Lecture 3

- More on Maxwell
- Wave Equations
- Boundary Conditions
- Poynting Vector
- Transmission Line

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Maxwell's equations in differential form

$$\nabla . \mathrm{D} = \rho \quad \textit{Gauss' law for electrostatics}$$

$$\nabla . \mathrm{B} = 0 \quad \textit{Gauss' law for magnetostatics}$$

$$\nabla \times \mathrm{H} = \mathrm{J} + \frac{\partial \mathrm{D}}{dt} \quad \textit{Ampere's law}$$

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{dt} \quad \textit{Faraday's law}$$

$$\nabla . \mathrm{J} = -\frac{\partial \rho}{\partial t} \quad \textit{Equation of continuity}$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

- Varying E and H fields are coupled

Electromagnetic waves in lossless media - Maxwell's equations

Maxwell

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{dt}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}$$

$$\nabla . \mathbf{D} = \rho$$

$$\nabla . \mathbf{B} = 0$$

Equation of continuity

$$\nabla . \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_o \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_r \mu_o \mathbf{H}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

SI Units

- J Amp/ metre^2
- D Coulomb/metre^2
- H Amps/metre
- B Tesla
  Weber/metre^2
  Volt-Second/metre^2
- E Volt/metre
- ε Farad/metre
- μ Henry/metre
- σ Siemen/metre

Wave equations in free space

- In free space
  - $\sigma = 0 \Rightarrow \mathbf{J} = 0$
  - Hence: $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{dt} = \frac{\partial \mathbf{D}}{dt}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}$$

  - Taking curl of both sides of latter equation:

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\mu_o \frac{\partial}{\partial t} \nabla \times \mathbf{H}$$

$$= -\mu_o \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Wave equations in free space cont.

$$\nabla \times \nabla \times \mathbf{E} = -\mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- It has been shown (last week) that for any vector **A**

$$\nabla \times \nabla \times \mathbf{A} = \nabla \nabla . \mathbf{A} - \nabla^2 \mathbf{A}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the *Laplacian* operator

Thus:

$$\nabla \nabla . \mathbf{E} - \nabla^2 \mathbf{E} = -\mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- There are no free charges in free space so $\nabla . \mathbf{E} = \rho = 0$ and we get

$$\boxed{\nabla^2 \mathbf{E} = \mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}}$$

A three dimensional wave equation

Wave equations in free space cont.

- Both **E** and **H** obey second order partial differential wave equations:

$$\nabla^2 \mathbf{E} = \mu_o \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{H} = \mu_o \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

- What does this mean
  - dimensional analysis ?

$$\frac{\text{Volts/metre}}{\text{metre}^2} = \mu_o \varepsilon \frac{\text{Volts/metre}}{\text{seconds}^2}$$

  - $\mu_o\varepsilon$ has units of velocity^{-2}
  - Why is this a wave with velocity $1/\sqrt{\mu_o\varepsilon}$?

Uniform plane waves - transverse relation of E and H

- Consider a uniform plane wave, propagating in the z direction. **E** is independent of x and y

$$\frac{\partial \mathbf{E}}{\partial x}=0 \qquad \frac{\partial \mathbf{E}}{\partial y}=0$$

In a source free region, $\nabla.\mathbf{D}=\rho=0$ (Gauss' law) :

$$\nabla.\mathbf{E}=\frac{\partial \mathbf{E}_x}{\partial x}+\frac{\partial \mathbf{E}_y}{\partial y}+\frac{\partial \mathbf{E}_z}{\partial z}=0$$

E is independent of x and y, so

$$\frac{\partial E_x}{\partial x}=0,\ \frac{\partial E_y}{\partial y}=0 \quad \Rightarrow \quad \frac{\partial E_z}{\partial z}=0 \quad \boxed{\Rightarrow E_z=0} \quad (E_z=\text{const is not a wave})$$

- So for a plane wave, **E** has no component in the direction of propagation. Similarly for **H.**
- Plane waves have only transverse **E** and **H** components.

Orthogonal relationship between E and H:

- For a plane z-directed wave there are no variations along x and y:

$$\nabla \times \mathbf{H} = -\mathbf{a}_x \frac{\partial H_y}{\partial z} + \mathbf{a}_y \frac{\partial H_x}{\partial z}$$

$$= \frac{\partial \mathbf{D}}{\partial t}$$

$$= \varepsilon \left(\mathbf{a}_x \frac{\partial E_x}{\partial t} + \mathbf{a}_y \frac{\partial E_y}{\partial t} + \cancel{\mathbf{a}_y \frac{\partial E_z}{\partial t}} \right)$$

$$\nabla \times A = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla \times \mathbf{H} = \cancel{\mathbf{J}} + \frac{\partial \mathbf{D}}{dt}$$

- Equating terms:

$$-\frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t}$$

- and likewise for $\nabla \times \mathbf{E} = -\mu_o \, \partial \mathbf{H} / \partial t$:

$$\frac{\partial E_y}{\partial z} = \mu_o \frac{\partial H_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = \mu_o \frac{\partial H_y}{\partial t}$$

- **Spatial rate of change of H is proportionate to the temporal rate of change of the orthogonal component of *E* & v.v. *at the same point in space***

Orthogonal and phase relationship between E and H:

- Consider a linearly polarised wave that has a transverse component in (say) the y direction only:

$$-\frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t}$$

$$\frac{\partial H_x}{\partial z} = \varepsilon \frac{\partial E_y}{\partial t}$$

$$E_y = E_o f(z - vt)$$

$$\Rightarrow \varepsilon \frac{\partial E_y}{\partial t} = -\varepsilon v E_o f'(z - vt) = \frac{\partial H_x}{\partial z}$$

$$\Rightarrow \quad H_x = -\varepsilon v E_o \int f'(z - vt) dz + const = -\varepsilon v E_o f(z - vt)$$

$$= -\varepsilon v E_y$$

$$H_x = -\sqrt{\frac{\varepsilon}{\mu_o}} E_y$$

- Similarly

$$\frac{\partial E_y}{\partial z} = \mu_o \frac{\partial H_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = \mu_o \frac{\partial H_y}{\partial t}$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu_o}} E_x$$

