-Hole Directional Coupler



Let a wave of value $1\angle 0$ be injected at Port 1. If the holes are small, there is only a small fraction of the power coupled through to the upper guide so that we can assume that the wave amplitude incident on all holes is essentially unity. The hole n causes a scattered wave F_n to propagate in the forward direction, and another scattered wave B_n to propagate in the backward direction. Thus the output signals are:

Bethe-Hole Directional Coupler

Port 1 (input)
and Port 4 (isolated)

$$B^{(1)} = B^{(4)} = \sum_{n=0}^{N} B_n e^{-j2n\beta d}$$

Port 2 (through)

$$F_{Total}^{(2)} = \underbrace{e^{-jN\beta d}}_{main\ incident\ wave} + \underbrace{e^{-jN\beta d} \sum_{n=0}^{N} F_n}_{forward\ scattered\ waves}$$

Port 3 (coupled)

$$F^{(3)} = e^{-j2N\beta d} \sum_{n=0}^{N} F_n$$

All of these waves are phase referenced to the n = 0 hole.

$$C = -20\,log\left|F^{(3)}\right| = -20\,log\left|\sum_{n=0}^{N} F_n\right| \ (dB)$$

$$D = -20\,log\left|\frac{B^{(4)}}{F^{(2)}}\right| = -20\,log\left|\frac{\sum_{n=0}^{N} B_n e^{-j2\beta nd}}{\sum_{n=0}^{N} F_n}\right| \ (dB)$$

Bethe-Hole Directional Coupler

We can rewrite this as

$$D = -20\,log\left|\sum_{n=0}^{N} B_n e^{-j2\beta nd}\right| + 20\,log\left|\sum_{n=0}^{N} F_n\right|$$

$$= -C - 20\,log\left|\sum_{n=0}^{N} B_n e^{-j2\beta nd}\right|$$

The coupling coefficients are proportional to the polarizability α_e and α_m of the coupling holes. Let r_n = radius of the n^{th} hole. Then the forward scattering coefficient from the n^{th} hole is

$$F_n = A_{10}^{+}(n)$$

And the backward scattering from the hole is

$$B_n = A_{10}^{-}(n)$$

Bethe-Hole Directional Coupler

Now let us assume the coupling holes are located at the midpoint across common broad wall, i.e. s = a/2. Then for circular holed, we have

$$F_n = A_{10}^{+} = -j\frac{2\omega\varepsilon_0 A}{3P_{10}}\left[1-\frac{2\mu_0}{\varepsilon_0 Z_{10}^2}\right]r_n^3$$

But

$$\omega\varepsilon_0 = \frac{k_0}{\eta_0} \quad and \quad Z_{10}^2 = \frac{\eta_0^2}{\left(\sqrt{1-(\lambda_0/2a)^2}\right)} = \frac{\eta_0^2}{1-\left(f_c/f\right)^2}$$

$$\therefore F_n = K_f r_n^3 \quad \text{where} \quad K_f = \frac{-j2k_0 A}{3\eta_0 P_{10}}\left[1-2\left(1-\left(f_c/f\right)^2\right)\right]$$

Let A = 1 v/m. Then

$$K_f = \frac{-j2k_0}{3\eta_0 P_{10}}\left[2\left(\frac{f_c}{f}\right)^2-1\right]$$

Bethe-Hole Directional Coupler

Likewise the backward scattering coefficient is

$$K_b = \frac{2k_0}{3\eta_0 P_{10}}\left[2\left(\frac{f_c}{f}\right)^2 - 3\right] \qquad \text{and} \qquad \beta_n = K_b r_n^3$$

Note that K_f and K_b are frequency-dependent constants that are the same for all aperture. Thus,

$$C = -20\,log\left|K_f\right| - 20\,log\sum_{n=0}^{N} r_n^3 \quad (dB)$$

$$D = -C - 20\,log\left|K_b\right| - 20\,log\left|\sum_{n=0}^{N} r_n^3 e^{-j2\beta nd}\right| \quad (dB)$$

Bethe-Hole Directional Coupler

Consider the following design problem:

Given a desired coupling level C, how do we design the coupler so that the directivity D is above a value D_{min} over a specified frequency band?

Note that if the coupling C is specified, then $\left|\sum_{n=0}^{N} F_n\right|$ **is known.**

We now assume that either (1) the holes scatter symmetrically (e.g. they are on the common narrow wall between two identical rectangular guides) or (2) holes scatter asymmetrically (e.g. they are on the centerline of the common broad wall, i.e. *s=a/2*). Thus:

$$B_n = F_n$$

or

$$B_n = -F_n$$

In either case, we have

$$D = 20\,log\frac{\left|\sum_{n=0}^{N} F_n\right|}{\left|\sum_{n=0}^{N} F_n e^{-j2\beta nd}\right|}$$

Bethe-Hole Directional Coupler

Thus, keeping the directivity D > D_{min} is equivalent to keeping $\sum_{n=0}^{N} F_n e^{-j2\beta nd}$ below a related minimum value. Let

$$\varphi = -2\beta d \quad and\ w = e^{j\phi} = e^{-j2\beta d}$$

We also introduce the function

$$g(\beta d) = \left| \sum_{n=0}^{N} F_n e^{-j2\beta nd} \right| \Rightarrow g(\phi) = \left| \sum_{n=0}^{N} F_n e^{jn\phi} \right|$$

$$\therefore g(w) = \left| \sum_{n=0}^{N} F_n w^n \right| = F_N \sum_{n=0}^{N} \frac{F_n}{F_N} w^n = F_N \prod_{n=1}^{N} (w - w_n)$$

Thus we have

$$D = 20\,log \frac{|g(1)|}{|g(w)|}$$

$$|g(1)| = \left| \sum_{n=0}^{N} F_n \right| = 10^{-C/20}$$

Coupling factor (dB)

Bethe-Hole Directional Coupler

From the previous two equations we can deduce that

$$\left|g\right|_{max} = \left|g(1)\right| \times 10^{-D_{min}/20}$$

The multi-hole coupler design problem thus reduces to finding a set of roots w_n that will produce a satisfactory *g(w),* and thus a satisfactory *D(f)* in the desired frequency band under the constraint that $\left|g(w)\right| \le \left|g\right|_{max}$.

Example: Design a 7-hole directional coupler in C-band waveguide with a binomial directivity response to provide 15 dB coupling and with D_{min} =30 dB. Assume an operating center frequency of 6.45 GHz and a hole spacing d = $\lambda_g/4$ (or $\lambda_g + \lambda_g/4$). Also assume broad-wall coupling with s = a/2.

Solution:

From $g(w) = F_N \prod_{n=1}^{N} (w - w_n)$, we have

$$g(w) = F_6 (w - w_n)^6 \quad where \quad w_n = e^{-j2\beta d} = -1$$

$$\therefore g(w) = F_6 (w+1)^6 = F_6 \left(w^6 + 6w^5 + 15w^4 + 20w^3 + 15w^2 + 6w + 1\right)$$

Bethe-Hole Directional Coupler

Thus,

$$\left|g(1)\right| = \left|F_6\right|(1+1)^6 = 64\left|F_6\right| = 10^{-15/20} = 0.1778 \qquad \therefore \left|F_6\right| = 0.00278 = \left|F_0\right|$$

By the binomial expansion we have

$$(w+1)^6 = \sum_{n=0}^{6} C_n^{(6)} w^n$$

where, $C_n^{(6)} = \dfrac{N!}{(N-n)!\,n!} = \dfrac{6!}{(6-n)!\,n!}$ is the set of binomial coefficients

Thus

$$\left|F_5\right| = \left|F_1\right| = 6\left|F_6\right| = 0.01667$$

$$\left|F_4\right| = \left|F_2\right| = 15\left|F_6\right| = 0.04168$$

$$\left|F_3\right| = 20\left|F_6\right| = 0.05557$$

Bethe-Hole Directional Coupler

We now can compute the radii of the coupling holed from

$$F_n = K_f r_n^3 \qquad where \qquad K_f = \frac{-j2k_0 A}{3\eta_0 P_{10}}\left[1-2\left(1-\left({f_c}/{f}\right)^2\right)\right]$$

and

$$K_f = \frac{-j2k_0}{3\eta_0 P_{10}}\left[2\left(\frac{f_c}{f}\right)^2 - 1\right]$$

We have – with f_c = 4. 30 GHz **for C-Band guide,** f =6.45 GHz, $k_0 = 2\pi/\lambda_0 = 135.1\ m^{-1}$, $\eta_0 = 376.7\ \Omega$, $P_{10} = ab/Z_{10}$,

$$Z_{10} = \eta_0 \Big/ \sqrt{1-\left(\frac{f_c}{f}\right)^2} = 505.4\Omega,\ \ P_{10} = 1.08\times 10^{-6}\ \ m^2/\Omega$$

$$\left|K_f\right| = \frac{2\times 135.1}{3\times 376.7\times 1.08\times 10^{-6}}\left|\left(2\left(\frac{4.30}{6.45}\right)^2 - 1\right)\right| = 24598$$

Bethe-Hole Directional Coupler

The hole radii are:

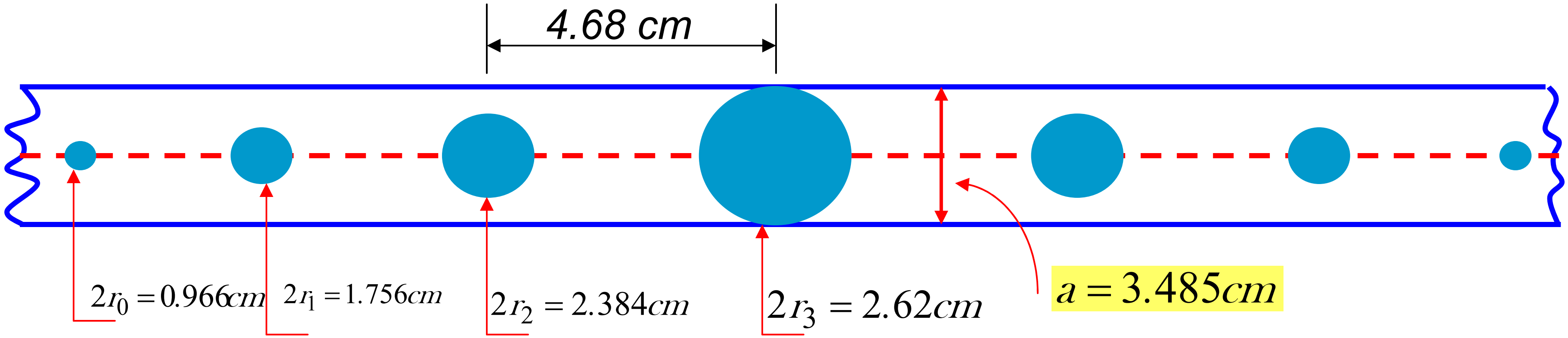
$$r_0 = \left(\frac{0.00278}{|K_f|} \right)^{1/3} = 0.00483m = r_6 \leftarrow \; 0.483cm$$

$$r_1 = \left(\frac{0.01667}{|K_f|} \right)^{1/3} = 0.00878m = r_5 \leftarrow \; 0.878cm$$

$$r_2 = \left(\frac{0.04168}{|K_f|} \right)^{1/3} = 0.011921m = r_4 \leftarrow \; 1.192cm$$

$$r_3 = \left(\frac{0.05557}{|K_f|} \right)^{1/3} = 0.0131m \leftarrow 1.31cm$$

Bethe-Hole Directional Coupler



Top view of C-Band guide common broad wall with coupling holes

The guide wavelength is $$\lambda_g = \frac{\lambda_0}{\sqrt{1-\left(\frac{4.3}{6.45}\right)^2}} = 0.624m$$

The nominal hole spacing is $d = \frac{\lambda_g}{4} = 1.56cm$ **. However, the center hole has a diameter of 2.62 cm, so it would overlap with adjacent holes. We can increase the hole spacing to** $d = \frac{3\lambda_g}{4} = 4.68cm$ **with no effect on electrical performance.**

The total length of the common broad wall section with coupling holes is ~ 30 cm, which is fairly large WG section.

