Lecture 7

Waveguides

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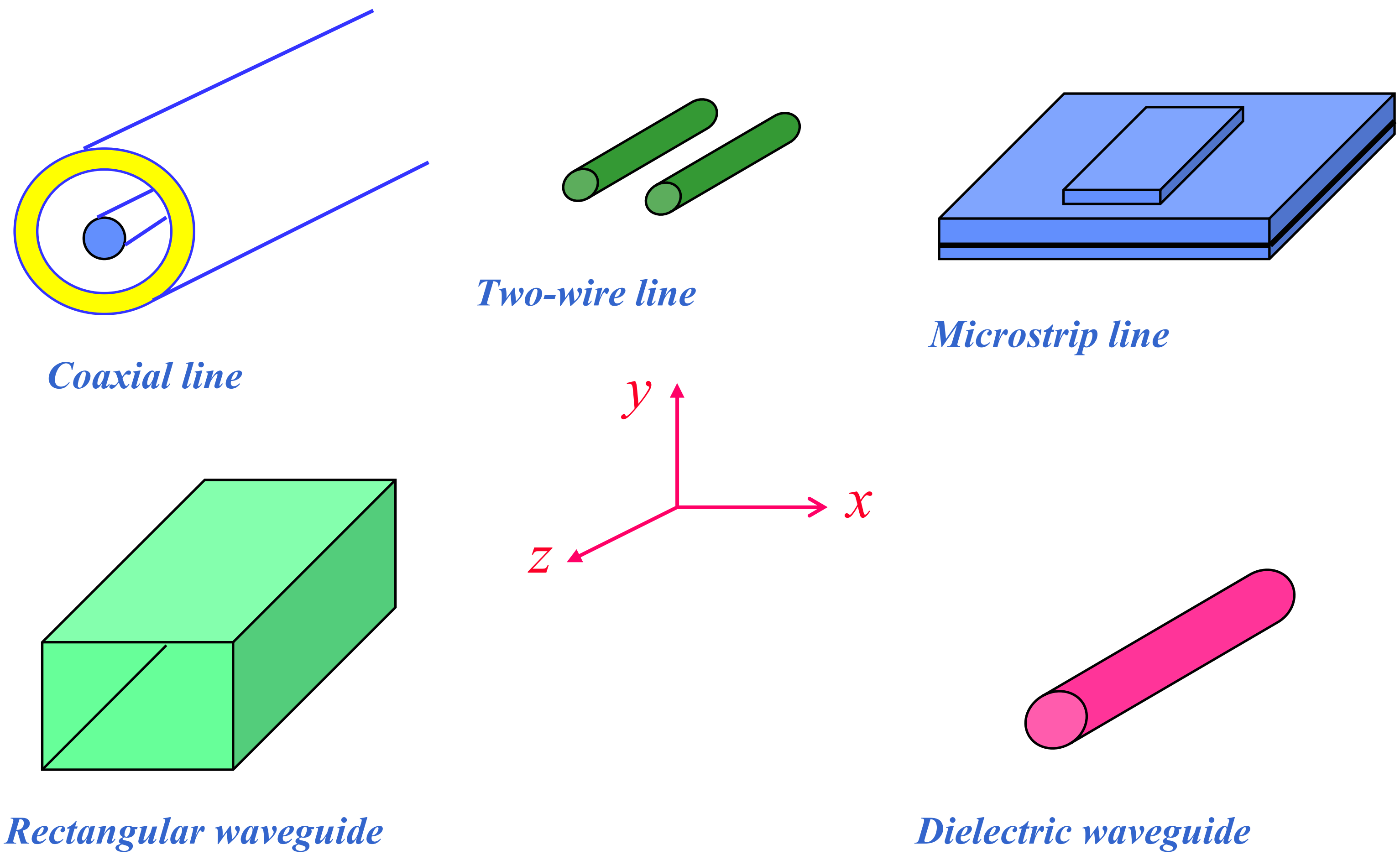
Waveguides

Waveguides are used to transfer electromagnetic power efficiently from one point in space to another.



Waveguides

In practice, the choice of structure is dictated by: (a) the desired operating frequency band, (b) the amount of power to be transferred, and (c) the amount of transmission losses that can be tolerated.

Coaxial cables are widely used to connect RF components. Their operation is practical for frequencies below 3 GHz. Above that the losses are too excessive. For example, the attenuation might be 3 dB per 100 m at 100 MHz, but 10 dB/100 m at 1 GHz, and 50 dB/100 m at 10 GHz. Their power rating is typically of the order of one kilowatt at 100 MHz, but only 200 W at 2 GHz, being limited primarily because of the heating of the coaxial conductors and of the dielectric between the conductors (dielectric voltage breakdown is usually a secondary factor.)

Another issue is the single-mode operation of the line. At higher frequencies, in order to prevent higher modes from being launched, the diameters of the coaxial conductors must be reduced, diminishing the amount of power that can be transmitted. Two-wire lines are not used at microwave frequencies because they are not shielded and can radiate. One typical use is for connecting indoor antennas to TV sets. Microstrip lines are used widely in microwave integrated circuits.

Waveguides

In a waveguide system, we are looking for solutions of Maxwell's equations that are propagating along the guiding direction (the z direction) and are confined in the near vicinity of the guiding structure. Thus, the electric and magnetic fields are assumed to have the form:

$$E(x,y,z;t) = E(x,y)e^{j\omega t - j\beta z}$$

$$H(x,y,z;t) = H(x,y)e^{j\omega t - j\beta z}$$

Where β is the propagation wave number along the guide direction. The corresponding wavelength, called the guide wavelength, is denoted by $\lambda_g = 2\pi/\beta$.

The precise relationship between ω and β depends on the type of waveguide structure and the particular propagating mode. Because the fields are confined in the transverse directions (the x, y directions,) they cannot be uniform (except in very simple structures) and will have a non-trivial dependence on the transverse coordinates x and y. Next, we derive the equations for the phasor amplitudes E (x, y) and H (x, y).

Waveguides

Because of the preferential role played by the guiding direction z, it proves convenient to decompose Maxwell's equations into components that are longitudinal, that is, along the z-direction, and components that are transverse, along the x, y directions. Thus, we decompose:

$$E(x,y)=\underbrace{\hat{x}E_x(x,y)+\hat{y}E_y(x,y)}_{transverse}+\underbrace{\hat{z}E_z(x,y)}_{longitudinal}\equiv E_T(x,y)+\hat{z}E_z(x,y)$$

In a similar fashion we may decompose the gradient operator:

$$\nabla=\hat{x}\partial_x+\hat{y}\partial_y+\hat{z}\partial_z=\nabla_T+\hat{z}\partial_z=\nabla_T-j\beta\hat{z}$$

Where we made the replacement $\partial_z \rightarrow -j\beta$ because of the assumed z-dependence. Introducing these decompositions into the source-free Maxwell's equation we have:

$$\nabla\times E=-j\omega\mu H \qquad (\nabla_T-j\beta\hat{z})\times(E_T+\hat{z}E_z)=-j\omega\mu(H_T+\hat{z}H_z)$$

$$\nabla\times H=j\omega\varepsilon E \qquad (\nabla_T-j\beta\hat{z})\times(H_T+\hat{z}H_z)=j\omega\varepsilon(E_T+\hat{z}E_z)$$

$$\nabla\cdot E=0 \qquad (\nabla_T-j\beta\hat{z})\cdot(E_T+\hat{z}E_z)=0$$

$$\nabla\cdot H=0 \qquad (\nabla_T-j\beta\hat{z})\cdot(H_T+\hat{z}H_z)=0$$

Transverse and Longitudinal Components

$$\nabla^2 \vec{E} = \left(\nabla_{xy}^2 + \nabla_z^2\right)\vec{E}$$

$$= \left(\nabla_{xy}^2 + \frac{\partial^2}{\partial z^2}\right)\vec{E}$$

$$= \left(\nabla_{xy}^2 + \gamma^2\right)\vec{E}$$

break up the operator
into transverse &
longitudinal components

The wave equations become now

$$\nabla_{xy}^2 \vec{E} + \left(\gamma^2 + k^2\right)\vec{E} = 0$$

$$\nabla_{xy}^2 \vec{H} + \left(\gamma^2 + k^2\right)\vec{H} = 0$$

Solution strategy

We still have (seemingly) six simultaneous equations to solve. In fact, the 6 are NOT independent. This looks complicated! Adopt a strategy of expressing the transverse fields (the E_x,E_y, H_x,H_y components in terms of the longitudinal components E_z and H_z only. If we can do this we only need find E_z and H_z from the wave equations….Too easy eh!

The first step can be carried out directly from the two curl equations from the original Maxwell's eqns. Writing these out:

First step

$$\frac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x \quad (1) \qquad \frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x \quad (4)$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \quad (2) \qquad -\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad (5)$$

$$\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad (3) \qquad \frac{\partial H_y}{\partial x} + \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \quad (6)$$

All $\frac{\partial}{\partial z}$ replaced by -γ. All fields are functions of x and y only.