- H and E are in phase and orthogonal

$$H_x = -\sqrt{\frac{\varepsilon}{\mu_o}}E_y \qquad H_y = \sqrt{\frac{\varepsilon}{\mu_o}}E_x$$

- The ratio of the magnetic to electric fields strengths is:

$$\frac{\sqrt{E_x^2 + E_y^2}}{\sqrt{H_x^2 + H_y^2}} = \frac{E}{H} = \sqrt{\frac{\mu_o}{\varepsilon}} = \eta$$

Note:

$$\frac{E}{B} = \frac{E}{\mu_o H} = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c$$

which has units of impedance

$$\frac{Volts\,/\,metre}{amps\,/\,metre} = \Omega$$

- and the *impedance of free space* is:

$$\sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi} \times 10^{-9}}} = 120\pi = 377\Omega$$

Orientation of **E** and **H**

- For any medium the intrinsic impedance is denoted by η

$$\eta = -\frac{E_y}{H_x} = \frac{E_x}{H_y}$$

and taking the scalar product

$$\mathbf{E}.\mathbf{H} = E_x H_x + E_y H_y$$
$$= \eta H_y H_x - \eta H_x H_y = 0$$

so **E** and **H** are mutually orthogonal

- Taking the cross product of **E** and **H** we get the direction of wave propagation

$$\mathbf{E} \times \mathbf{H} = \mathbf{a}_z \left(E_x H_y - E_y H_x \right)$$
$$= \mathbf{a}_z \left(\eta H_y^2 - \eta H_x^2 \right)$$

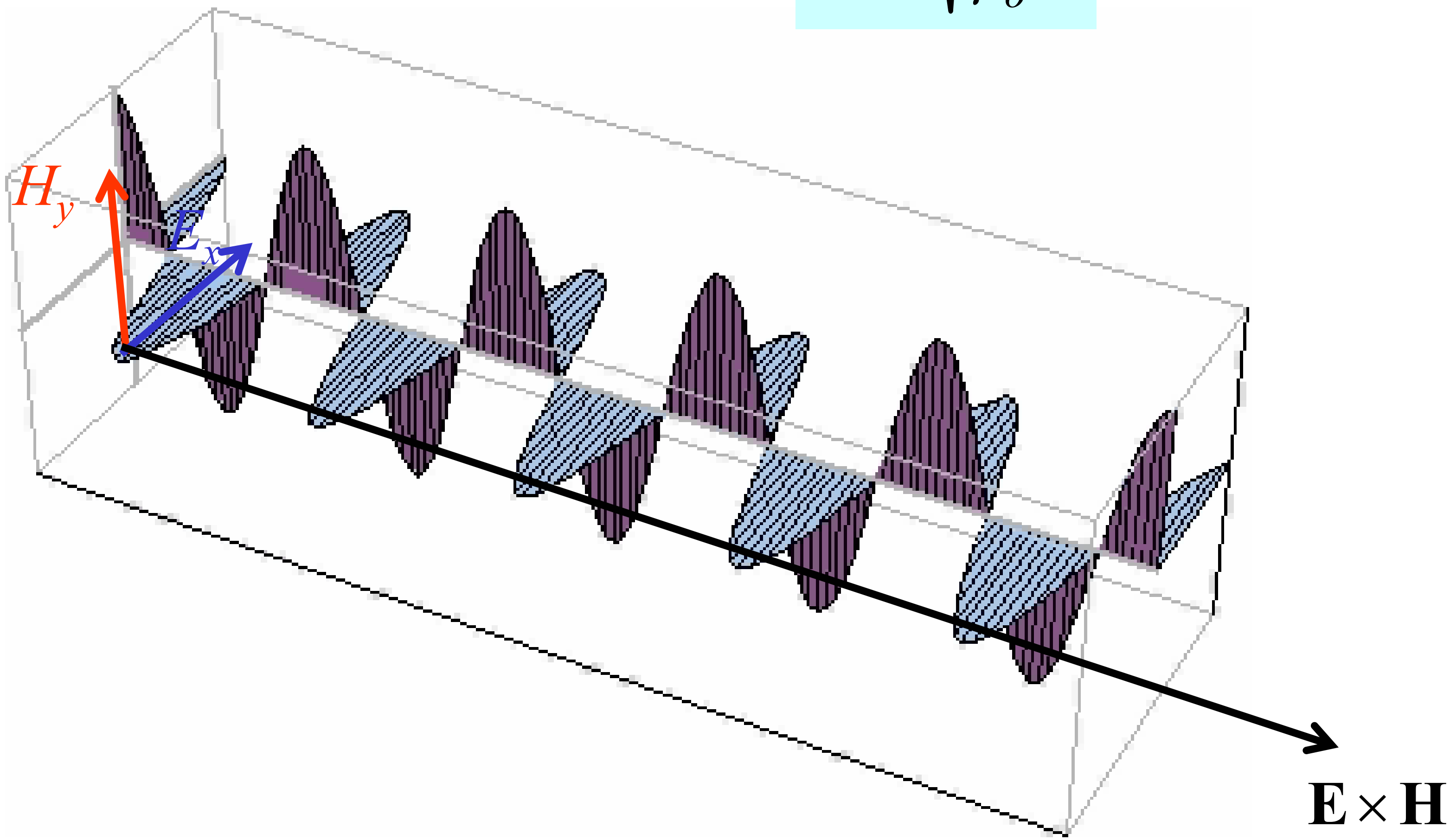
$$\mathbf{E} \times \mathbf{H} = \mathbf{a}_z \eta H^2$$

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_x \left(A_y B_z - A_z B_y \right) + \mathbf{a}_y \left(A_z B_x - A_x B_z \right) + \mathbf{a}_z \left(A_x B_y - A_y B_x \right)$$