Bethe-Hole Directional Coupler

We now plot the coupling and directivity vs. frequency

$$g(w) = F_6 (w+1)^6 = F_6 \left(e^{j\phi}+1\right)^6 = F_6 \left\{ e^{j\frac{\phi}{2}} \left(e^{j\frac{\phi}{2}} + e^{-j\frac{\phi}{2}} \right) \right\}^6 = F_6 \left[2e^{j\frac{\phi}{2}} \cos\frac{\phi}{2} \right]^6$$

$$\therefore \left| g(w) \right| = 2^6 \left| F_6 \right| \left| \cos\frac{\phi}{2} \right|^6 = 0.1778 \left| \cos\frac{\phi}{2} \right|^6$$

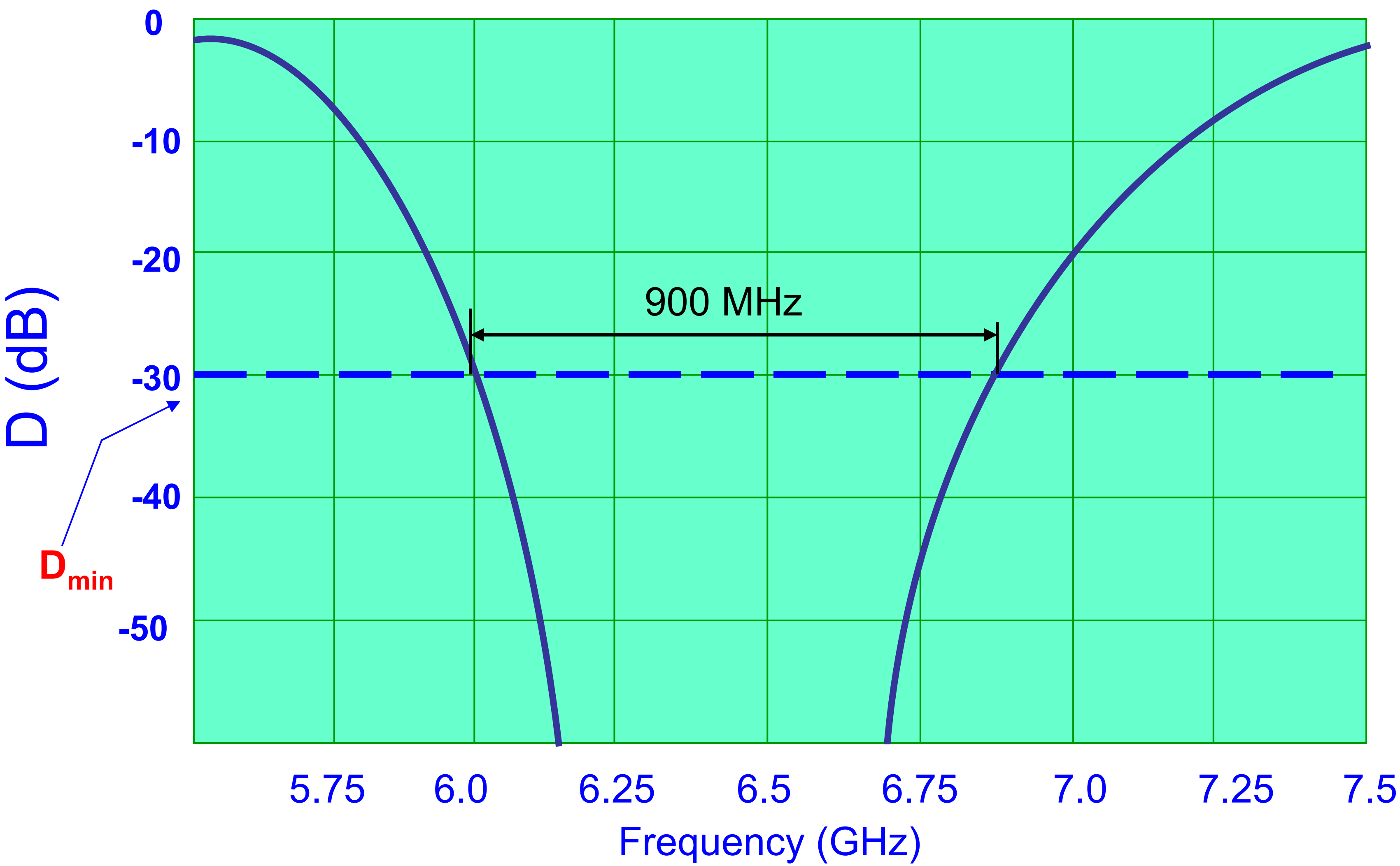
We then have

$$D(dB) = -20\, \log \frac{\left| g(w) \right|}{\left| g(1) \right|} = -120\, \log \left| \cos \frac{2\pi d}{\lambda_g} \right|$$

where *d=4.68 cm*

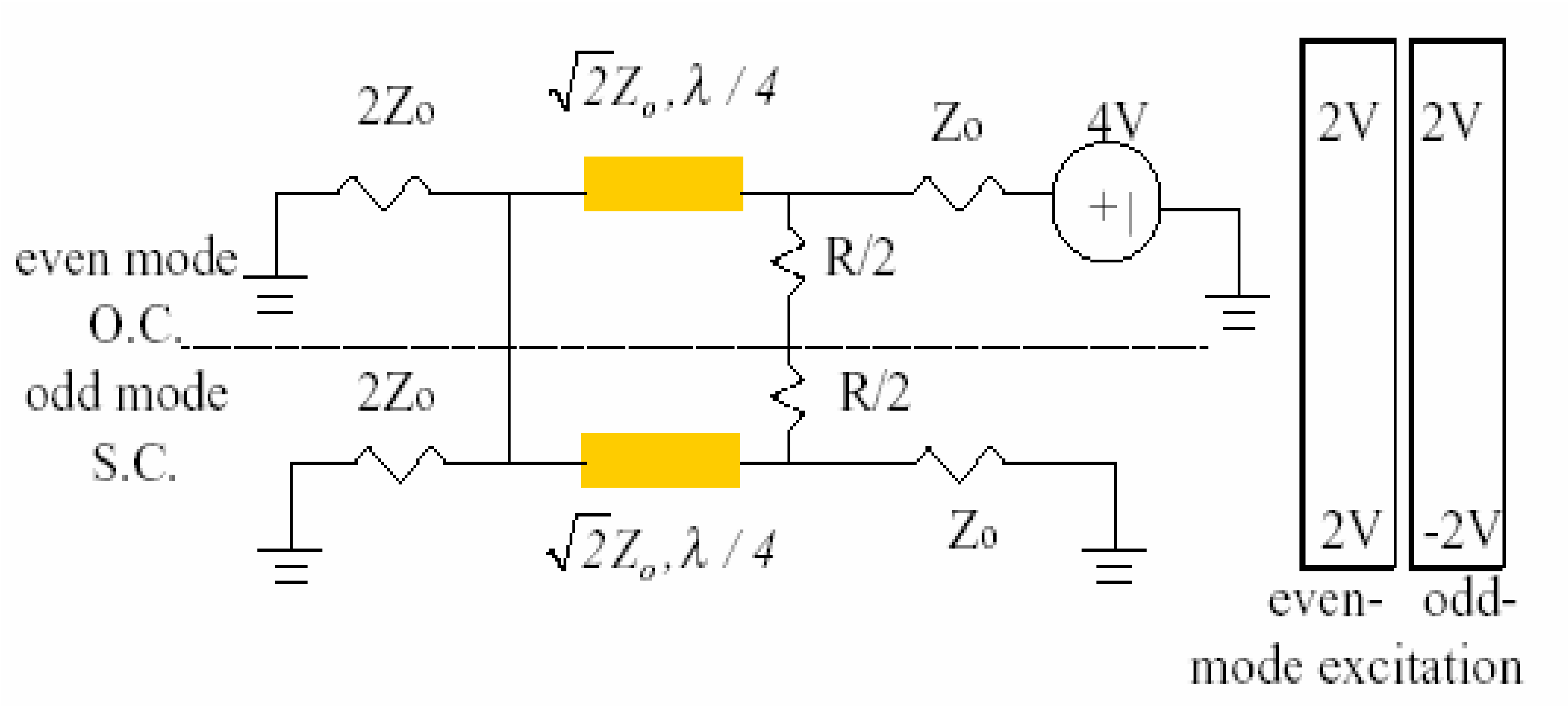
$$\lambda_g = \frac{\lambda_0}{\sqrt{1-(\lambda_0/2a)^2}} = \frac{\left(3\times 10^8/f\right)}{\sqrt{1-(f_c/f)^2}}$$

Bethe-Hole Directional Coupler

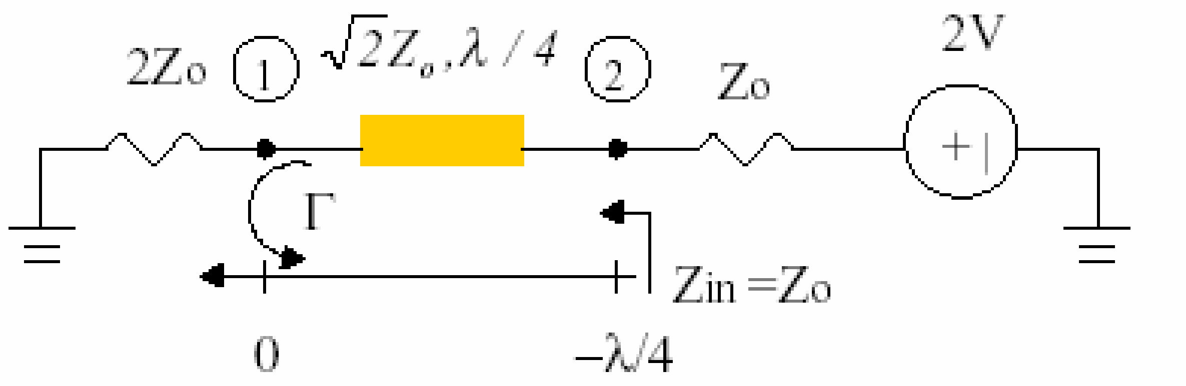


Note that the directivity is better than D_{min}= -30 dB over a bandwidth of 900 MHz centered about 6.45 GHz.