Result

Now, manipulate to express the transverse in terms of the longitudinal. E.g. From (1) and (5) eliminate E_y

$$\frac{\partial E_z}{\partial y} + \frac{\gamma}{j\omega\varepsilon}\left(-\gamma H_x - \frac{\partial H_z}{\partial x}\right) = -j\omega\mu H_x$$

longitudinal

transverse

$$\Rightarrow \quad H_x = \frac{-1}{k_c^2}\left(\gamma\frac{\partial H_z}{\partial x} - j\omega\varepsilon\frac{\partial E_z}{\partial y}\right)$$

where $\boxed{k_c^2 = \gamma^2 + k^2}$

k_c is an eigenvalue (to be discussed)

The other components

$$H_x = \frac{-1}{k_c^2}\left(\gamma \frac{\partial H_z}{\partial x} - j\omega\varepsilon \frac{\partial E_z}{\partial y}\right)$$

$$H_y = \frac{-1}{k_c^2}\left(\gamma \frac{\partial H_z}{\partial y} + j\omega\varepsilon \frac{\partial E_z}{\partial x}\right)$$

$$E_x = \frac{-1}{k_c^2}\left(\gamma \frac{\partial E_z}{\partial x} + j\omega\mu \frac{\partial H_z}{\partial y}\right)$$

$$E_y = \frac{-1}{k_c^2}\left(\gamma \frac{\partial E_z}{\partial y} - j\omega\mu \frac{\partial H_z}{\partial x}\right)$$

So find solutions for E_z and H_z and then use these 4 eqns to find all the transverse components

We only need to find E_z and H_z now!

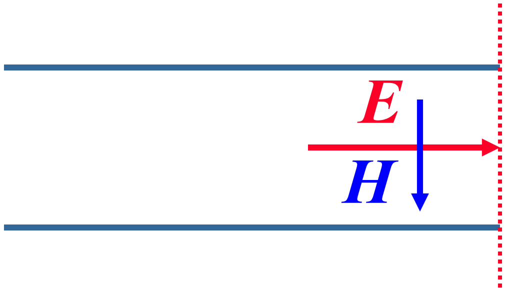
Wave type classification

It is convenient to to classify as to whether E_z or H_z exists according to:

TEM:	$E_z = 0$	$H_z = 0$
TE:	$E_z = 0$	$H_z \neq 0$
TM	$E_z \neq 0$	$H_z = 0$

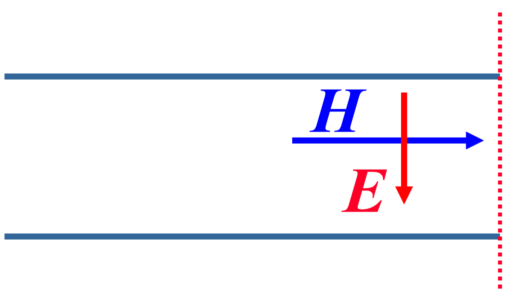
We will first see how TM wave types propagate in waveguide
Then we will infer the properties of TE waves.

The TE modes of a parallel plate wave guide are preserved if perfectly conducting walls are added perpendicularly to the electric field.



The added metal plate does not disturb normal electric field and tangent magnetic field.

On the other hand, TM modes of a parallel wave guide disappear if perfectly conducting walls are added perpendicularly to the magnetic field.



The magnetic field cannot be normal and the electric field cannot be tangent to a perfectly conducting plate.

TM waves (H_z=0)

$$\nabla^2_{xy} E_z + \left(\gamma^2 + k^2\right) E_z = 0$$

Longitudinal: 2nd order PDE for E_z. we defer solution until we have defined a geometry plus b/c.

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y}$$

$$H_y = -\frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$E_x = -\frac{\gamma}{k_c^2}\frac{\partial E_z}{\partial x}$$

$$E_y = -\frac{\gamma}{k_c^2}\frac{\partial E_z}{\partial y}$$

Transverse solutions once E_z is found

Further Simplification

The two E-components can be combined. If we use the notation:

$$\vec{E}_t = E_x \hat{x} + E_y \hat{y} = -\frac{\gamma}{k_c^2} \nabla_{xy} E_z \quad \text{where } \nabla_{xy} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$$

$$\boxed{\begin{aligned} E_t &= -\frac{\gamma}{k_c^2} \nabla_{xy} E_z \\ Z_{TM} &= \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\gamma}{j\omega\varepsilon}\,\Omega \\ \vec{H} &= \frac{\hat{z} \times \vec{E}}{Z_{TM}} \end{aligned}}$$

Eigenvalues

We will discover that in closed systems, solutions are possible only for discrete values of $k_{c.}$ There may be an infinity of values for k_c, but solutions are not possible for all k_c. Thus k_c are known as eigenvalues. Each eigenvalue will determine the properties of a particular TM mode. The eigenvalues will be geometry dependent.

Assume for the moment we have determined an appropriate value for k_c, we now wish to determine the propagation conditions for a particular mode.

We have the following propagation vector components for the modes in a rectangular wave guide

$$\beta^2 = \omega^2\mu\varepsilon = \beta_x^2 + \beta_y^2 + \beta_z^2$$

$$\beta_x = \frac{m\pi}{a}; \beta_y = \frac{n\pi}{a}$$

$$\beta_z^2 = \left(\frac{2\pi}{\lambda_z}\right)^2 = \left(\frac{2\pi}{\lambda_g}\right)^2 = \omega^2\mu\varepsilon - \beta_x^2 - \beta_y^2$$

$$\beta_z^2 = \omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{a}\right)^2$$

At the cut-off, we have

$$\beta_z^2 = 0 = (2\pi f_c)^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{a}\right)^2$$

Operating bandwidth

All waveguide systems are operated in a frequency range that ensures that only the lowest mode can propagate. If several modes can propagate simultaneously, one has no control over which modes will actually be carrying the transmitted signal. This may cause undue amounts of dispersion, distortion, and erratic operation.

A mode with cutoff frequency ω_c will propagate only if its frequency is $\omega \geq \omega_c$, or $\lambda < \lambda_c$. If $\omega < \omega_c$, the wave will attenuate exponentially along the guide direction. This follows from the ω,β relationship

$$\omega^2 = \omega_c^2 + \beta^2 c^2 \Rightarrow \beta^2 = \frac{\omega^2 - \omega_c^2}{c^2}$$

If $\omega \geq \omega_c$, the wavenumber β is real-valued and the wave will propagate. But if $\omega < \omega_c$, β becomes imaginary, say, $\beta = -j\alpha$, and the wave will attenuate in the z-direction, with a penetration depth $\delta = 1/\alpha$:

$$e^{-j\beta z} = e^{-\alpha z}$$

Operating bandwidth

If the frequency ω is greater than the cutoff frequencies of several modes, then all of these modes can propagate. Conversely, if ω is less than all cutoff frequencies, then none of the modes can propagate.

If we arrange the cutoff frequencies in increasing order, $\omega_{c1} < \omega_{c2} < \omega_{c3} < \cdots$, then, to ensure single-mode operation, the frequency must be restricted to the interval $\omega_{c1} < \omega < \omega_{c2}$, so that only the lowest mode will propagate. This interval defines the operating bandwidth of the guide.

This applies to all waveguide systems, not just hollow conducting waveguides. For example, in coaxial cables the lowest mode is the TEM mode having no cutoff frequency, $\omega_{c1} = 0$. However, TE and TM modes with non-zero cutoff frequencies do exist and place an upper limit on the usable bandwidth of the TEM mode. Similarly, in optical fibers, the lowest mode has no cutoff, and the single-mode bandwidth is determined by the next cutoff frequency.

The cut-off frequencies for all modes are

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2}$$

With cut-off wavelengths

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2}}$$

With indices

TE modes m=0,1,2,3,… TM modes m=1,2,3,…

n=0,1,2,3,… n=1,2,3,…

(but m=n=0 not allowed)

Cut-off

Since the wave propagates according to $e^{\pm\gamma z}$. Then propagation ceases when $\gamma = 0$.

since $\gamma = \sqrt{k_c^2 - \omega^2 \mu\varepsilon}$ then $\gamma = 0$ implies $\omega_c^2 \mu\varepsilon = k_c^2$

Or $$\boxed{f_c = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}}}$$

Cut-off frequency

Write γ in terms of f_c

It is usual, now to write γ in terms of the cut-off frequency. This allows us to physically interpret the result.

$$\gamma = \sqrt{k_c^2 - \omega^2 \mu\varepsilon} = \sqrt{k_c^2 - \frac{\omega^2 k_c^2}{\omega_c^2}} = k_c \sqrt{1 - \frac{f^2}{f_c^2}}$$

This part from the definition see slide 8/5.