A ‘horizontally’ polarised wave

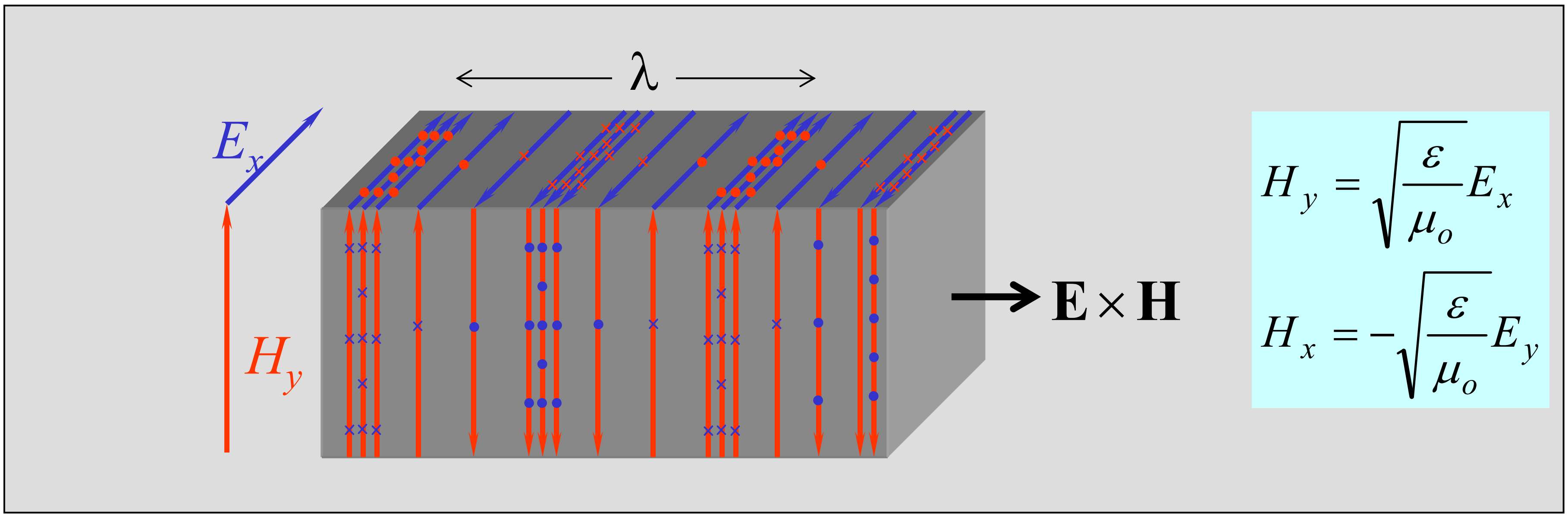
- Sinusoidal variation of E and H
- E and H in phase and orthogonal

$$H_y = \sqrt{\frac{\varepsilon}{\mu_o}} E_x$$



A block of space containing an EM plane wave

- Every point in 3D space is characterised by
  - E_x, E_y, E_z
  - Which determine
    - H_x, H_y, H_z
    - and vice versa
  - 3 degrees of freedom



Power flow of EM radiation

- Energy stored in the EM field in the thin box is:

$$\mathrm{d}U = \mathrm{d}U_E + \mathrm{d}U_H = \left(u_E + u_H\right)A\mathrm{d}x$$

$$\mathrm{d}U = \left(\frac{\varepsilon E^2}{2} + \frac{\mu_o H^2}{2}\right)A\mathrm{d}x$$

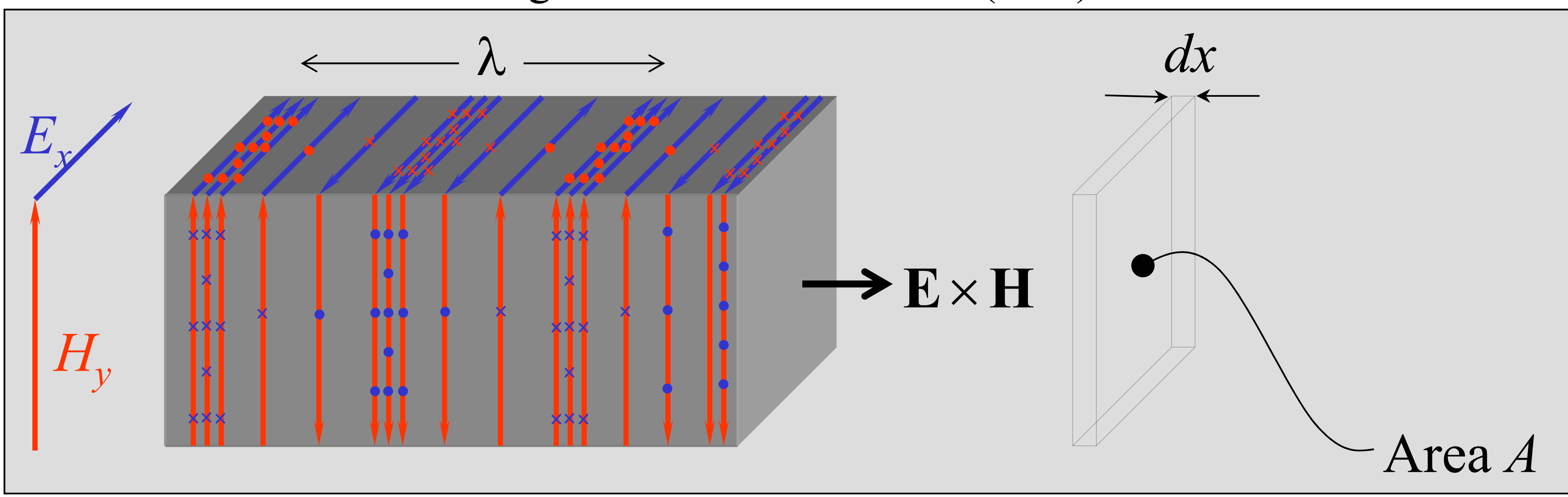
$$= \varepsilon E^2 A\mathrm{d}x$$

$$u_E = \frac{\varepsilon E^2}{2}$$

$$u_H = \frac{\mu_o H^2}{2}$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu_o}}E_x$$

- Power transmitted through the box is dU/dt=dU/(dx/c)....