Even-odd mode analysis



Even-mode

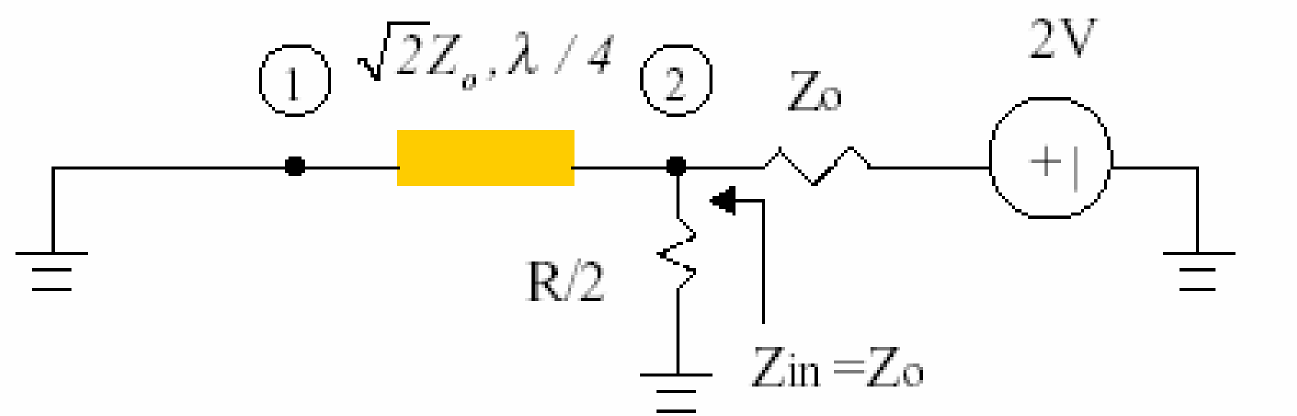


$$V_{2_e} = V \rightarrow S_{22_e} = 0$$

$$\tilde{A} = \frac{2-\sqrt{2}}{2+\sqrt{2}} \rightarrow V_{1_e} = jV\frac{\tilde{A}+1}{\tilde{A}-1} = -j\sqrt{2}V \rightarrow S_{12_e} = -j\sqrt{2}$$

Symmetry of port 2 and 3 → $V_{3_e} = V \rightarrow S_{33_e} = 0, S_{13_e} = -j\sqrt{2}$

Odd-mode



$$\frac{R}{2} = Z_o \rightarrow R = 2Z_o \rightarrow S_{22o} = 0 \Rightarrow S_{22} = \frac{1}{2}(S_{22e} + S_{22o}) = 0$$

$$V_{1o} = 0, V_{2o} = V \rightarrow S_{12o} = 0 \Rightarrow S_{12} = \frac{1}{2}(S_{12e} + S_{12o}) = -j\frac{1}{\sqrt{2}} = S_{21}$$

symmetry of port 2 and 3, $V_{3o} = -V \Rightarrow S_{13} = \frac{1}{2}(S_{12e} - S_{12o}) = -j\frac{1}{\sqrt{2}} = S_{31},$

$$S_{33} = 0$$

open, short at bisection $\Rightarrow S_{32} = S_{23}$

port 1 matched $\Rightarrow S_{11} = 0$

Discussion

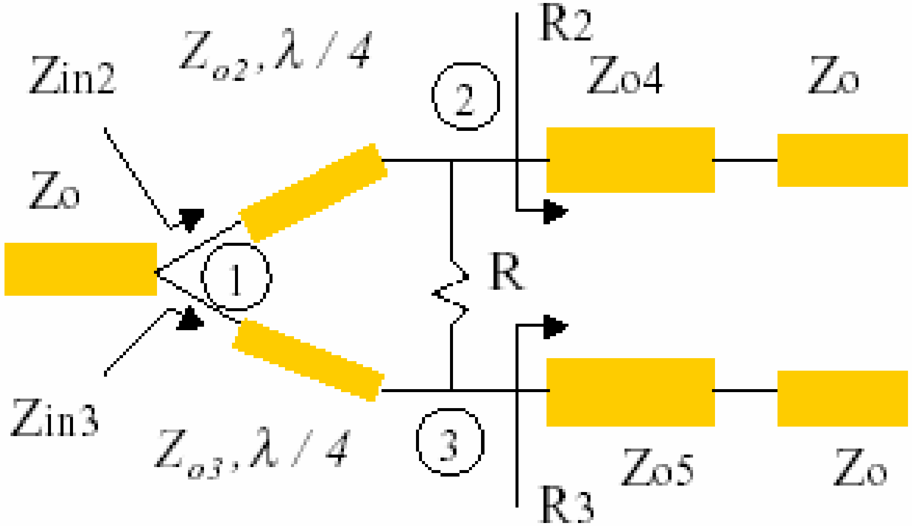
3dB Wilkinson power divider has equal amplitude and phase outputs at port 2 and port 3.

3dB Wilkinson power combiner

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -j & -j \\ -j & 0 & 0 \\ -j & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -j\frac{1}{\sqrt{2}}(a_2 + a_3) \\ 0 \\ 0 \end{bmatrix}$$

$$a_2 = a_3 \rightarrow P_1 = 2P_2$$

Unequal power division Wilkinson power divider



$$(1)\ \text{port1 match} \rightarrow Z_o = Z_{in2} // Z_{in3}$$

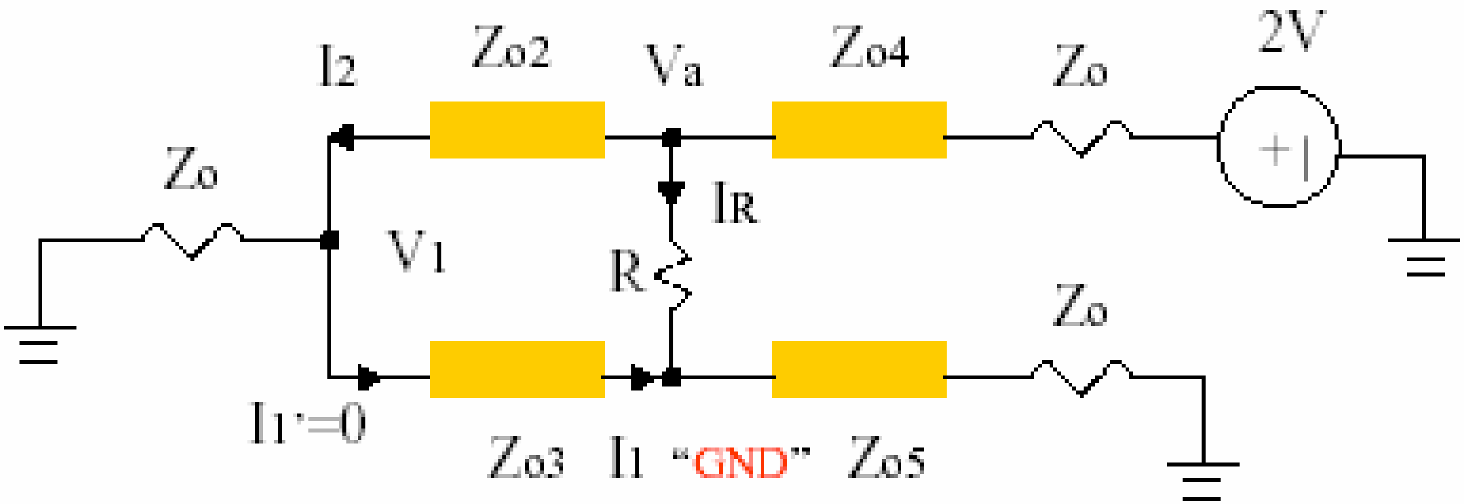
$$(2)\ \frac{P_3}{P_2} = K^2 \rightarrow \frac{V_3^2}{Z_{in3}} = K^2 \frac{V_2^2}{Z_{in2}}$$

$$(3)\ V_2 = V_3 \rightarrow Z_{in2} = K^2 Z_{in3}$$

$$(1),(3) \rightarrow Z_{in2} = (1+K^2)Z_o, Z_{in3} = \frac{1+K^2}{K^2} Z_o$$

$$R_2 = K^2 R_3, R_2 = KZ_o \rightarrow R_3 = \frac{Z_o}{K}, Z_{o4} = \sqrt{K} Z_o, Z_{o5} = \frac{Z_o}{\sqrt{K}}$$

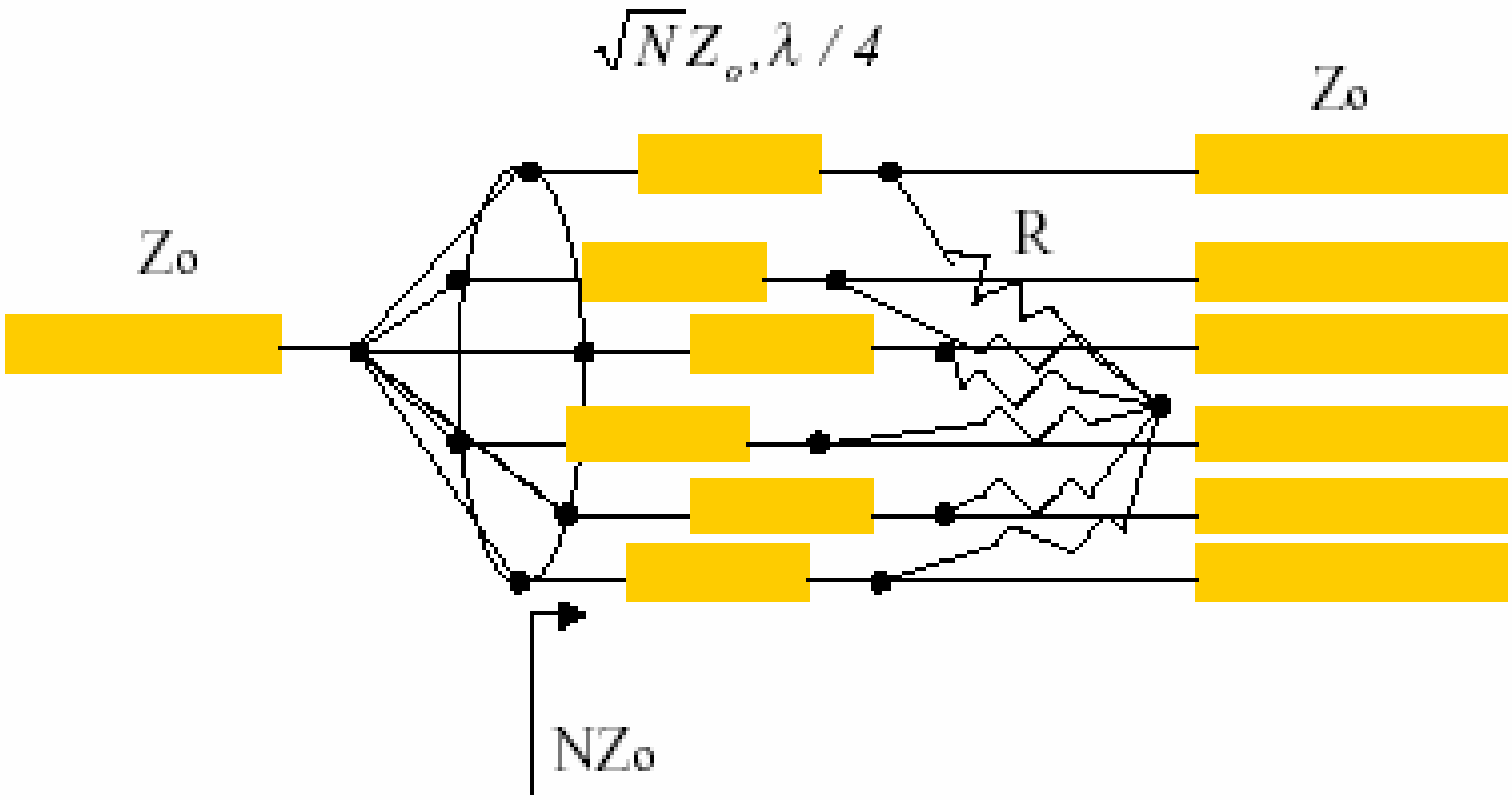
$$Z_{o2} = \sqrt{Z_{in2}R_2} = \sqrt{K(1+K^2)} Z_o, Z_{o3} = \sqrt{Z_{in3}R_3} = \sqrt{\frac{1+K^2}{K^3}} Z_o$$



$$V_1 = jI_1 Z_{o3} \rightarrow I_1 = \frac{V_1}{jZ_{o3}}$$

$$V_a = jZ_{o2} I_2, I_R = \frac{V_a}{R}, I_R + I_1 = 0 \rightarrow R = \frac{Z_{o2} Z_{o3}}{Z_o} = \frac{1+K^2}{K} Z_o$$