Substitute for $\mu\varepsilon$ from definition of f_c

Recall similarity of this result with β for an ionized gas. see slide 7/12

$$\gamma^2 = k_c^2 - k^2$$

Conditions for Propagation

There are two possibilities here:

1 $\boxed{f > f_c}$ → γ is imaginary

$$\text{with } \gamma = j\beta = j\sqrt{k^2 - k_c^2}$$

$$= jk\sqrt{1 - \frac{k_c^2}{k^2}}$$

$$= jk\sqrt{1 - \frac{f_c^2}{f^2}}$$

This says now that γ becomes $j\beta$ with

$$\beta = \sqrt{k^2 - k_c^2}$$

this is a special case of the result in the previous slide

We conclude that if the operational frequency is above cut-off then the wave is propagating with the form $e^{-j\beta z}$

Different wavelengths

The corresponding wavelength inside the guide is

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - \frac{f_c^2}{f^2}}} > \lambda$$

g for guide

This is the "free space" wavelength

The free space wavelength may be written alternatively

$$\lambda = \frac{2\pi}{k}$$

Now if we introduce a cut-off wavelength $\lambda = v/f_c$ where v is the corresponding velocity (=c, in air) in an unbounded medium. We can derive:

$$\boxed{\frac{1}{\lambda^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}}$$

Dispersion in waveguides

The previous relationship showed that β was a function of frequency i.e. waveguides are dispersive. Hence we expect the phase velocity to also be a function of frequency. In fact:

$$v_p = \frac{\omega}{\beta} = \frac{v}{\sqrt{1 - \frac{f_c^2}{f^2}}} = \frac{\lambda_g}{\lambda} v > v$$

This can be > c!

So, as expected the phase velocity is always higher than in an unbounded medium (fast wave) and is frequency dependent. So we conclude waveguides are dispersive.

Group velocity

This is similar to as discussed previously.

$$v_g = \frac{1}{\dfrac{\partial \beta}{\partial \omega}} = v\sqrt{1 - \frac{f_c^2}{f^2}} = \frac{\lambda}{\lambda_g} v < v$$

So the group velocity is always less than in an unbounded medium. And if the medium is free space then $v_g v_p = v^2 = c^2$ which is also as previously discussed. Finally, recall that the energy transport velocity is the group velocity.

Wave Impedance

Wave impedance can also be written in terms of the radical:

$$\sqrt{1-\frac{f_c^2}{f^2}}$$

For $f>f_c$ Then the impedance is real and less than the surrounding medium dielectric

In particular:

$$\frac{\gamma}{j\omega\varepsilon}=\frac{k\sqrt{1-\frac{f_c^2}{f^2}}}{\omega\varepsilon}=\boxed{Z_{TM}=\eta\sqrt{1-\frac{f_c^2}{f^2}}}$$

The factor $k/\omega\varepsilon$ can be shown to be:

$$\sqrt{\frac{\mu_0}{\varepsilon}}=\eta=377\Omega\ \text{(if air)}$$

Evanescent waves

2 $\boxed{f < f_c}$ $\longrightarrow$ γ is real

$$\text{with } \gamma = \alpha = k\sqrt{1-\frac{k_c^2}{k^2}} = k\sqrt{1-\frac{f_c^2}{f^2}}$$

We conclude that the propagation is of the form $e^{-\alpha z}$ i.e. the wave is attenuating or is evanescent as it propagates in the $+z$ direction. This is happening for frequencies below the cut-off frequency. At $f=f_c$ the wave is said to be cut-off. Finally, note that there is no loss mechanism contributing to the attenuation.

Impedance for evanescent waves

A similar derivation to that for the propagating case produces:

$$Z_{TM} = -j\frac{k_c}{\omega\varepsilon}\sqrt{1-\frac{f^2}{f_c^2}}$$

This says that for TM waves, the wave impedance is capacitive and that no power flow occurs if the frequency is below cut-off. Thus evanescent waves are associated with reactive power only.

TE Waves

A completely parallel treatment can be made for the case of TE propagation, $E_z = 0, H_z \neq 0$. We only give the parallel results.

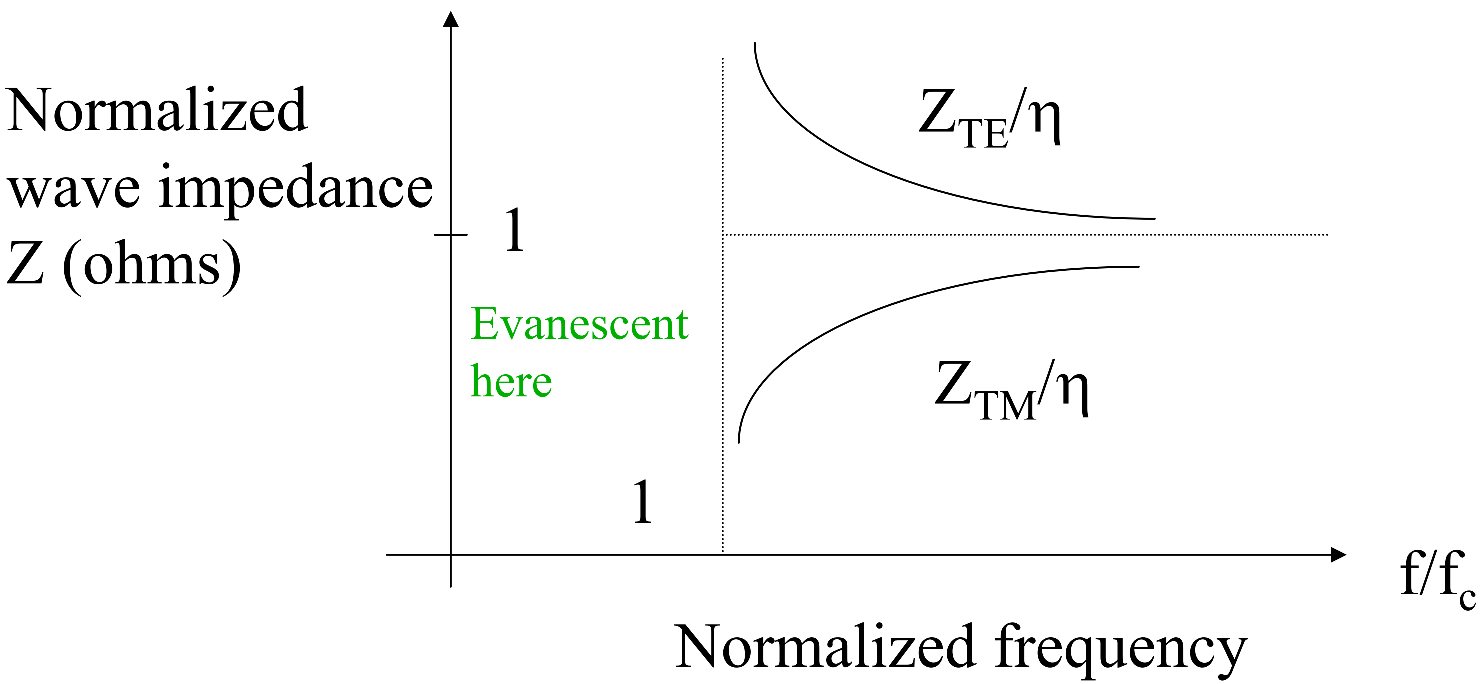
$$\nabla_{xy}^2 H_z + \left(\gamma^2 + k^2\right) H_z = 0$$

$$(H_t)_{TE} = -\frac{\gamma}{k_c^2} \nabla_{xy} H_z$$

$$Z_{TE} = \frac{j\omega\mu}{\gamma} = \frac{\eta}{\sqrt{1 - \frac{f_c^2}{f^2}}}$$

$$\vec{E} = -Z_{TE}\left(\hat{z} \times \vec{H}\right)$$

Wave Impedance



Dispersion

For propagating modes ($\gamma = j\beta$), we may graph the variation of β with frequency (for either TM or TE) and this determines the dispersion characteristic.

$$\beta = k\sqrt{1-\frac{f_c^2}{f^2}}$$ where v is the velocity in the unbounded medium

or alternatively $$\omega = \frac{\beta v}{\sqrt{1-\frac{\omega_c^2}{\omega^2}}}$$