Power flow of EM radiation cont.

$$\mathrm{d}U = \varepsilon E^2 A\,\mathrm{d}x$$

$$S = \frac{\mathrm{d}U}{A\,\mathrm{d}t} = \frac{\varepsilon E^2}{A(\mathrm{d}x/c)} A\,\mathrm{d}x = \sqrt{\frac{\varepsilon}{\mu_o}} = \frac{E^2}{\eta} \qquad \mathrm{W/m^2}$$

- This is the instantaneous power flow
  - Half is contained in the electric component
  - Half is contained in the magnetic component
- E varies sinusoidal, so the average value of S is obtained as:

$$\mathrm{E} = E_o \sin\frac{2\pi}{\lambda}(z - vt)$$

$$S = \frac{E_o^2 \sin^2(z - vt)}{\eta}$$

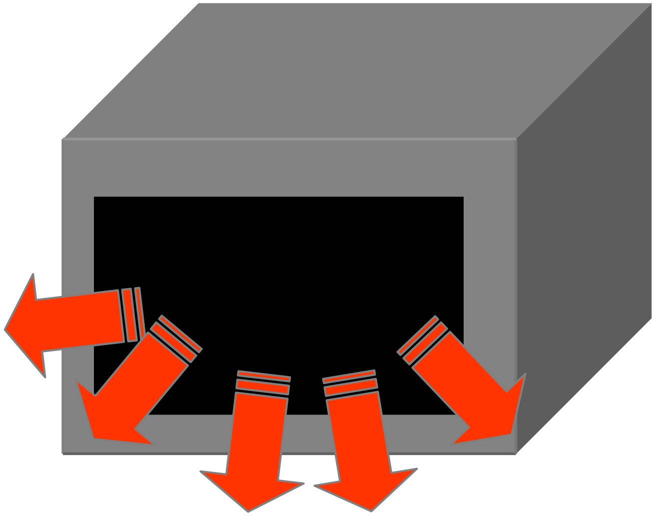
$$\bar{S} = \frac{E_o^2}{\eta}\mathrm{RMS}\left(E_o^2 \sin^2(z - vt)\right) = \frac{E_o^2}{2\eta}$$

- S is the Poynting vector and indicates the direction and magnitude of power flow in the EM field.

Example problem

- The door of a microwave oven is left open
  - estimate the peak *E* and *H* strengths in the aperture of the door.
  - Which plane contains both E and H vectors ?
  - What parameters and equations are required?
    - Power-750 W
    - Area of aperture - 0.3 x 0.2 m
    - impedance of free space - 377 Ω
    - Poynting vector:

$$S = \frac{E^2}{\eta} = \eta H^2 \quad \mathrm{W/m^2}$$



Solution

$$Power = SA = \frac{E^2}{\eta} A \quad = \eta H^2 \text{ A} \quad \text{Watts}$$

$$E = \sqrt{\eta \frac{Power}{A}} = \sqrt{377 \frac{750}{0.3.0.2}} = 2{,}171 \text{ V/m}$$

$$\text{H} = \frac{E}{\eta} = \frac{2170}{377} = 5.75 \text{A/m}$$

$$\text{B} = \mu_o H = 4\pi \times 10^{-7} \times 5.75 = 7.2 \mu\text{Tesla}$$

Constitutive relations

- permittivity of free space ε_0=8.85 x 10^{-12} F/m
- permeability of free space μ_o=4πx10^{-7} H/m
- Normally ε_r (dielectric constant) and μ_r
  - vary with material
  - are frequency dependant
    - For non-magnetic materials μ_r ~1 and for Fe is ~200,000
    - ε_r is normally a few ~2.25 for glass at optical frequencies
  - are normally simple scalars (i.e. for *isotropic* materials) so that **D** and **E** are parallel and **B** and **H** are parallel
    - For ferroelectrics and ferromagnetics ε_r and μ_r depend on the relative orientation of the material and the applied field:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_r\varepsilon_o\mathbf{E}$$

$$\mathbf{B} = \mu\mathbf{H} = \mu_r\mu_o\mathbf{H}$$

$$\mathbf{J} = \sigma\mathbf{E}$$

$$\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}$$

At microwave frequencies:

$$\mu_{ij} = \begin{pmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & \mu_o \end{pmatrix}$$

Constitutive relations cont...

- What is the relationship between ε and refractive index for non magnetic materials ?
  - $v=c/n$ is the speed of light in a material of refractive index n

$$v = \frac{1}{\sqrt{\mu_o \varepsilon_o \varepsilon_r}} = \frac{c}{n}$$

$$n = \sqrt{\varepsilon_r}$$

  - For glass and many plastics at optical frequencies
    - n~1.5
    - ε_r~2.25
- Impedance is lower within a dielectric

$$\eta = \sqrt{\frac{\mu_o \mu_r}{\varepsilon_o \varepsilon_r}}$$

What happens at the boundary between materials of different n, μ_r, ε_r ?

Why are boundary conditions important ?

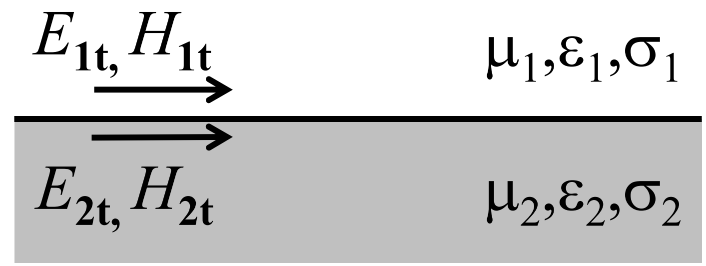
- When a free-space electromagnetic wave is incident upon a medium secondary waves are
  - transmitted wave
  - reflected wave
- The transmitted wave is due to the **E** and **H** fields at the boundary as seen from the incident side
- The reflected wave is due to the **E** and **H** fields at the boundary as seen from the transmitted side
- To calculate the transmitted and reflected fields we need to know the fields at the boundary
  - These are determined by the boundary conditions

Boundary Conditions cont.