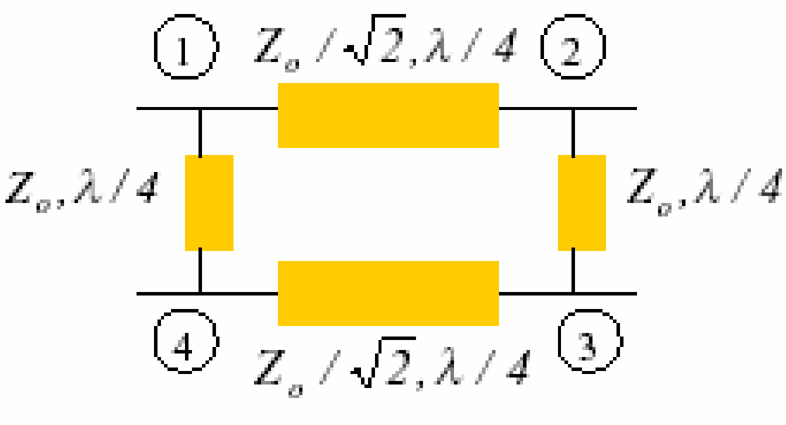
N-way Wilkinson power divider

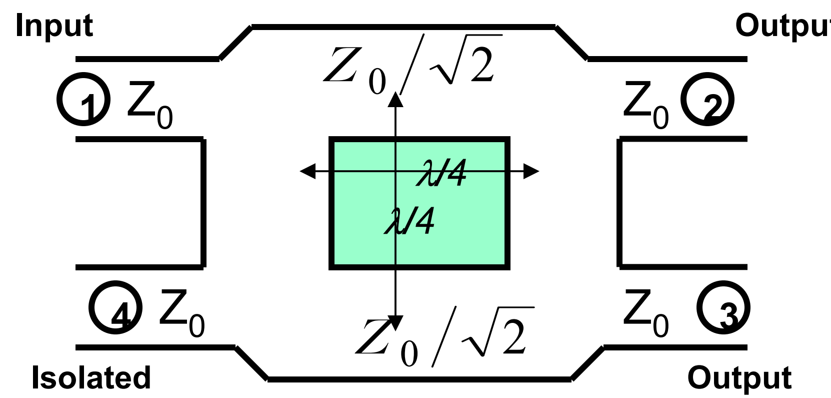


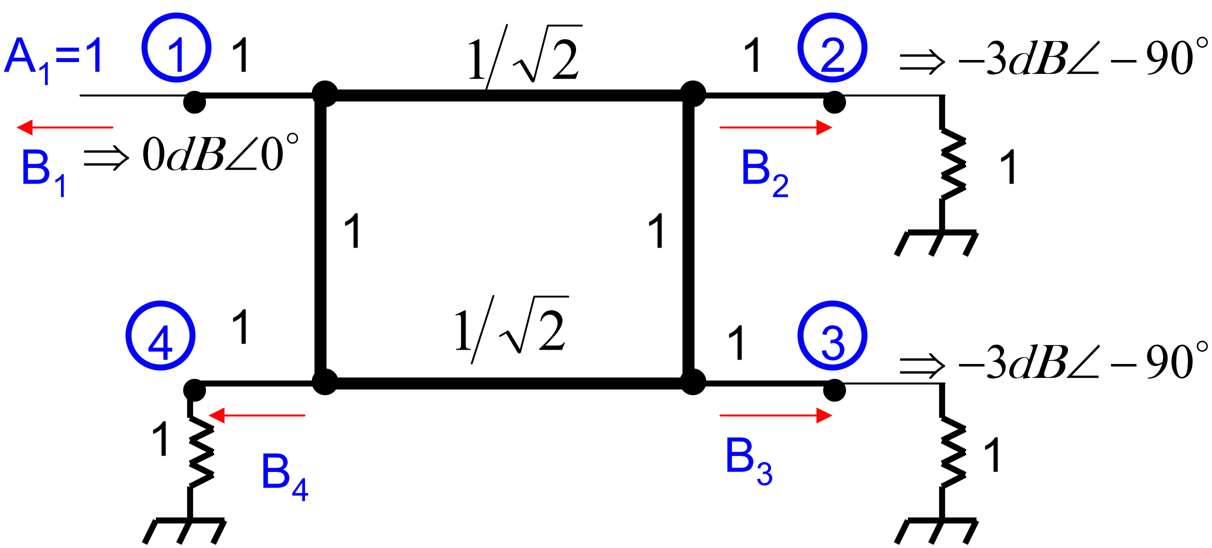
The quadrature (90°) hybrid

- **Branch-line coupler**

Port 2 and port 3 have equal amplitude, but 90° phase different



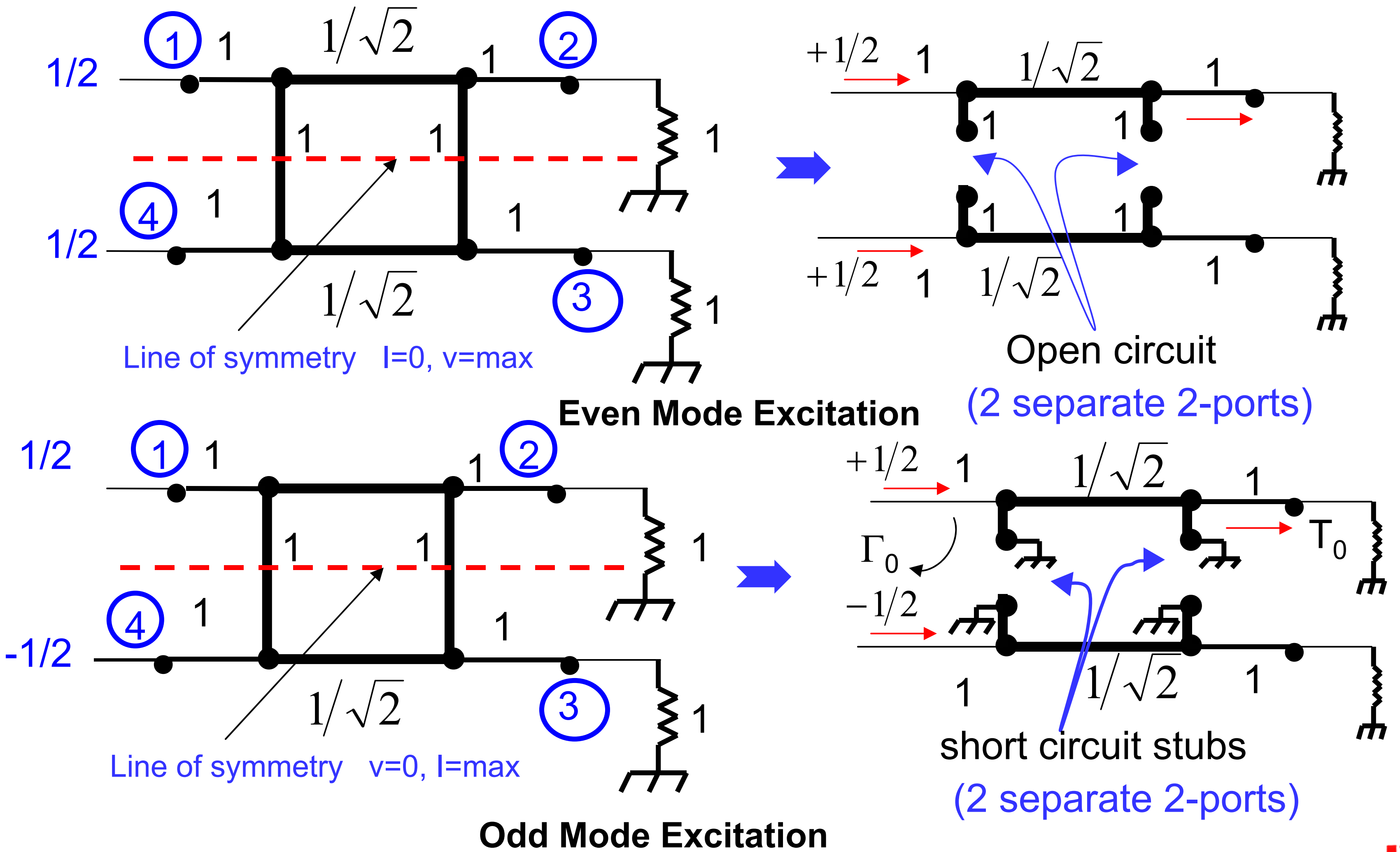




$$[S] = \frac{-1}{\sqrt{2}} \begin{bmatrix} 0 & j & 1 & 0 \\ j & 0 & 0 & 1 \\ 1 & 0 & 0 & j \\ 0 & 1 & j & 0 \end{bmatrix}$$

Quadrature Hybrids

We can analyze this circuit by using superposition of even-modes and 0dd-modes. We add the even-mode excitation to the odd-mode excitation to produce the original excitation of A_1=1 volt at port 1 (and no excitation at the other ports.)



Quadrature Hybrids

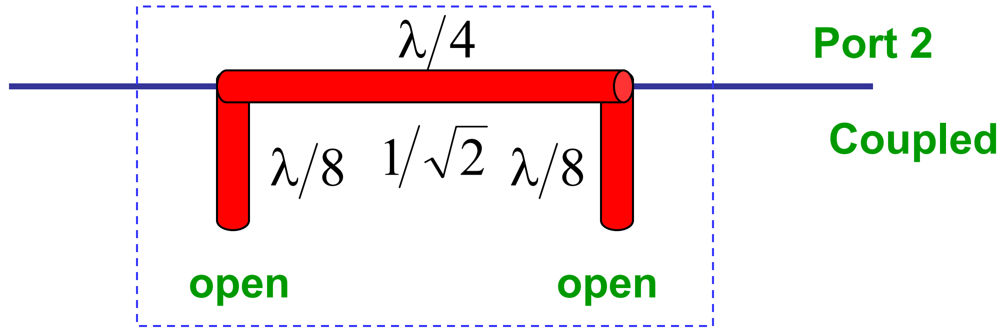
We now have a set of two decoupled 2-port networks. Let Γ_e and T_e be the reflection and transmission coefficients of the even-mode excitation. Similarly Γ_o and T_e for the odd-mode excitation.