This is more useful form for plotting

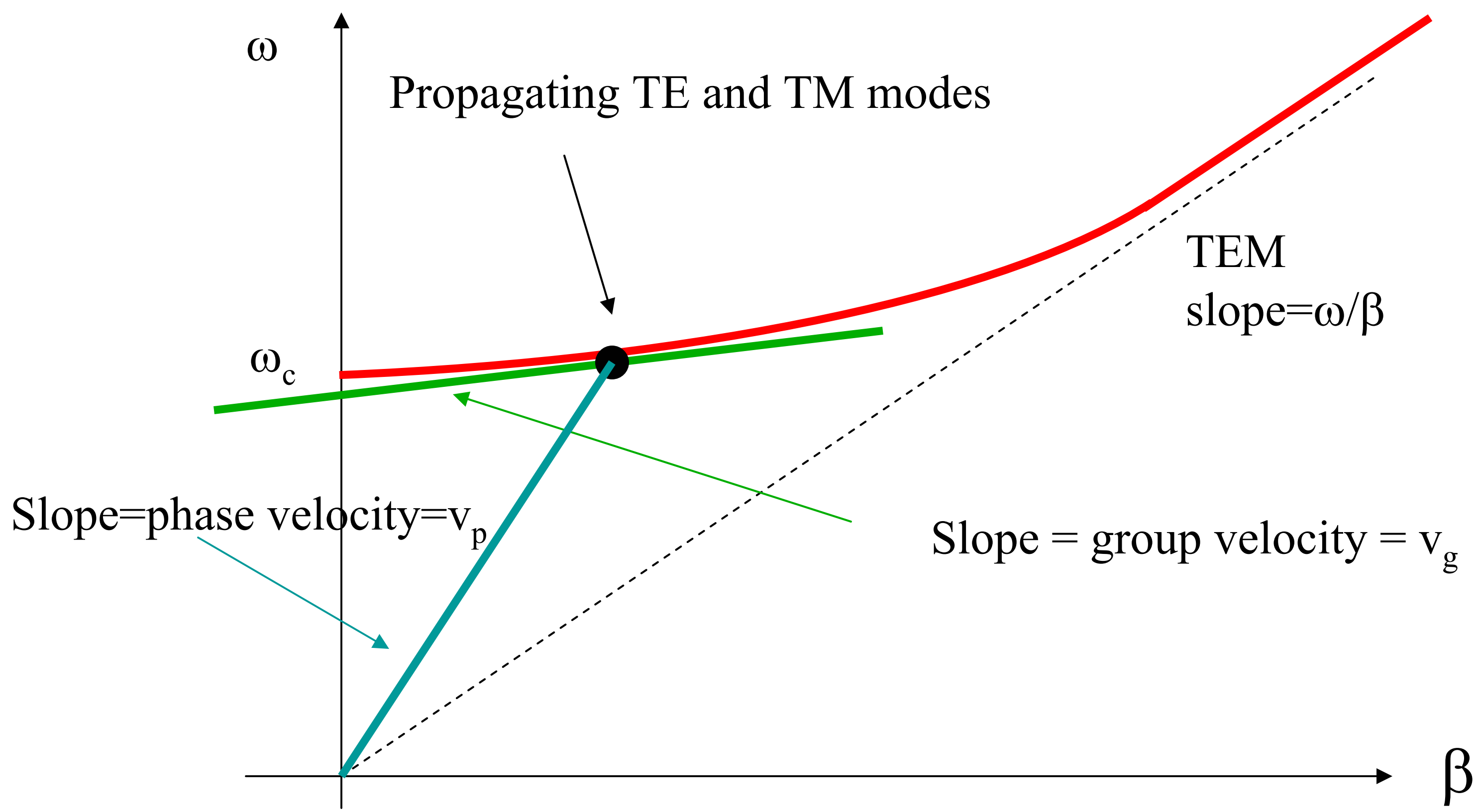
Equation of red plot

Note that $v_p > v$

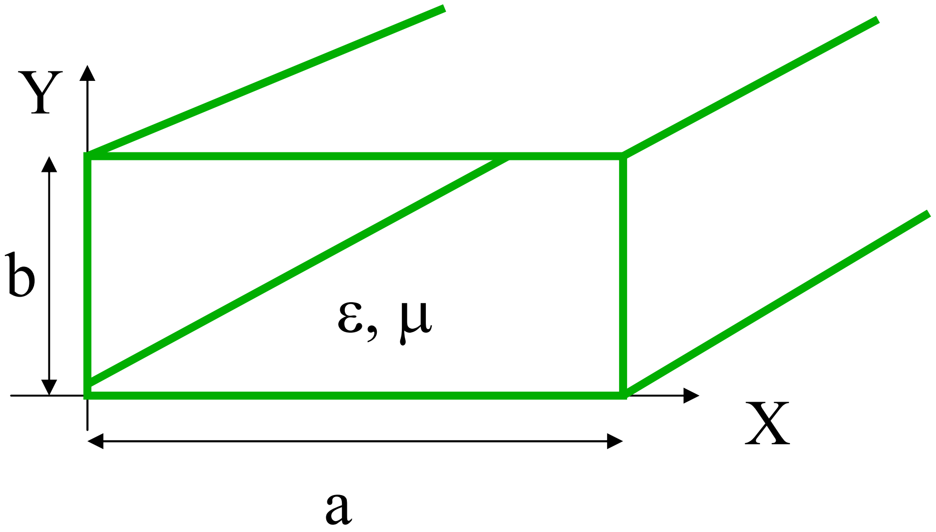
$v_g < v$

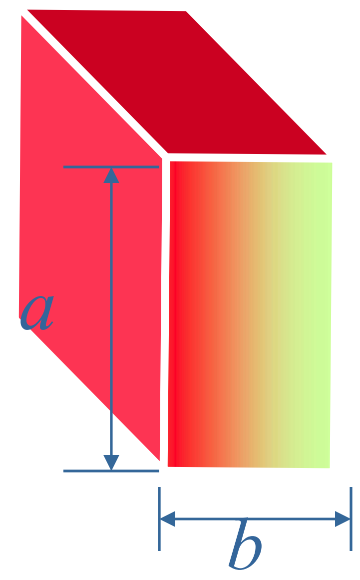
$v_p v_g = v^2$

Dispersion for Waveguide



Rectangular waveguide





Assume perfectly conducting walls and perfect dielectric filling the wave guide.

Convention always says that *a* is the long side.

TM waves

TM waves have $E_z \neq 0$. We write $E_z(x,y,z)$ as $E_z(x,y)e^{-\gamma z}$. The wave equation to solve is then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right)E_z(x,y)=0$$

Plus some boundary conditions on the walls of the waveguide. The standard method of solving this PDE is to use separation of variables. I.e..

$$E_z(x,y)=X(x)Y(y)$$

Possible Solutions

If we substitute into the original equation we get two more equations. But this time we have full derivatives and we can easily write solutions.

$$\frac{d^2X}{dx^2} + k_x^2 X = 0 \qquad \text{and} \qquad \frac{d^2Y}{dy^2} + k_y^2 Y = 0$$

$$\text{with} \qquad k_x^2 + k_y^2 = k_c^2$$

Mathematics tells us that the solutions depend on the sign of $k_x^{\,2}$

$k_x^{\,2}$	k_x	Appropriate X(x)	
0	0	A_0x+B_0	
+	k	A_1**sin** kx+B_1**cos** kx;	$C_1e^{jkx} + D_1e^{-jkx}$
-	jk	A_2**sinh** kx +B_2**cosh** kx;	$C_2e^{kx} + D_2e^{-kx}$

Boundary conditions

Boundary conditions say that the tangential components of E_z vanish on the walls of the guide :

$$\left.\begin{array}{l} E_z(0,y)=0 \\ E_z(a,y)=0 \end{array}\right\} \text{ left and right hand walls.}$$

$$\left.\begin{array}{l} \mathrm{E}_z(x,0)=0 \\ E_z(x,b)=0 \end{array}\right\} \text{ top and bottom walls.}$$

We choose the *sin/cos* form (why?) and directly write:

$$E_z(x,y)=\left(A_1 \sin k_x x + B_1 \cos k_x x\right)\left(A_2 \sin k_y y + B_2 \cos k_y y\right)$$

Final solution

Using the boundary conditions, we find:

X(x) must be in the form **sin**$k_x x$

Y(y) must be in the form **sin**$k_y y$

$$\left.\begin{array}{l} k_x = \dfrac{m\pi}{a} \\ \\ k_y = \dfrac{n\pi}{b} \end{array}\right\} \text{with m, n integer and } m, n = 1,2,3,\ldots..$$

Do not start from 0

$$\boxed{\begin{array}{l} E_z(x,y) = E_0 \,\mathbf{sin}\, \dfrac{m\pi x}{a} \,\mathbf{sin}\, \dfrac{n\pi y}{b} \\ \\ k_c^2 = \left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2 \end{array}}$$

This satisfies all the boundary conditions

We can only get discrete values of k_c -eigenvalues!

Mode numbers (m,n)

The *m,n* numbers will give different solutions for E_z (as well as all the other transverse components. Each *m,n* combination will correspond to a mode which will satisfy all boundary and wave equations. Notice how the modes depend on the geometry (*a,b*)!

We usually refer to the modes as TM_{mn} or TE_{mn} eg $TM_{2,3}$ Thus each mode will specify a unique field distribution in the guide. We now have a formula for the parameter k_c once we specify the mode numbers.

The concept of a mode is fundamental to many E/M problems.

Mode cut-off

From previous formulas, we have directly upon using the value k_c

$$\boxed{f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}}$$

Note this!

Neither m nor n can be zero therefore the TM mode with the lowest cut-off frequency is TM_{11}

TE Modes

For TE modes, we have $E_z = 0$, $H_z \neq 0$ as before.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right)H_z(x,y)=0$$

Boundary conditions

$$\left.\begin{array}{l}
\left.\dfrac{\partial H_z}{\partial x}\right|_{x=0} \Rightarrow E_y = 0 \text{ at } x = 0 \\
\left.\dfrac{\partial H_z}{\partial x}\right|_{x=0} \Rightarrow E_y = 0 \text{ at } x = a
\end{array}\right\}$$

$$\left.\begin{array}{l}
\left.\dfrac{\partial H_z}{\partial y}\right|_{y=0} \Rightarrow E_x = 0 \text{ at } \mathrm{y} = 0 \\
\left.\dfrac{\partial H_z}{\partial y}\right|_{y=0} \Rightarrow E_x = 0 \text{ at } \mathrm{y} = b
\end{array}\right\}$$

Boundary conditions for H_z (longitudinal) are equivalently expressed in terms of E_x and E_y (transverse)

TE_{mn} Results

The expressions for f_c and λ_c are identical to the TM case. But this time we have that the TE dominant mode (ie. the TE mode with the lowest cut-off frequency) is TE_{10} This mode has an even lower cut-off frequency than TM_{11} and is said to be the Dominant Mode for a rectangular waveguide.