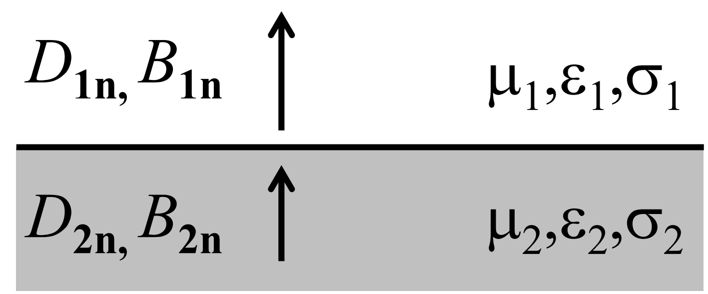
- At a boundary between two media, $\mu_r, \varepsilon_r \sigma$ are different on either side.
- An abrupt change in these values changes the characteristic impedance experienced by propagating waves
- Discontinuities results in partial reflection and transmission of EM waves
- The characteristics of the reflected and transmitted waves can be determined from a solution of Maxwells equations along the boundary

Boundary conditions

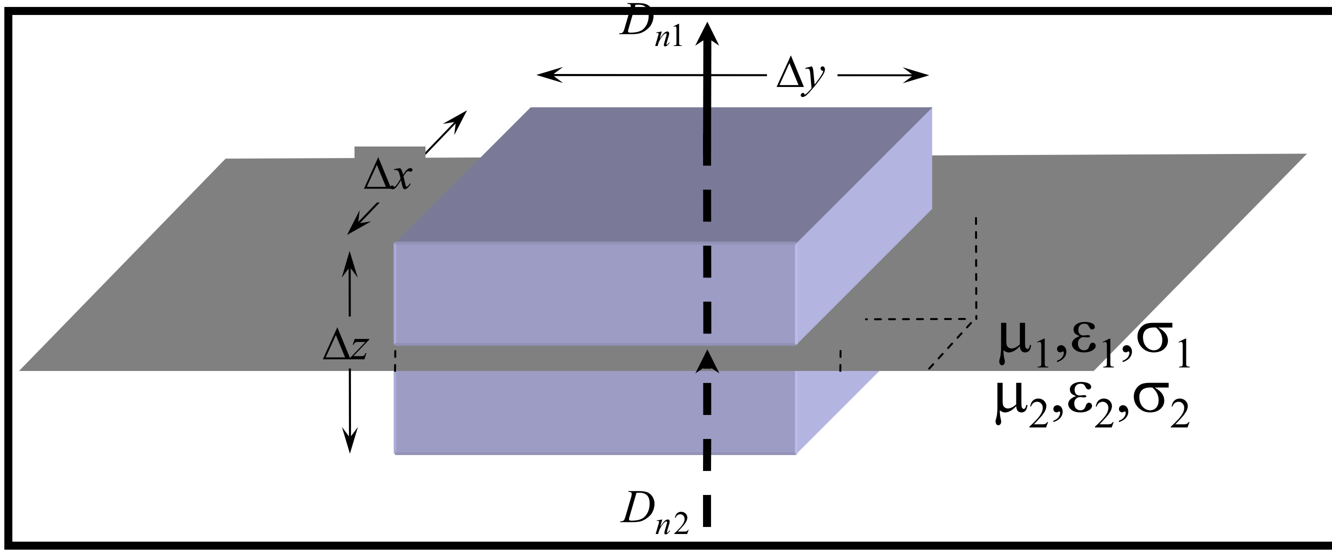
- The tangential component of **E** is continuous at a surface of discontinuity
  - $E_{\mathbf{1t},} = E_{\mathbf{2t}}$
- **Except for a perfect conductor**, the tangential component of **H** is continuous at a surface of discontinuity
  - $H_{\mathbf{1t},} = H_{\mathbf{2t}}$



- The normal component of **D** is continuous at the surface of a discontinuity if there is no surface charge density. If there is surface charge density **D** is discontinuous by an amount equal to the surface charge density.
  - $D_{\mathbf{1n},} = D_{\mathbf{2n}} + \rho_s$
- The normal component of B is continuous at the surface of discontinuity
  - $B_{\mathbf{1n},} = B_{\mathbf{2n}}$



Proof of boundary conditions - $\underline{D}_n$

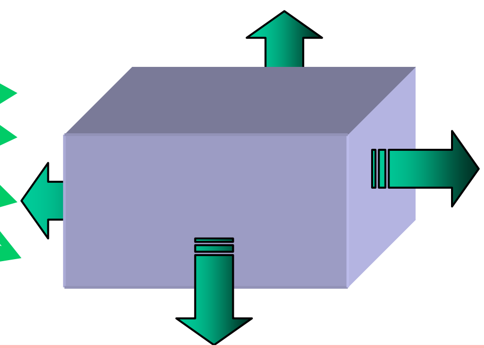


- The integral form of Gauss' law for electrostatics is:

$$\oiint \mathbf{D}.\mathrm{d}\mathbf{A} = \iiint_V \rho dV$$

applied to the box gives

$$D_{n1}\Delta x \Delta y - D_{n2}\Delta x \Delta y + \Psi_{\text{edge}} = \rho_s \Delta x \Delta y$$



As $\mathrm{d}z \to 0, \Psi_{\text{edge}} \to 0$ hence

$$\boxed{D_{n1} - D_{n2} = \rho_s}$$

The change in the normal component of **D** at a boundary is equal to the surface charge density

Proof of boundary conditions - $\underline{\mathbf{D}}_n$ cont.

$$D_{n1} - D_{n2} = \rho_s$$

- For an insulator with no static electric charge ρ_s=0

$$D_{n1} = D_{n2}$$

- For a conductor all charge flows to the surface and for an infinite, plane surface is uniformly distributed with area charge density ρ_s