Superposition: →

$$\left.\begin{aligned}
&\textit{Input} && B_1 = \frac{1}{2}\Gamma_e + \frac{1}{2}\Gamma_o \\
&\textit{Through} && B_2 = \frac{1}{2}\mathrm{T}_e + \frac{1}{2}\mathrm{T}_o \\
&\textit{Coupled} && B_3 = \frac{1}{2}\mathrm{T}_e - \frac{1}{2}\mathrm{T}_o \\
&\textit{Isolated} && B_4 = \frac{1}{2}\Gamma_e - \frac{1}{2}\Gamma_o
\end{aligned}\right\} \textit{Reflected waves}$$

Quadrature Hybrids

Consider the even-mode 2-port circuit:



We can represent the two $\lambda/8$ open circuit stubs by their admittance:

$$y = \lim_{z_L \to \infty} \frac{\dfrac{1}{z_L} + j\tan\dfrac{\pi}{4}}{1 + \dfrac{j}{z_L}\tan\dfrac{\pi}{4}} = j$$

The $\lambda/4$ transmission line, with characteristic impedance $1/\sqrt{2}$ has ab ABCD matrix

$$\begin{bmatrix} 0 & j/\sqrt{2} \\ j/\sqrt{2} & 0 \end{bmatrix}$$

Quadrature Hybrids

Thus, the ABCD matrix for the cascade is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_e = \underbrace{\begin{bmatrix} 1 & 0 \\ j & 1 \end{bmatrix}}_{\lambda/8\ stub} \underbrace{\begin{bmatrix} 0 & j/\sqrt{2} \\ \sqrt{j}2 & 0 \end{bmatrix}}_{\lambda/4\ line} \underbrace{\begin{bmatrix} 1 & 0 \\ j & 1 \end{bmatrix}}_{\lambda/8\ stub} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & j \\ j & -1 \end{bmatrix}$$

Using the conversion table (next slide) to convert [S] parameters (with Z_0=1 as the reference characteristic impedance).

$$\textbf{denominator} = A + \frac{B}{Z_0} + CZ_0 + D = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} + \frac{j}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$\Gamma_e = S_{11} = \frac{A + \frac{B}{Z_0} - CZ_0 - D}{A + \frac{B}{Z_0} + CZ_0 + D} = \frac{(-1 + j + j + 1)/\sqrt{2}}{(-1 + j + j - 1)/\sqrt{2}} = 0$$

$$\mathrm{T}_e = S_{21} = \frac{2}{A + \frac{B}{Z_0} + CZ_0 + D} = \frac{2}{(-1 + j + j - 1)/\sqrt{2}} = -\frac{1}{\sqrt{2}}(1 + j)$$

Similarly for odd mode we have: $S_{11} = \Gamma_0 = 0 \ \ and \ \ S_{12} = \mathrm{T}_0 = \frac{1}{\sqrt{2}}(1 - j)$

Conversion between two-port network parameters

	S	Z	Y	ABCD
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\|Y\|}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\|Y\|}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\|Y\|}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\|Y\|}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\|Z\|}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\|Z\|}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\|Z\|}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\|Z\|}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\|Z\|}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\|Y\|}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

$|Z| = Z_{11}Z_{22} - Z_{12}Z_{21}$; $|Y| = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$

D. Pozar

Quadrature Hybrids

Therefore we have

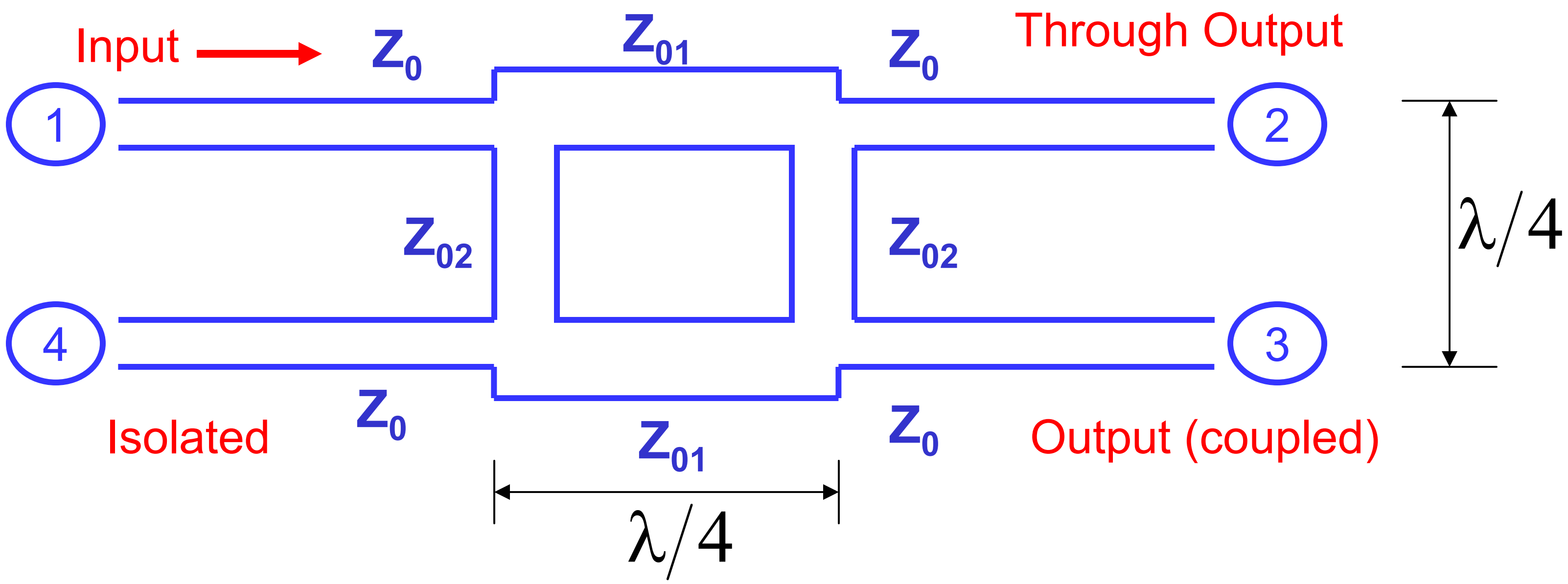
Scattered wave voltages

$$\begin{cases} B_1 = 0 & \text{(port 1 is matched)} \\ B_2 = -\dfrac{j}{\sqrt{2}} & \text{Through (half power, -90}^0\text{ phase shift port 1 to 2)} \\ B_3 = -\dfrac{1}{\sqrt{2}} & \text{(half power, -180}^0\text{ phase shift port 1 to 3)} \\ B_4 = 0 & \text{(no power to port 4)} \end{cases}$$

The bandwidth of a single branch-line hybrid is about 10% - 20%, due to the requirement that the top and bottom lines are λ/4 in length. We can obtain increased directivity bandwidth (with fairly constant coupling) by using three or more sections.

Quadrature Hybrids

Next we consider a more general single section branch-line coupler:



We can show that if the condition

$$\frac{Z_{02}}{Z_0} = \frac{Z_{01}/Z_0}{\sqrt{1-(Z_{01}/Z_0)^2}}$$

is satisfied, then port 1 is matched; port 4 is decoupled from port 1.

Quadrature Hybrids

Single section branch-line coupler

scattered wave voltages

$$\begin{cases} B_1 = 0 & \textit{matched } \angle 0^\circ \\ B_2 = -j\dfrac{Z_{01}}{Z_0} & \angle -90^\circ \\ B_3 = -\dfrac{Z_{01}/Z_0}{Z_{02}/Z_0} & \angle -180^\circ \\ B_4 = 0 & \textit{(matched)} \end{cases}$$

isolated

Thus, the directivity is theoretically infinite at the design frequency. We can also show that the coupling is given by

$$C = 10\,log\left[\frac{1}{1-\left(Z_{01}/Z_0\right)^2}\right]\ (dB)$$

For stripline + microstrip, we control Z_{01}/Z_0 by varying the strip width, in coax by adjusting the ratio b/a, and in the rectangular guide by changing the b dimension.

Quadrature Hybrids

Example:

Design a one-section branch-line directional coupler to provide a coupling of 6 dB. Assume the device is to be implemented in microstrip, with an 0.158 cm substrate thickness, a dielectric constant of 2.2, and that the operating frequency is 1.0 GHz.

Solution:

$$C = 10\,log\left[\frac{1}{1-\left(Z_{01}/Z_0\right)^2}\right] = 6 \quad (dB)$$

$$\therefore Z_{01}/Z_0 = 0.8653 \;\Rightarrow\; Z_{01} = 43.27\Omega$$

$$\frac{Z_{02}}{Z_0} = \frac{Z_{01}/Z_0}{\sqrt{1-\left(Z_{01}/Z_0\right)^2}} = 1.7263 \;\Rightarrow\; Z_{02} = 86.31\Omega$$

$$\lambda \cong \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{30}{\sqrt{2.2}} = 20.226\,cm \qquad \ell = \frac{\lambda}{4} = 5.0565\,cm$$

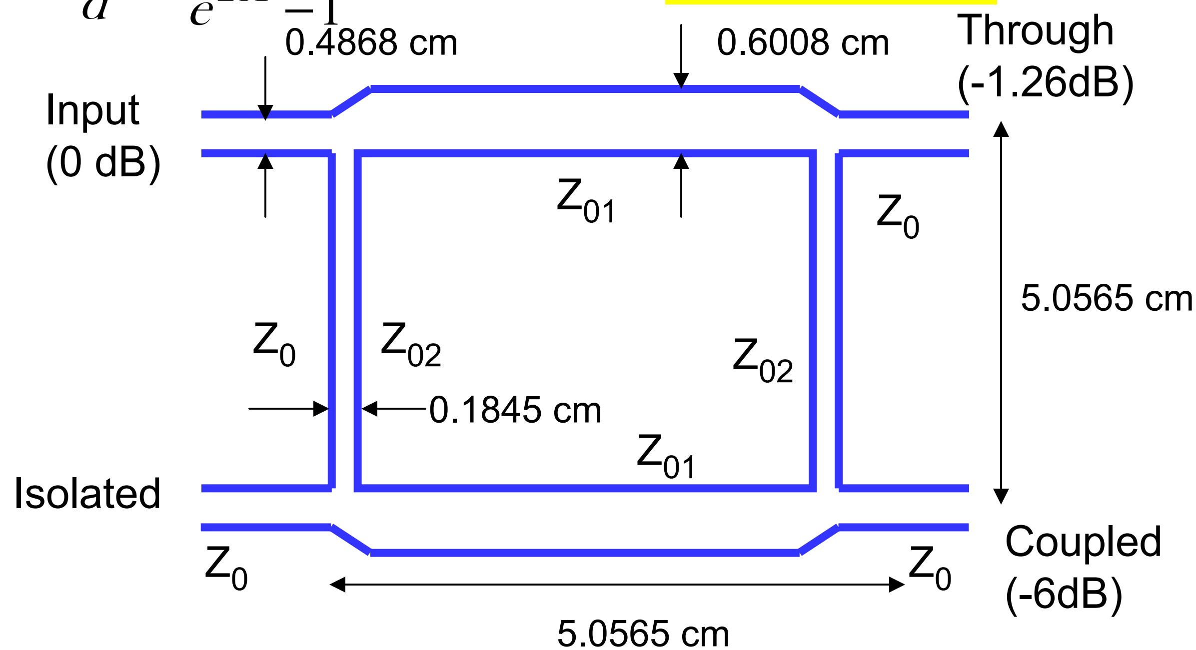
$$\frac{w_\circ}{d} = \frac{2}{\pi}\left\{B-1-ln(2B-1)+\frac{\varepsilon_r-1}{2\varepsilon_r}\left[ln\left(1B-1+.39-\frac{0.61}{\varepsilon_r}\right)\right]\right\} = 3.081$$

$$\Rightarrow w_\circ = 0.487 \;\; cm$$

Quadrature Hybrids

$$For \quad Z_{01} = 43.27\Omega, \qquad w_1 = 0.601cm$$

$$For \quad Z_{02} = 86.31\Omega, \qquad \frac{w_2}{d} = \frac{8e^{A}}{e^{2A}-1} = 1.1675 \longrightarrow w_2 = 0.185cm$$