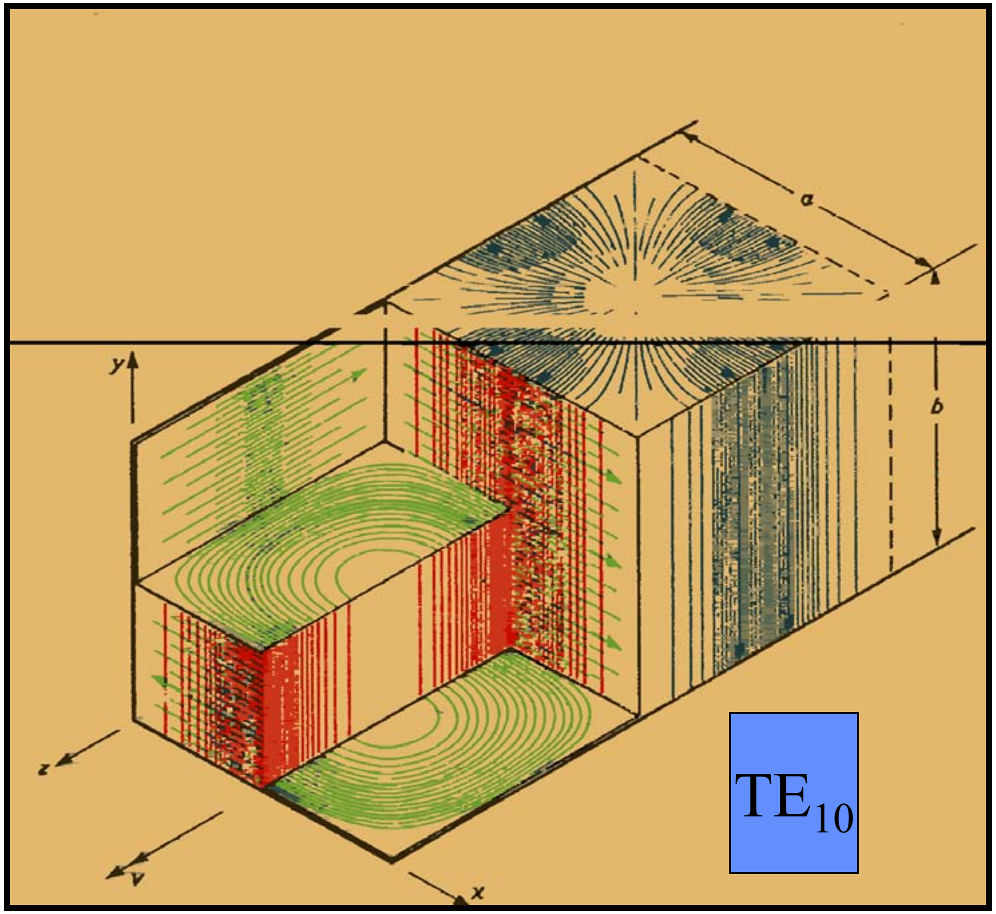
$$H_z(x,y) = H_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

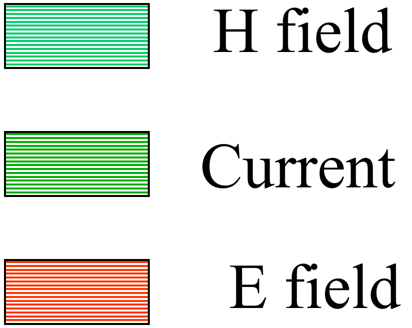
$$(f_c)_{TE_{10}} = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{v}{2a}$$

$$(\lambda_c)_{TE_{10}} = 2a$$

This is provided we label the large side 'a' and associate this side with the mode number 'm'

View of TE10 mode for waveguide.





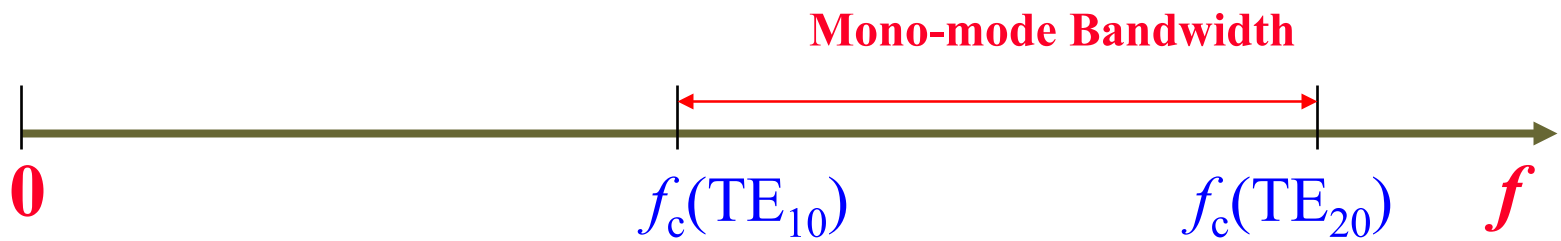
For mono-mode (or single-mode) operation, only the fundamental TE_{10} mode should be propagating over the frequency band of interest.

The mono-mode bandwidth depends on the cut-off frequency of the second propagating mode. We have <u>two</u> possible modes to consider, TE_{01} and TE_{20}.

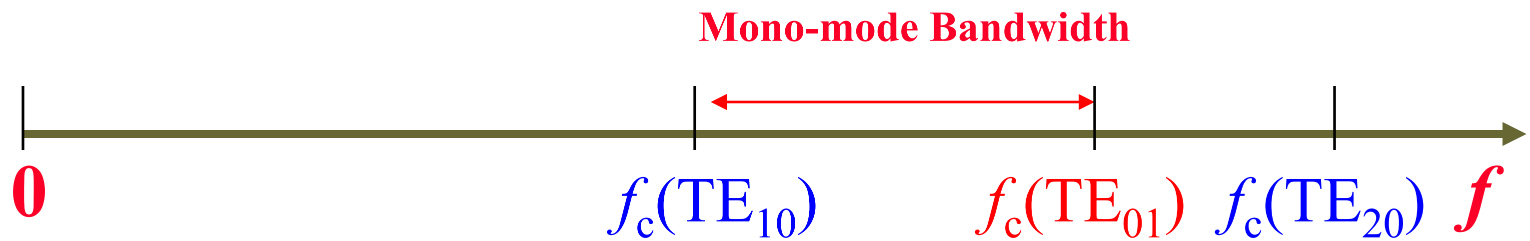
$$f_c(TE_{01}) = \frac{1}{2b\sqrt{\mu\varepsilon}}$$

$$f_c(TE_{20}) = \frac{1}{a\sqrt{\mu\varepsilon}} = 2f_c(TE_{10})$$

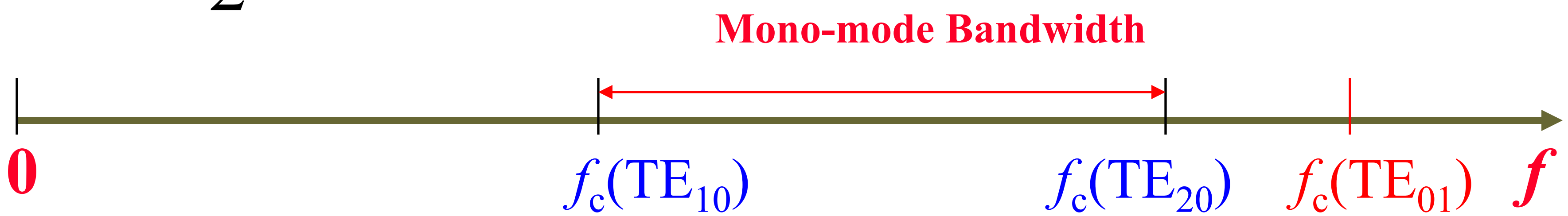
$$\mathcal{If}\ b = \frac{a}{2} \Rightarrow f_c(TE_{01}) = f_c(TE_{20}) = 2f_c(TE_{10}) = \frac{1}{a\sqrt{\mu\varepsilon}}$$