  In a good conductor, σ is large, $\mathbf{D}=\varepsilon\mathbf{E}\approx 0$ hence if medium 2 is a good conductor

$$D_{n1} = \rho_s$$

Proof of boundary conditions - $\underline{\mathbf{B}}_{\mathbf{n}}$

- Proof follows same argument as for D_n on page 47,
- The integral form of Gauss' law for magnetostatics is

$$\oiint \mathbf{B}.\mathrm{d}\mathbf{A} = 0$$

  - there are no isolated magnetic poles

$$B_{n1}\Delta x\Delta y - B_{n2}\Delta x\Delta y + \Psi_{\mathrm{edge}} = 0$$

$$\Rightarrow \qquad \boxed{B_{n1} = B_{n2}}$$

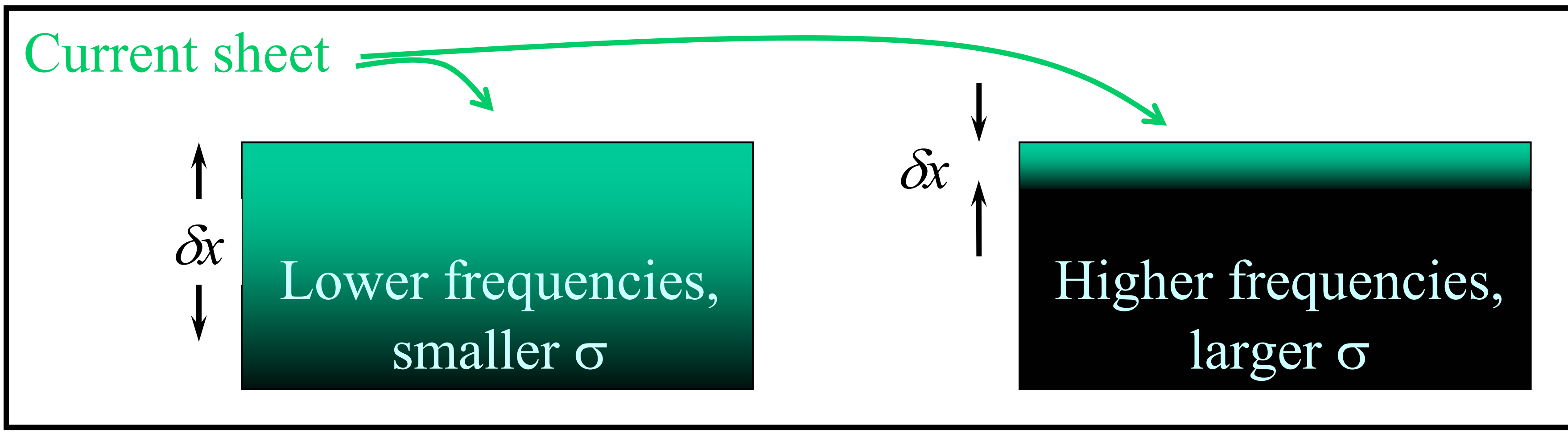
The normal component of **B** at a boundary is always continuous at a boundary

Conditions at a perfect conductor

- In a perfect conductor σ is infinite
- Practical conductors (copper, aluminium silver) have very large σ and field solutions assuming infinite σ can be accurate enough for many applications
  - Finite values of conductivity are important in calculating Ohmic loss
- For a conducting medium
  - $\mathbf{J}=\sigma\mathbf{E}$
    - infinite $\sigma \Rightarrow$ infinite $\mathbf{J}$
    - More practically, σ is very large, $\mathbf{E}$ is very small (≈ 0) and $\mathbf{J}$ is finite

Conditions at a perfect conductor

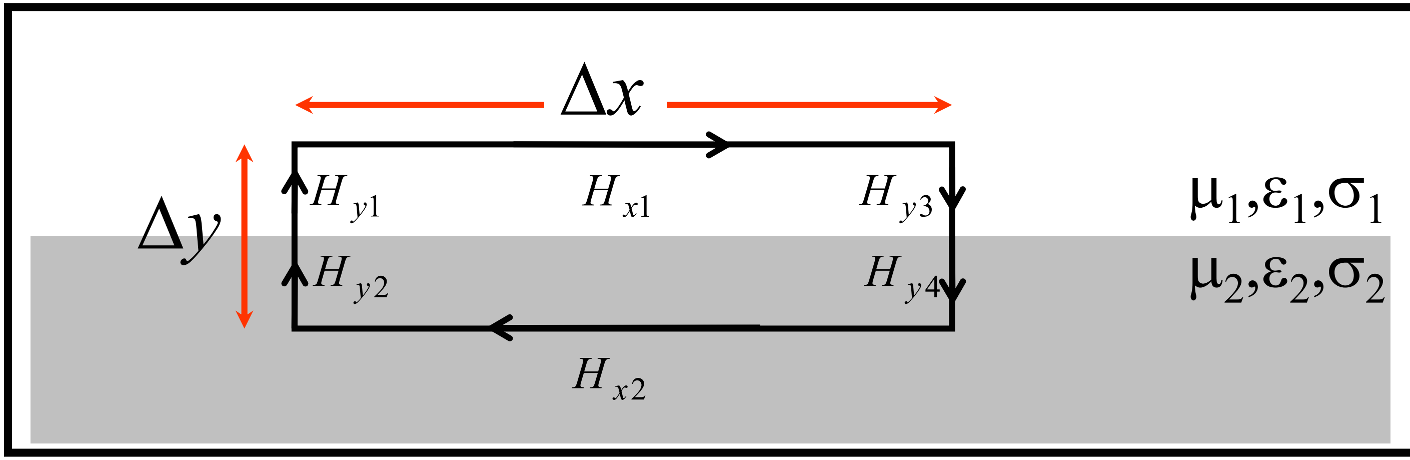
- It will be shown that at high frequencies **J** is confined to a surface layer with a depth known as the *skin depth*
- With increasing frequency and conductivity the skin depth, δx becomes thinner



- It becomes more appropriate to consider the current density in terms of current per unit with:

$$\lim_{\delta x \to 0} \mathbf{J}\,\delta x = \mathbf{J}_s \ \text{A/m}$$

Conditions at a perfect conductor cont.