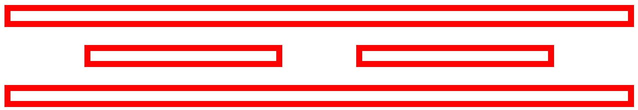
With 0 dB power input at the upper left arm, the power delivered to a matched load at the through arm is

$$P_2(dB) = -10\,log\frac{P_1^{(in)}}{P_2^{(out)}} = -10\,log\frac{1}{B_2B_2^*}$$

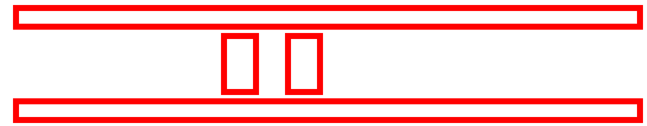
$$= -10\,log\left(\frac{Z_0}{Z_{01}}\right)^2 = -10\,log\left(\frac{50}{43.27}\right)^2 = -1.26dB$$

Coupled Line Directional Couplers

These are either stripline or microstrip 3-wire lines with close proximity of parallel lines providing the coupling.

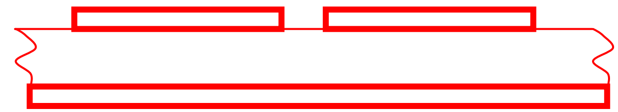


Co-planar stripline

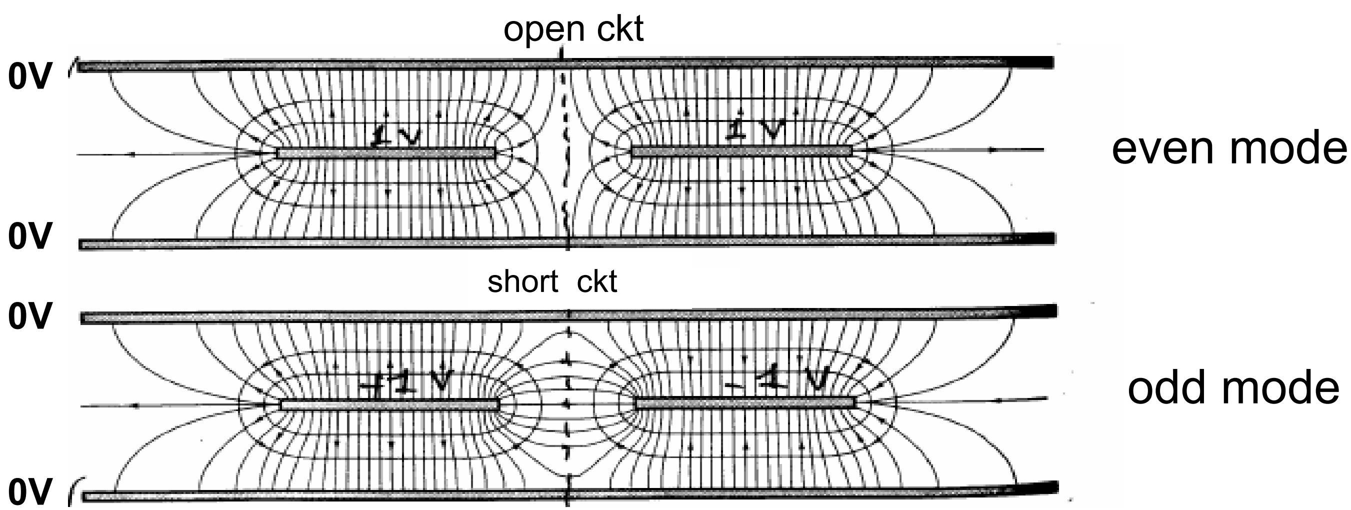


Side-stacked coupled stripline

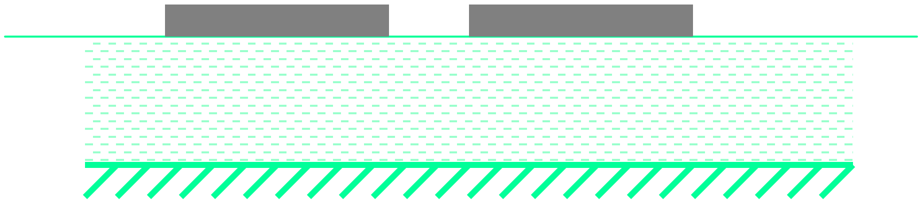
Broadside-stacked coupler stripline



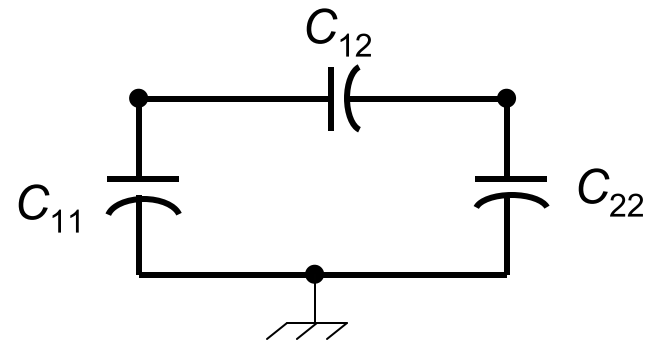
Co-planar microstrip



Theory of Coupled Lines



Three-wire coupled line



Equivalent network

C_{12} $\longrightarrow$ *capacitance between two strip conductors in absence of the ground conductor.*

C_{11}, C_{22} $\longrightarrow$ *capacitance between one conductor and ground*

In the even mode excitation, the currents in the strip conductors are equal in amplitude and in the same direction.

In the odd mode excitation, the currents in the strip conductors are equal in amplitude but are in opposite directions.

Theory of Coupled Lines

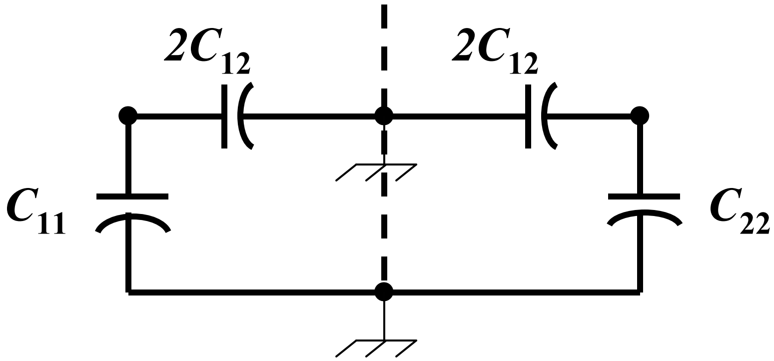
In the even mode, $\vec{E}$ fields have even symmetry about centerline, and no current flows between strip conductors. Thus C_{12} is effectively open-circuited. The resulting capacitance of either line to ground is

$$C_e = C_{11} = C_{22}$$

The characteristic impedance of the even modes is

$$Z_{0e} = \sqrt{\frac{L}{C_e}} = \frac{1}{\nu C_e}$$

In the odd mode, $\vec{E}$ fields have an odd symmetry about centerline, and a voltage null exists between the strip conductors.



The effective capacitance between either strip conductor and ground is

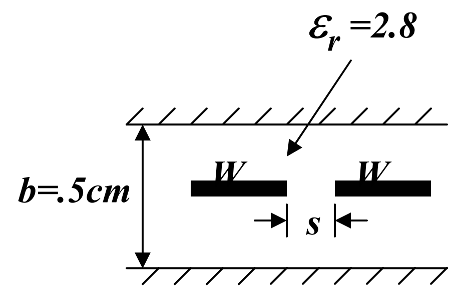
$$C_0 = C_{11} + 2\,C_{12} = C_{22} + 2\,C_{12}$$

$$Z_{0o} = \sqrt{\frac{L}{C_o}} = \frac{1}{\nu C_o}$$

Theory of Coupled Lines

Example: **An edge-coupled stripline with ε_r=2.8 and a ground plane spacing of 0.5 cm is required to have even- and odd-mode characteristic impedance of Z_{0e} 100 Ω and Z_{0o}=50 Ω. Find the necessary strip widths and spacing.**

Solution: $b = 0.50\ cm,\quad \varepsilon_r = 2.8,\quad Z_{0e}=100\ \Omega,\quad Z_{0o}=50\ \Omega$