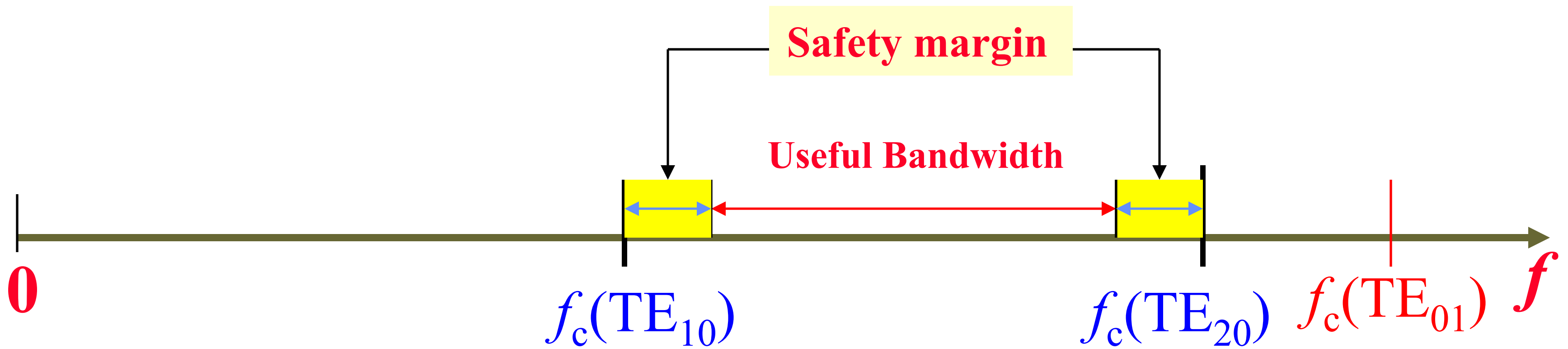
$$\mathcal{If}\ a > b > \frac{a}{2} \Rightarrow f_c(TE_{10}) < f_c(TE_{01}) < f_c(TE_{20})$$



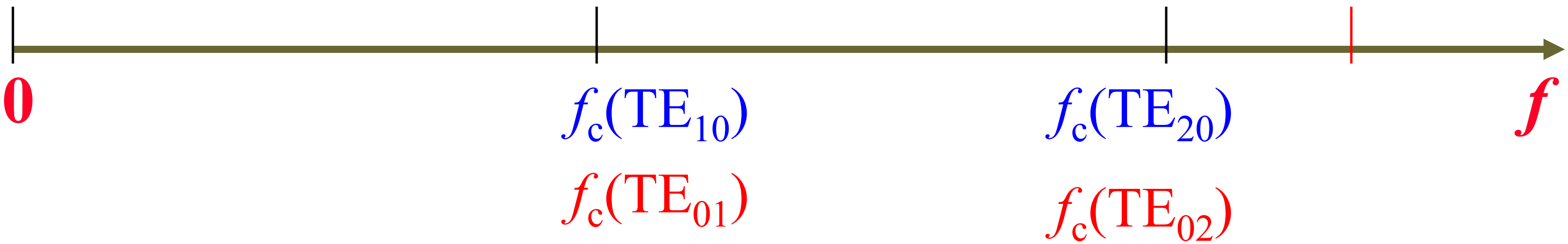
$$\text{If } b < \frac{a}{2} \Rightarrow f_c(TE_{20}) < f_c(TE_{01})$$



In practice , a safety margin of about 20% is considered, so that the useful bandwidth is less than the maximum mono-mode bandwidth. This is necessary to make sure that the first mode (TE_{10}) is well above cut-off, and the second mode (TE_{01} or TE_{20}) is strongly evanescent.



If a=b (square wave guide) $\Rightarrow \quad f_c(TE_{10}) = f_c(TE_{20})$



In the case of perfectly square wave guide, TE_{m0} and TE_{0n} modes with $m=n$ are are degenerate with the same cut-off frequency.

Except for orthogonal field orientation, all other properties of the degenerate modes are the same.

Example – Design an air-filled rectangular wave guide for the following operation conditions:

a. 10 GHz in the middle of the frequency band (single mode operation)

b. b=a/2

The fundamental mode is the TE_{10} with cut-off frequency

$$f_c(TE_{10}) = \frac{1}{2a\sqrt{\mu_\circ\varepsilon_\circ}} = \frac{c}{2a} \approx \frac{3\times10^8\, m/sec}{2a} Hz$$

For b=a/2, TE_{01} and TE_{20} have the same cut-off frequency

$$f_c(TE_{01}) = \frac{1}{2b\sqrt{\mu_\circ\varepsilon_\circ}} = \frac{c}{2b} = \frac{c}{2}\frac{2}{a} = \frac{c}{a} \approx \frac{3\times10^8\, m/sec}{a} Hz$$

$$f_c(TE_{20}) = \frac{1}{a\sqrt{\mu_\circ\varepsilon_\circ}} = \frac{c}{a} \approx \frac{3\times10^8\, m/sec}{a} Hz$$

The operation frequency can be expressed in terms of the cut-off frequencies

$$f = f_c(TE_{10}) + \frac{f_c(TE_{10}) - f_c(TE_{01})}{2}$$

$$= \frac{f_c(TE_{10}) + f_c(TE_{01})}{2} = 10.0GHz$$

$$\Rightarrow 10.0 \times 10^9 = \frac{1}{2}\left[\frac{3 \times 10^8}{2a} + \frac{3 \times 10^8}{a}\right]$$

$$\Rightarrow a = 2.25cm \quad b = \frac{a}{2} = 1.125cm$$

Example

We consider an air filled guide, so ε_r=1. The internal size of the guide is 0.9 x 0.4 inches (waveguides come in standard sizes). The cut-off frequency of the dominant mode:

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad \mathrm{a} = 0.9'' = 22.86\mathrm{mm}; \mathrm{b} = 0.4'' = 10.16\mathrm{mm}$$

$$(k_c)_{TE_{10}} = \frac{\pi}{a} = 137.43$$

$$(f_c)_{TE_{10}} = \frac{k_c}{2\pi\sqrt{\mu_0\varepsilon_0}} = \frac{137.43\times 3\times 10^8}{2\pi} = 6.56GHz$$

Example

The next few modes are:

$$\left.\begin{aligned}(k_c)_{11} &= 338.38\\(k_c)_{01} &= 309.21\\(k_c)_{20} &= 274.86\end{aligned}\right\}\text{the ascending order of mode is } 10, 20, 01, 11$$

The next cuff-off frequency after TE_{10} will then be

$$(f_c)_{TE_{20}} = \frac{274.86\times 3\times 10^8}{2\pi} = 13.12GHz$$

So for single mode operation we must operate the guide within the frequency range of 6.56<f <13.12GHz.

Example

It is not good to operate too close to cut-off for the reason that the wall losses increases very quickly as the frequency approaches cut-off. A good guideline is to operate between $1.25f_c$ and $1.9f_c$. This then would restrict the single mode operation to 8.2 to 12.5 GHz.

The propagation coefficient for the next higher mode is:

$$(\gamma)_{20} = \sqrt{k_c^2 - k^2} = (k_c)_{20}\sqrt{1 - \frac{f^2}{f_{c_{20}}^2}}$$

Specify an operating frequency f, half way in the original range of TE_{10} i.e.. 9.84GHz.

Example

$$(\gamma)_{20} = 274.86\sqrt{1-\left(\frac{9.84}{13.12}\right)^2} = 181.8 \ \text{(Real)}$$

So $\alpha = 181.8 Np/m$

or in dB $181.8 \text{ x } 8.7 = 1581 \text{ dB/m}$ ie. TE_{20} is very strongly evanescent.

In comparison for TE_{10}:

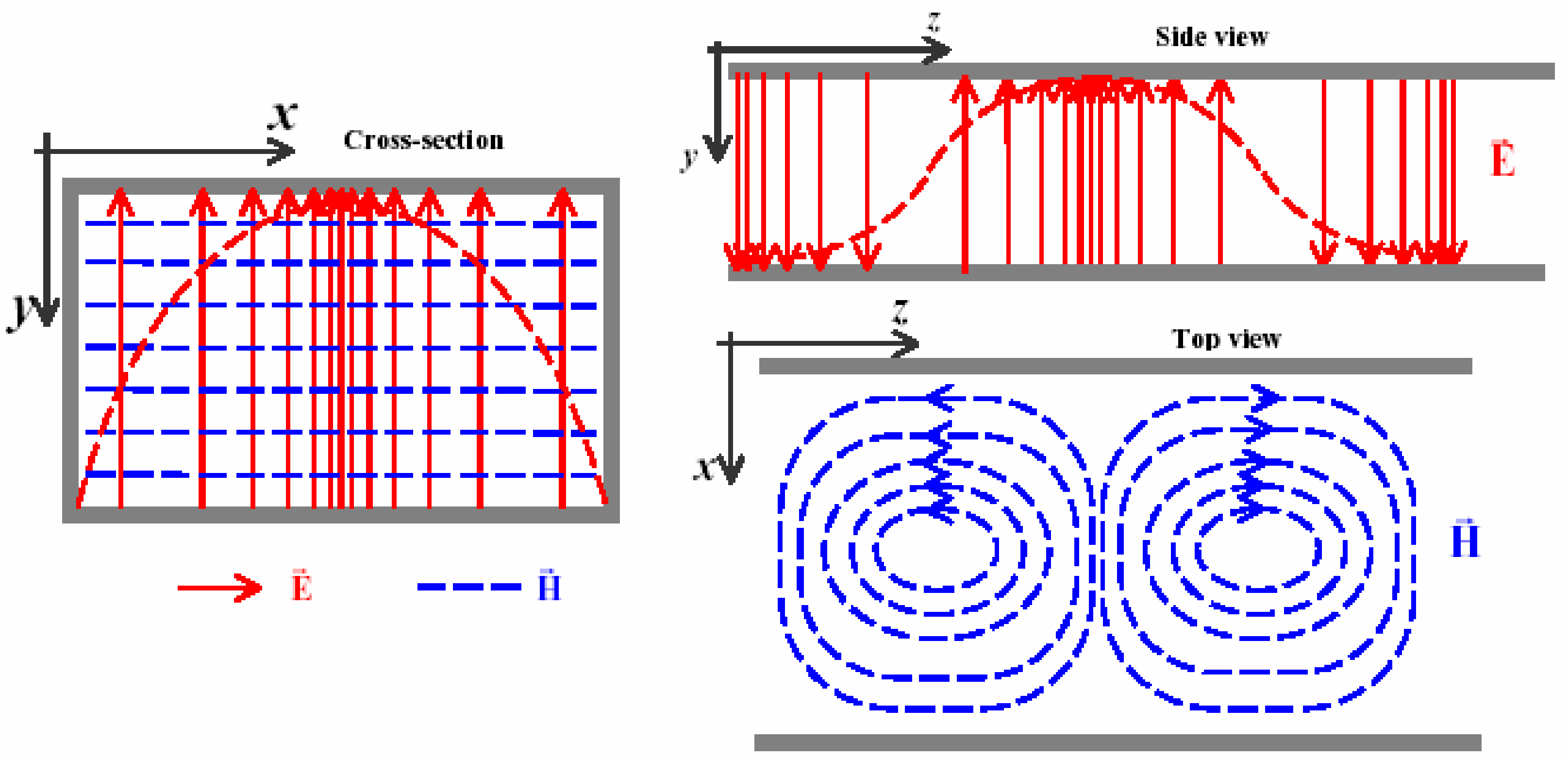
$$(\gamma)_{10} = (k_c)_{10}\sqrt{1-\frac{f^2}{f_{c10}^2}} = 137.43\sqrt{1-\left(\frac{9.84}{6.56}\right)^2} = 153.64 j \ \text{(Imaginary)}$$

So $\beta = 153.64 \text{ rad/m}$

All further higher order modes will be cut-off with higher rates of attenuation.

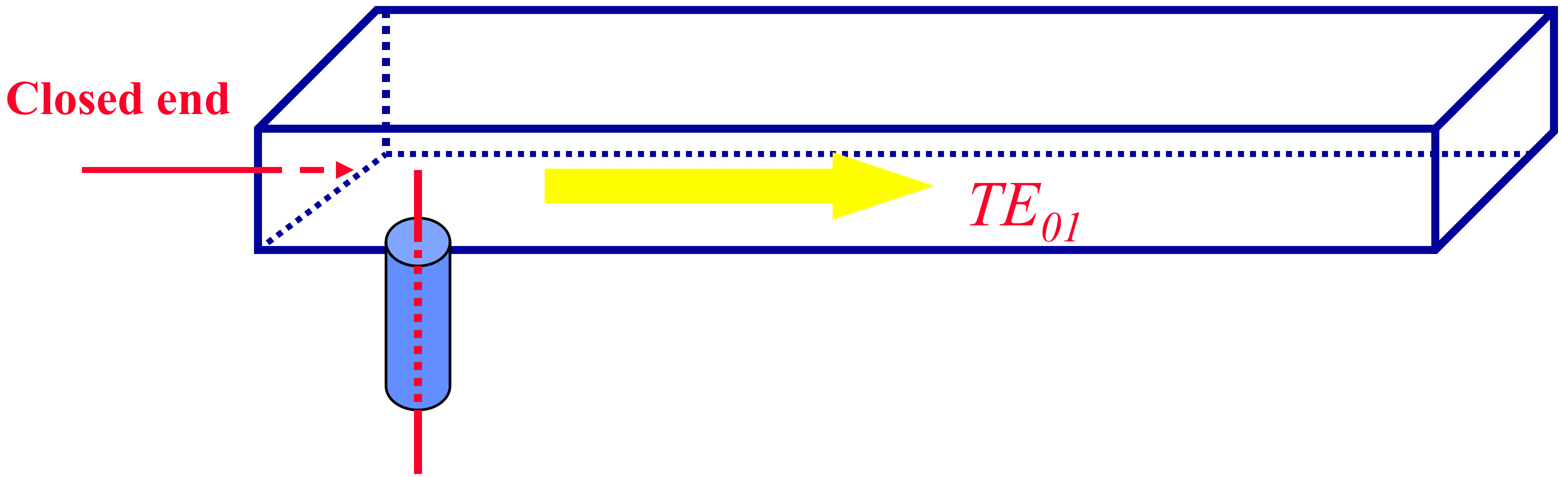
Field Patterns

Field patterns for the TE_{10} mode in rectangular wave guide



The simple arrangement below can be used to excite TE_{10} in a rectangular wave guide.

The inner conductor of the coaxial cable behaves like a dipole antenna and it creates a maximum electric field in the middle of the cross-section.



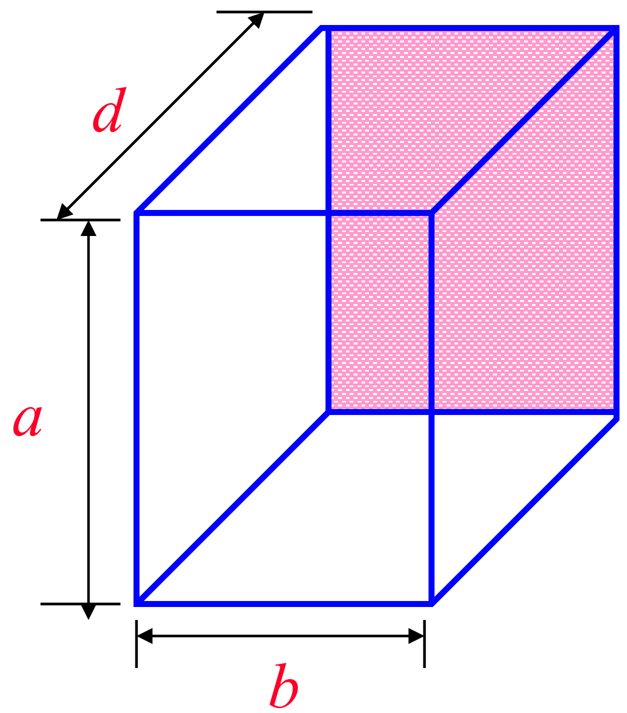
Waveguide Cavity Resonator

The cavity resonator is obtained from a section of rectangular wave guide, closed by two additional metal plates. We assume again perfectly conducting walls and loss-less dielectric.

$$\beta_x = \frac{m\pi}{a}$$

$$\beta_y = \frac{n\pi}{b}$$

$$\beta_z = \frac{p\pi}{d}$$



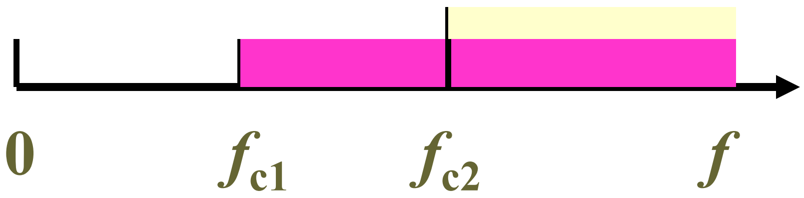
Waveguide Cavity Resonator

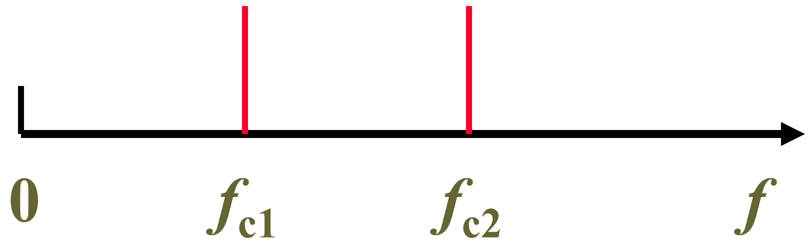
The addition of a new set of plates introduces a condition for standing waves in the z-direction which leads to the definition of oscillation frequencies

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

The high-pass behavior of the rectangular wave guide is modified into a very narrow pass-band behavior, since cut-off frequencies of the wave guide are transformed into oscillation frequencies of the resonator.

Waveguide Cavity Resonator





In the wave guide, each mode is associated with a band of frequencies larger the cut-off frequency.

In the resonator, resonant modes can only exist in correspondence of discrete resonance frequencies.