- Ampere's law: $$\oint \mathbf{H}.d\mathbf{s} = \iint_A \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right).d\mathbf{A}$$

$$\overbrace{H_{y2}\frac{\Delta y}{2}}^{0} + \overbrace{H_{y1}\frac{\Delta y}{2}}^{0} + H_{x1}\Delta x - \overbrace{H_{y3}\frac{\Delta y}{2}}^{0} - \underbrace{H_{y4}\frac{\Delta y}{2}}_{0} - H_{x2}\Delta x = \overbrace{\left(\frac{\partial D_z}{\partial t} + J_z\right)\Delta x\Delta y}^{J_{sz}\Delta x}$$

$$\text{As } \Delta y \to 0, \qquad \partial D_z / \partial t\, \Delta x\Delta y \to 0, \qquad J_z \Delta x\Delta y \to \Delta x J_{sz}$$

$$H_{x1} - H_{x2} = J_{sz}$$

That is, the tangential component of **H** is discontinuous by an amount equal to the surface current density

- From Maxwell's equations:
  - If in a conductor E=0 then dE/dT=0
  - Since $$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{dt}$$

H_{x2}=0 (it has no time-varying component and also cannot be established from zero)

$$H_{x1} = J_{sz}$$

The current per unit width, $\mathbf{J}_s$, along the surface of a perfect conductor is equal to the magnetic field just outside the surface:

- **H** and **J** and the surface normal, **n**, are mutually perpendicular: $$\mathbf{J}_s = \mathbf{n} \times \mathbf{H}$$

Summary of Boundary conditions

At a boundary between non-conducting media

$$\begin{aligned} E_{t1} &= E_{t2} \\ H_{t1} &= H_{t2} \\ D_{n1} &= D_{n2} \\ B_{n1} &= B_{n2} \end{aligned} \quad \equiv \quad \begin{aligned} n \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\ n \times (\mathbf{H}_1 - \mathbf{H}_2) &= 0 \\ n.(\mathbf{D}_1 - \mathbf{D}_2) &= 0 \\ n.(\mathbf{B}_1 - \mathbf{B}_2) &= 0 \end{aligned}$$

At a metallic boundary (large σ)

$$\begin{aligned} n \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\ n \times (\mathbf{H}_1 - \mathbf{H}_2) &= 0 \\ n.(\mathbf{D}_1 - \mathbf{D}_2) &= \rho_s \\ n.(\mathbf{B}_1 - \mathbf{B}_2) &= 0 \end{aligned}$$

At a perfectly conducting boundary

$$\begin{aligned} n \times \mathbf{E}_1 &= 0 \\ n \times \mathbf{H}_1 &= \mathbf{J}_s \\ n.\mathbf{D}_1 &= \rho_s \\ n.\mathbf{B}_1 &= 0 \end{aligned}$$

Reflection and refraction of plane waves

- At a discontinuity the change in μ, ε and σ results in partial reflection and transmission of a wave
- For example, consider normal incidence:

$$\text{Incident wave} = E_i e^{j(\omega t - \beta z)}$$

$$\text{Reflected wave} = E_r e^{j(\omega t + \beta z)}$$

- Where E_r is a complex number determined by the boundary conditions

Reflection at a perfect conductor

- Tangential **E** is continuous across the boundary
- For a perfect conductor **E** just inside the surface is zero
  - *E* just outside the conductor must be zero

$$E_i + E_r = 0$$
$$\Rightarrow E_i = -E_r$$

- Amplitude of reflected wave is equal to amplitude of incident wave, but reversed in phase

Standing waves

- Resultant wave at a distance -*z* from the interface is the sum of the incident and reflected waves

$$
\begin{aligned}
E_T(z,t) &= \text{incident wave} + \text{reflected wave} \\
&= E_i e^{j(\omega t - \beta z)} + E_r e^{j(\omega t + \beta z)} \\
&= E_i \left(e^{-j\beta z} - e^{j\beta z} \right) e^{j\omega t} \\
&= -2jE_i \sin \beta z \; e^{j\omega t}
\end{aligned}
$$