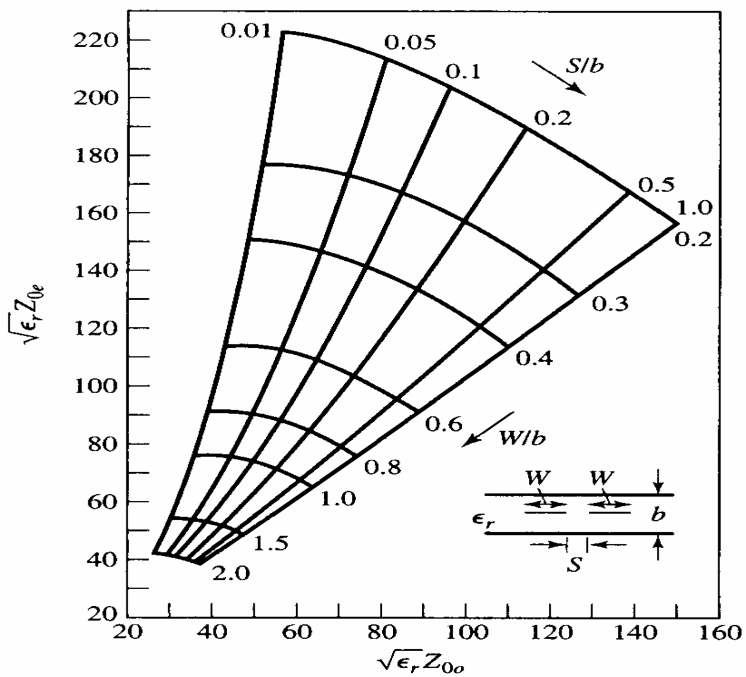


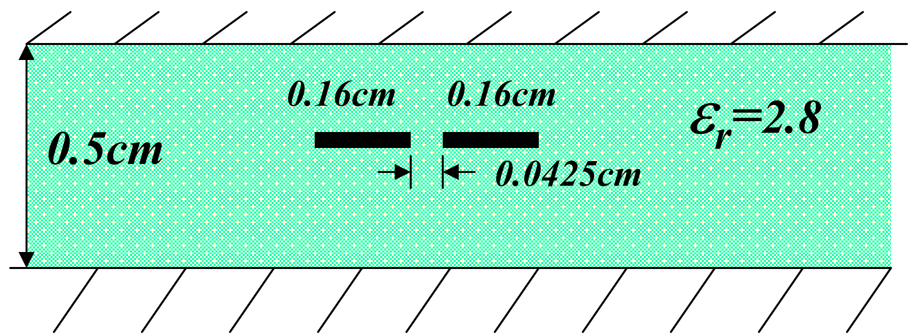
$$\therefore \underbrace{\sqrt{\varepsilon_r}Z_{0e} = 167.3}_{even} \qquad \underbrace{\sqrt{\varepsilon_r}Z_{0o} = 83.66}_{odd}$$

from the graph, $s/b \approx 0.095,\ W/b \approx 0.32$

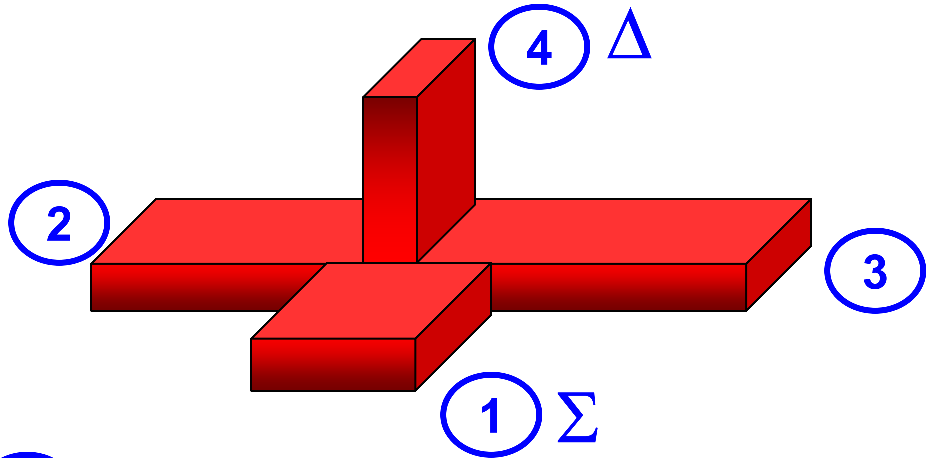
$$S=0.95\ b = 0.095 \times 0.5 = 0.0425\ cm$$

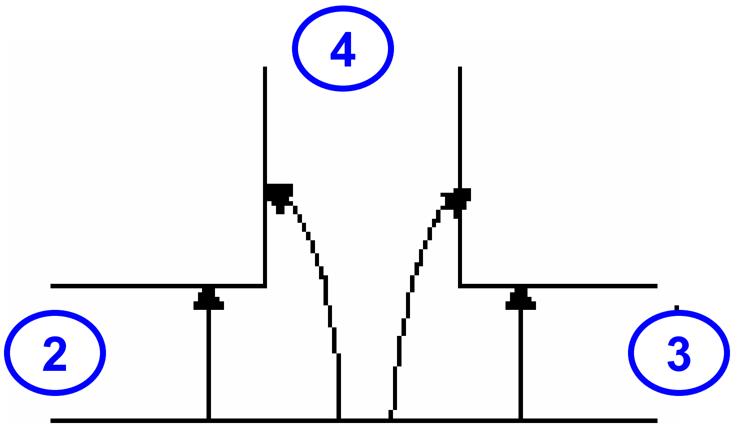
$$W=0.32b=0.32\times0.5=0.16\ cm$$





Waveguide magic-T

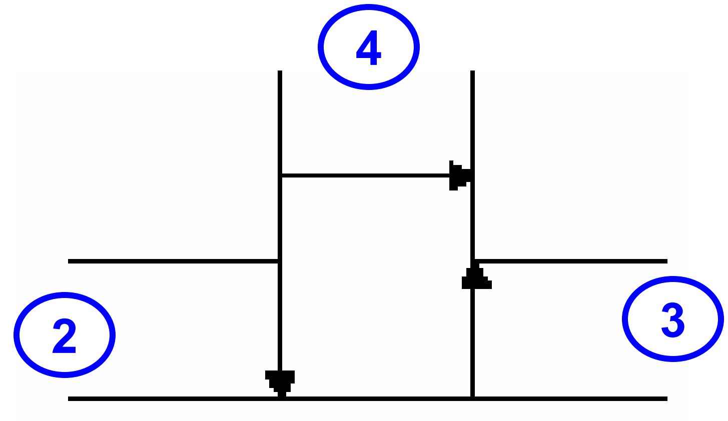




Input at port 1

Port 4:0

Ports 2 and 3: equal amplitude and phase



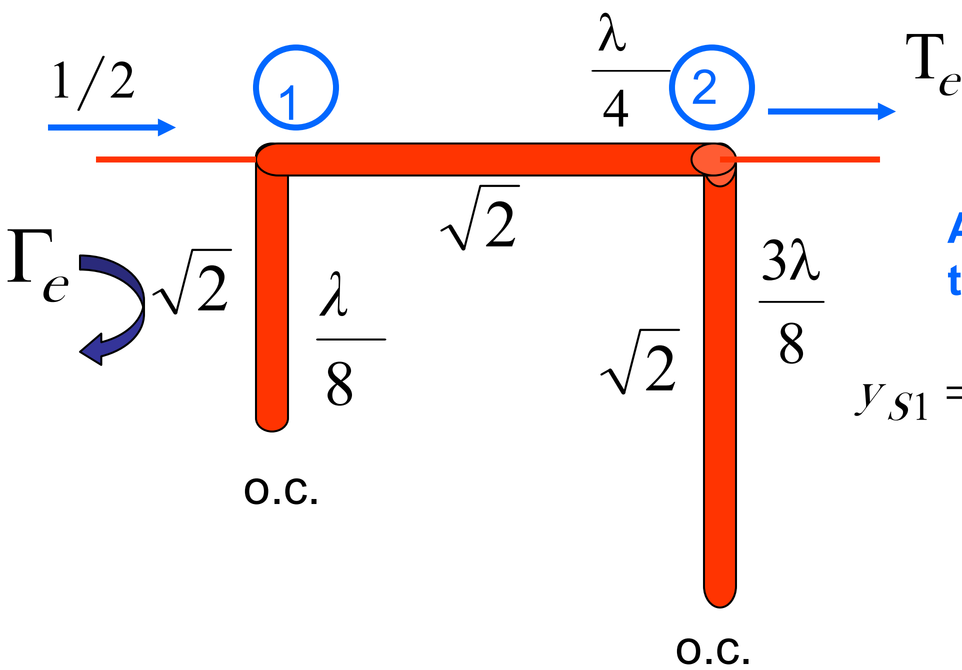
Input at port 4

Port 1:0

Ports 2 and 3: 180^0 phase difference

Even Mode ABCD Analysis

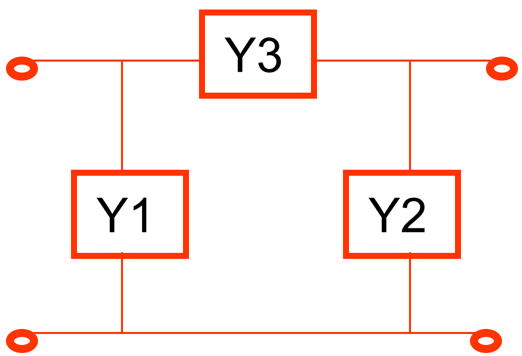
Consider the equivalent circuit for the even mode:



At port 1, the admittance looking into the λ/8 o.c. stub is

$$y_{S1} = y_0\left(j\tan\beta\ell\right) = \frac{1}{\sqrt{2}}\left(j\tan\frac{2\pi}{\lambda}\cdot\frac{\lambda}{8}\right) = \frac{j}{\sqrt{2}}$$

The ABCD matrix of a shunt admittance y_{S1} is



$A=1+Y_2/Y_3$

$B=1/Y_3$

$C=Y_1 + Y_2+Y_1Y_2/Y_3$

$D=1+Y_1/Y_3$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_1 = \begin{bmatrix} 1 & 0 \\ \frac{j}{\sqrt{2}} & 1 \end{bmatrix}$$

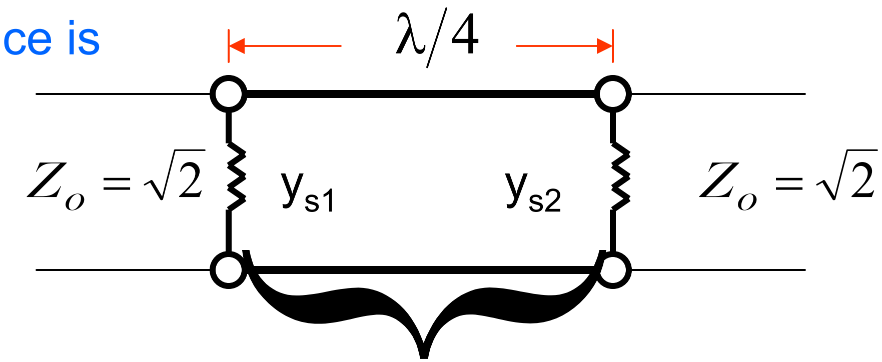
Even Mode ABCD Analysis

At port 2, the admittance looking into the $3\lambda/8$ o.c. stub is

$$y_{S2} = y_0\left(j\,tan\,\beta\ell\right) = \frac{1}{\sqrt{2}}\left(j\,tan\frac{2\pi}{\lambda}\cdot\frac{3\lambda}{8}\right) = -\frac{j}{\sqrt{2}}$$

The ABCD matrix of this shunt admittance is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_3 = \begin{bmatrix} 1 & 0 \\ -\dfrac{j}{\sqrt{2}} & 1 \end{bmatrix}$$



quarter-wave section

The ABCD matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_2 = \begin{bmatrix} cos\,\beta\ell & jZ_o\,sin\,\beta\ell \\ jy_o\,sin\,\beta\ell & cos\,\beta\ell \end{bmatrix}$$