Waveguide Cavity Resonator

The cavity resonator will have modes indicated as

$$TE_{mnp} \qquad TM_{mnp}$$

The values of the index corresponds to periodicity (number of sine or cosine waves) in three direction. Using z-direction as the reference for the definition of transverse electric or magnetic fields, the allowed indices are

$$TE\begin{cases} m = 0,1,2,3,\ldots \\ n = 0,1,2,3,\ldots \\ p = 0,1,2,3,\ldots \end{cases} \qquad TM\begin{cases} m = 0,1,2,3,\ldots \\ n = 0,1,2,3,\ldots \\ p = 0,1,2,3,\ldots \end{cases}$$

With only one zero index m or n allowed

The mode with lowest resonance frequency is called dominant mode. In case a ≥ d>b the dominant mode is the TE_{101}.

Waveguide Cavity Resonator

Note that a **TM** cavity mode, with magnetic field transverse to the z-direction, it is possible to have the **third index** equal **zero**. This is because the magnetic field is going to be parallel to the third set of plates, and it can therefore be uniform in the third direction, with no periodicity.

The **electric field** components will have the following form that satisfies the **boundary conditions** for perfectly conducting walls.

$$E_x = \mathcal{E}_x \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$E_y = \mathcal{E}_y \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)$$

$$E_y = \mathcal{E}_z \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right)$$

Waveguide Cavity Resonator

The amplitudes of the electric field components also must satisfy the divergence condition which, in absence of charge is

$$\nabla \cdot \vec{E} = 0 \Rightarrow \left(\frac{m\pi}{a}\right)E_x + \left(\frac{n\pi}{b}\right)E_y + \left(\frac{p\pi}{d}\right)E_z = 0$$

The magnetic field intensities are obtained from Ampere's law:

$$H_x = \frac{\beta_z E_y - \beta_y E_z}{j\omega\mu} sin\left(\frac{m\pi}{a}x\right)cos\left(\frac{n\pi}{b}y\right)cos\left(\frac{p\pi}{d}z\right)$$

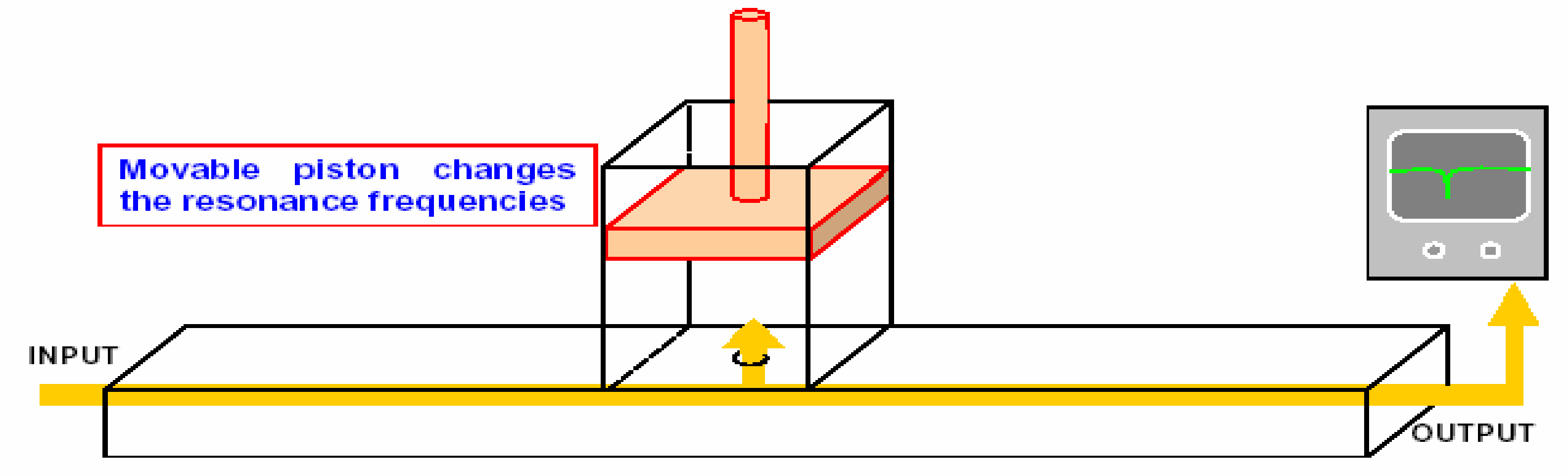
$$H_y = \frac{\beta_x E_z - \beta_z E_x}{j\omega\mu} cos\left(\frac{m\pi}{a}x\right)sin\left(\frac{n\pi}{b}y\right)cos\left(\frac{p\pi}{d}z\right)$$

$$H_z = \frac{\beta_y E_x - \beta_x E_y}{j\omega\mu} cos\left(\frac{m\pi}{a}x\right)cos\left(\frac{n\pi}{b}y\right)sin\left(\frac{p\pi}{d}z\right)$$

Waveguide Cavity Resonator

Similar considerations for modes and indices can be made if the other axes are used as a reference for the transverse field, leading to analogous resonant field configurations.

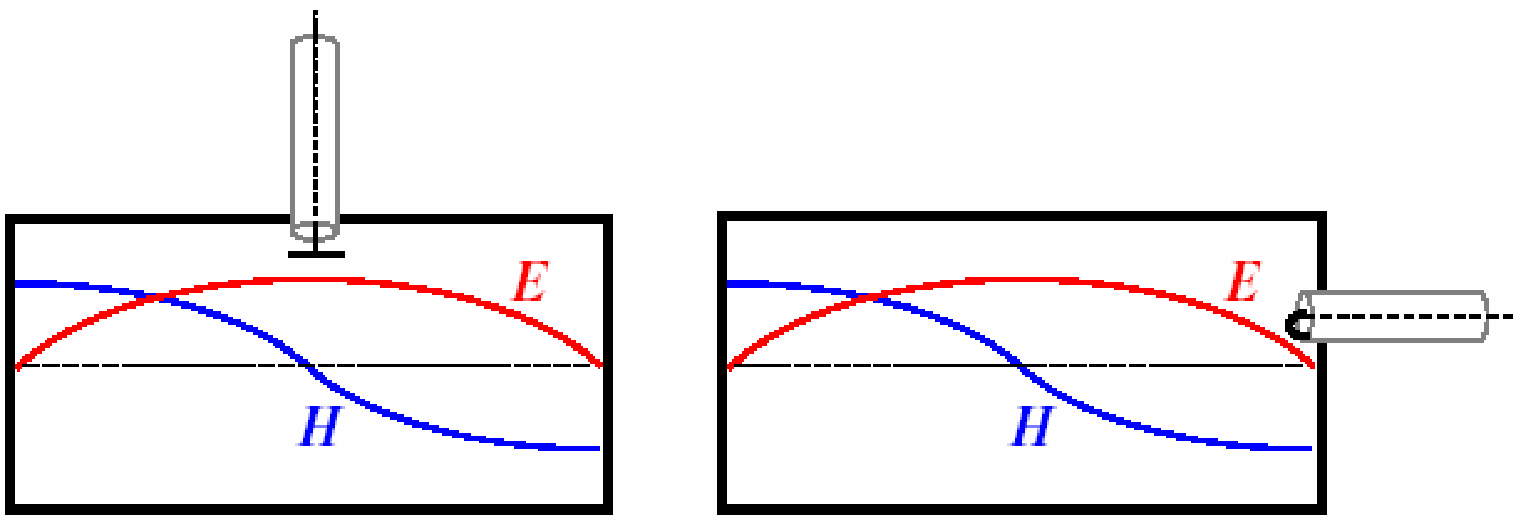
A cavity resonator can be coupled to a wave guide through a small opening. When the input frequency resonates with the cavity, electromagnetic radiation enters the resonator and a lowering in the output is detected. By using carefully tuned cavities, this scheme can be used for frequency measurements.



Waveguide Cavity Resonator

Example of resonant cavity excited by using coaxial cables.

The termination of the inner conductor of the cable acts like an elementary dipole (left) or an elementary loop (right) antenna.



Excitation with a dipole antenna **Excitation with a loop antenna**

Waveguides

Here are some standard air-filled rectangular waveguides with their naming designations, inner side dimensions a, b in inches, cutoff frequencies in GHz, minimum and maximum recommended operating frequencies in GHz, power ratings, and attenuations in dB/m (the power ratings and attenuations are representative over each operating band.) We have chosen one example from each microwave band.

name	a	b	f_c	$f_{\min}$	$f_{\max}$	band	P	α
WR-510	5.10	2.55	1.16	1.45	2.20	L	9 MW	0.007
WR-284	2.84	1.34	2.08	2.60	3.95	S	2.7 MW	0.019
WR-159	1.59	0.795	3.71	4.64	7.05	C	0.9 MW	0.043
WR-90	0.90	0.40	6.56	8.20	12.50	X	250 kW	0.110
WR-62	0.622	0.311	9.49	11.90	18.00	Ku	140 kW	0.176
WR-42	0.42	0.17	14.05	17.60	26.70	K	50 kW	0.370
WR-28	0.28	0.14	21.08	26.40	40.00	Ka	27 kW	0.583
WR-15	0.148	0.074	39.87	49.80	75.80	V	7.5 kW	1.52
WR-10	0.10	0.05	59.01	73.80	112.00	W	3.5 kW	2.74

Characteristics of some standard air-filled rectangular waveguides.