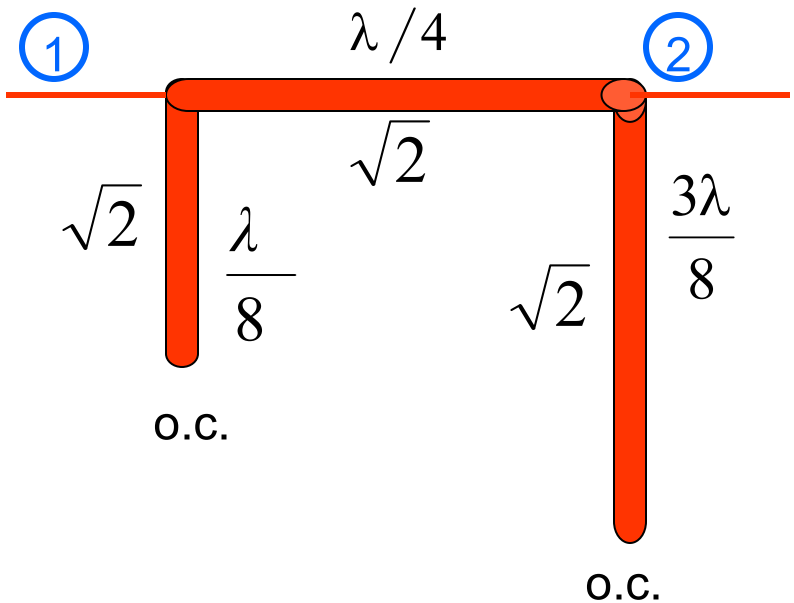
where $\beta\ell = 2\pi/\lambda \cdot \pi/4 = \pi/2$

$$Z_o = \sqrt{2},\, y_o = 1/\sqrt{2}$$

$$\therefore \begin{bmatrix} A & B \\ C & D \end{bmatrix}_2 = \begin{bmatrix} 0 & j\sqrt{2} \\ \dfrac{j}{\sqrt{2}} & 0 \end{bmatrix}$$

Even Mode ABCD Analysis

We can now compute the ABCD matrix of the even mode cascade



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_e = \begin{bmatrix} 1 & 0 \\ \frac{j}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 0 & j\sqrt{2} \\ \frac{j}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{j}{\sqrt{2}} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{j}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 1 & j\sqrt{2} \\ \frac{j}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} 1 & j\sqrt{2} \\ j\sqrt{2} & -1 \end{bmatrix}$$

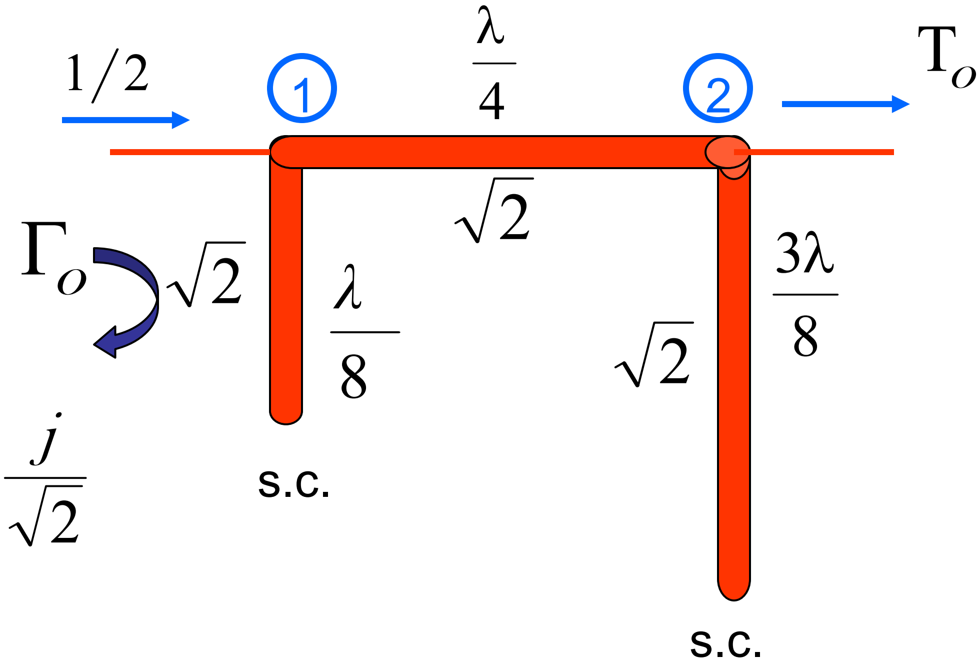
Odd Mode ABCD Analysis

The input admittance to a short-circuited lossless stub is

$$y_{in} = y_0\left(-j\cot\beta\ell\right)$$

Thus the input admittance to the s.c. $\lambda/8$ tub is

$$y_{S1} = \frac{1}{\sqrt{2}}\left(-j\cot\frac{2\pi}{\lambda}\cdot\frac{\lambda}{8}\right) = -\frac{j}{\sqrt{2}}$$



ABCD matrix:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_1 = \begin{bmatrix} 1 & 0 \\ \frac{-j}{\sqrt{2}} & 1 \end{bmatrix}$$

$$y_{S2} = \frac{1}{\sqrt{2}}\left(-j\cot\frac{2\pi}{\lambda}\cdot\frac{3\lambda}{8}\right) = +\frac{j}{\sqrt{2}}$$

(Input admittance to s.c. $3\lambda/8$ stub)

Odd Mode ABCD Analysis

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_3 = \begin{bmatrix} 1 & 0 \\ \dfrac{j}{\sqrt{2}} & 1 \end{bmatrix}$$

So the ABCD matrix of the odd-mode cascade is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_o = \begin{bmatrix} 1 & 0 \\ \dfrac{-j}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 0 & j\sqrt{2} \\ \dfrac{j}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{j}{\sqrt{2}} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & j\sqrt{2} \\ j\sqrt{2} & 1 \end{bmatrix}$$

$$\Gamma_e = -j/\sqrt{2}, \mathrm{T}_e = -j/\sqrt{2}$$

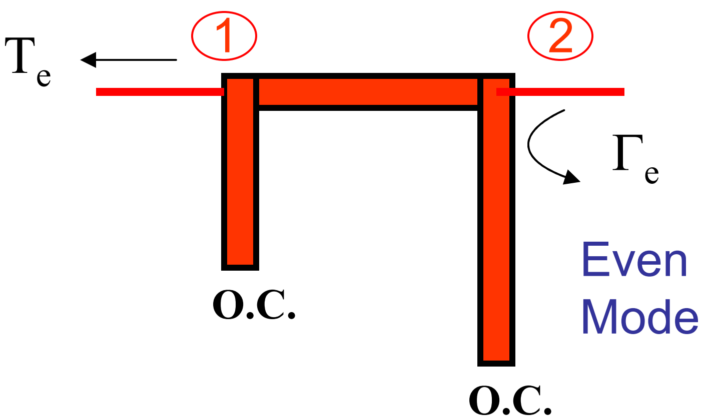
$$\Gamma_o = j/\sqrt{2}, \mathrm{T}_o = -j/\sqrt{2}$$

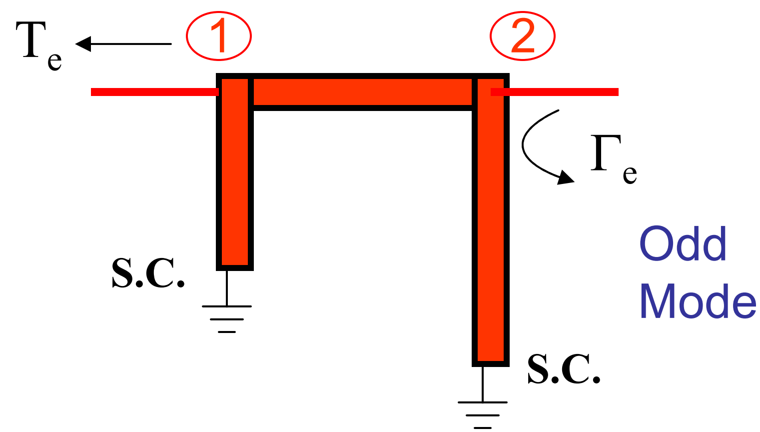
Excitation at Port 4

We have derived the ABCD matrices for the Even (e) and Odd (o) modes:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_e = \begin{bmatrix} 1 & j\sqrt{2} \\ j\sqrt{2} & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}_o = \begin{bmatrix} -1 & j\sqrt{2} \\ j\sqrt{2} & 1 \end{bmatrix}$$

For excitation at Port 4 instead of Port 1 the ABCD matrices remain the same. What changes are the definitions of Γ and T for each mode and their relations to $B_1 - B_4$.





Excitation at Port 4

Even mode: $$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_e = \begin{bmatrix} 1 & j\sqrt{2} \\ j\sqrt{2} & -1 \end{bmatrix}$$

$$\Gamma_e = S_{22} = \frac{-A + B/Z_o - CZ_o + D}{A + B/Z_o + CZ_o + D} \rightarrow \Gamma_e = \frac{-1 + j\sqrt{2} - j\sqrt{2} - 1}{1 + j\sqrt{2} + j\sqrt{2} - 1} = \frac{-2}{j2\sqrt{2}} = \frac{j}{\sqrt{2}}$$

$$\mathrm{T}_e = S_{12} = \frac{2(AD - BC)}{A + B/Z_o + CZ_0 + D} \rightarrow \mathrm{T}_e = \frac{2(-1+2)}{1 + j\sqrt{2} + j\sqrt{2} - 1} = \frac{2}{j2\sqrt{2}} = -\frac{j}{\sqrt{2}}$$

Odd mode: $$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_o = \begin{bmatrix} -1 & j\sqrt{2} \\ j\sqrt{2} & 1 \end{bmatrix}$$

$$\Gamma_o = S_{22} = \frac{1 + j\sqrt{2} - j\sqrt{2} + 1}{-1 + j\sqrt{2} + j\sqrt{2} + 1} = \frac{2}{j2\sqrt{2}} = -\frac{j}{\sqrt{2}}$$

$$\mathrm{T}_o = S_{12} = \frac{2(-1+2)}{-1 + j\sqrt{2} + j\sqrt{2} + 1} = \frac{2}{j2\sqrt{2}} = -\frac{j}{\sqrt{2}}$$

Excitation at Port 4 is expressed as :

Even $V_2^+ = 1/2$, $V_4^+ = 1/2$

Odd $V_2^+ = -1/2$, $V_4^+ = 1/2$

Excitation at Port 4 of Rat-Race Coupler (cont.)

$$\Gamma_e = \frac{j}{\sqrt{2}} \qquad \mathrm{T}_e = -\frac{j}{\sqrt{2}}$$

$$\Gamma_o = -\frac{j}{\sqrt{2}} \qquad \mathrm{T}_o = -\frac{j}{\sqrt{2}}$$

Output waves

$$B_1 = \frac{1}{2}\mathrm{T}_e - \frac{1}{2}\mathrm{T}_o$$

$$B_2 = \frac{1}{2}\Gamma_e - \frac{1}{2}\Gamma_o$$

$$B_3 = \frac{1}{2}\mathrm{T}_e + \frac{1}{2}\mathrm{T}_o$$

$$B_4 = \frac{1}{2}\Gamma_e + \frac{1}{2}\Gamma_o$$

The resulting output vector for unit excitation at Port 4 is :

$$[B_i]_4 = \begin{bmatrix} 0 \\ j/\sqrt{2} \\ -j/\sqrt{2} \\ 0 \end{bmatrix}$$