$$\sin \phi = \frac{e^{j\phi} - e^{j\phi}}{2j}$$

and if E_i is chosen to be real

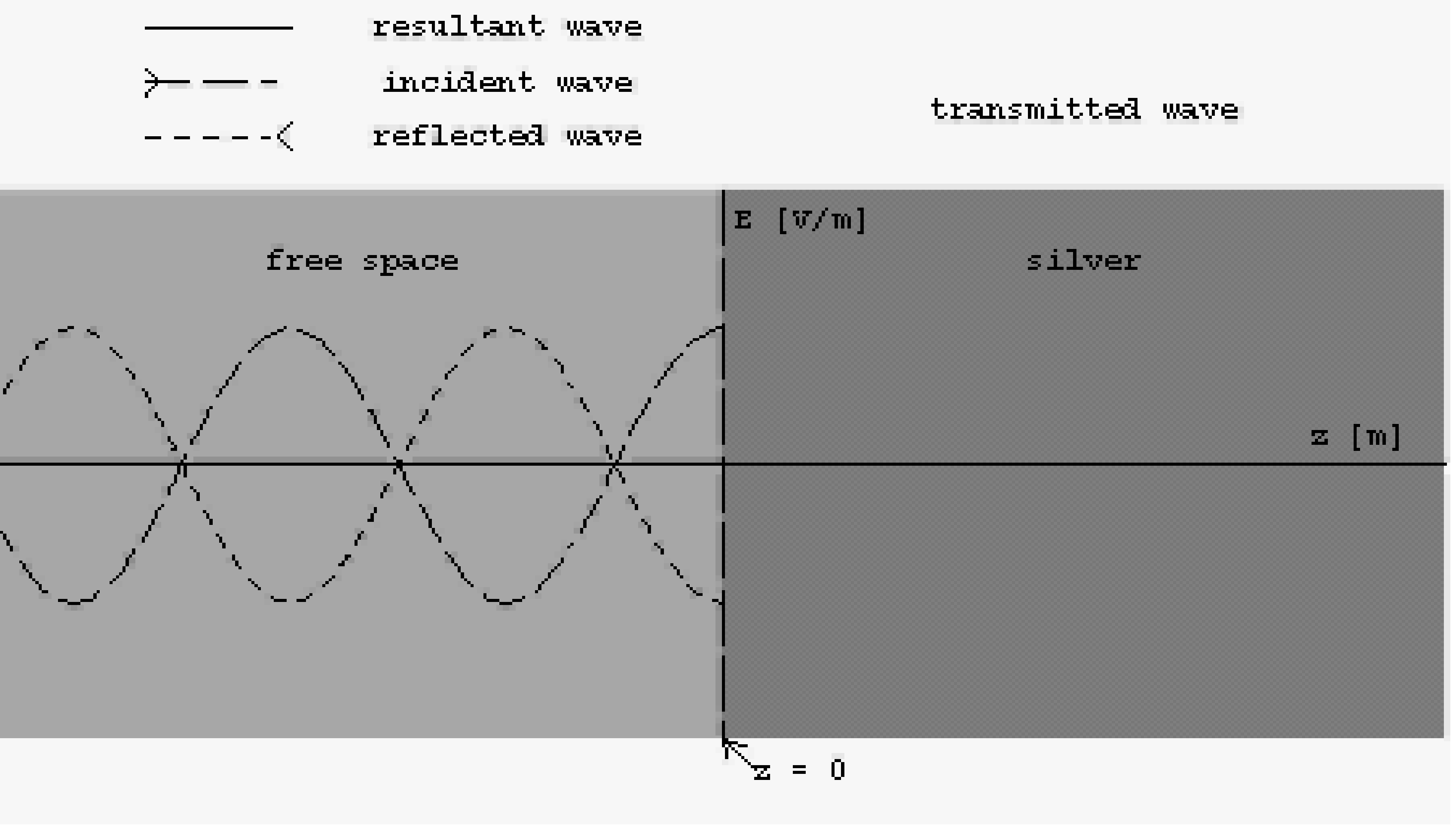
$$
\begin{aligned}
E_T(z,t) &= \operatorname{Re}\left\{ -2jE_i \sin \beta z \left(\cos \omega t + j \sin \omega t \right) \right\} \\
&= 2E_i \sin \beta z \sin \omega t
\end{aligned}
$$

Standing waves cont...

$$E_T(z,t) = 2E_i \sin \beta z \sin \omega t$$

- Incident and reflected wave combine to produce a standing wave whose amplitude varies as a function ($\sin \beta z$) of displacement from the interface
- Maximum amplitude is twice that of incident fields

Reflection from a perfect conductor



Reflection from a perfect conductor

- Direction of propagation is given by **E**×**H**

If the incident wave is polarised along the *y* axis:

$$E_i = \mathbf{a}_y E_{yi}$$

$$\Rightarrow H_i = -\mathbf{a}_x H_{xi}$$

then $$\mathbf{E} \times \mathbf{H} = \left(-\mathbf{a}_y \times \mathbf{a}_x\right) E_{yi} H_{xi}$$

$$= +\mathbf{a}_z E_{yi} H_{xi}$$

That is, a z-directed wave.

For the reflected wave $\mathbf{E} \times \mathbf{H} = -\mathbf{a}_z E_{yi} H_{xi}$ and $E_r = -\mathbf{a}_y E_{yi}$

So $H_r = -\mathbf{a}_x H_{xi} = H_i$ and the magnetic field is reflected without change in phase

Reflection from a perfect conductor

- Given that $\cos\phi = \dfrac{e^{j\phi} + e^{-j\phi}}{2}$

$$
\begin{aligned}
H_T(z,t) &= H_i e^{j(\omega t - \beta z)} + H_r e^{j(\omega t + \beta z)} \\
&= H_i \left(e^{j\beta z} + e^{-j\beta z}\right) e^{j\omega t} \\
&= 2H_i \cos\beta z \; e^{j\omega t}
\end{aligned}
$$

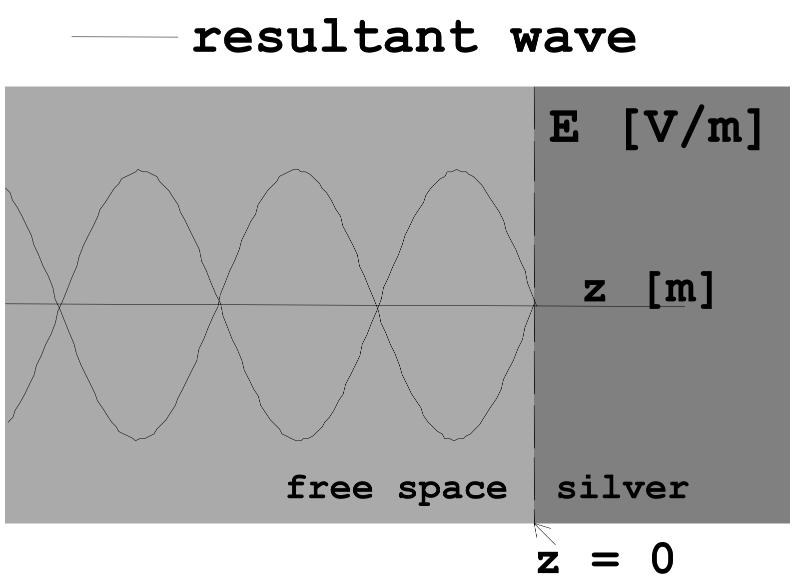
As for E_i, H_i is real (they are in phase), therefore

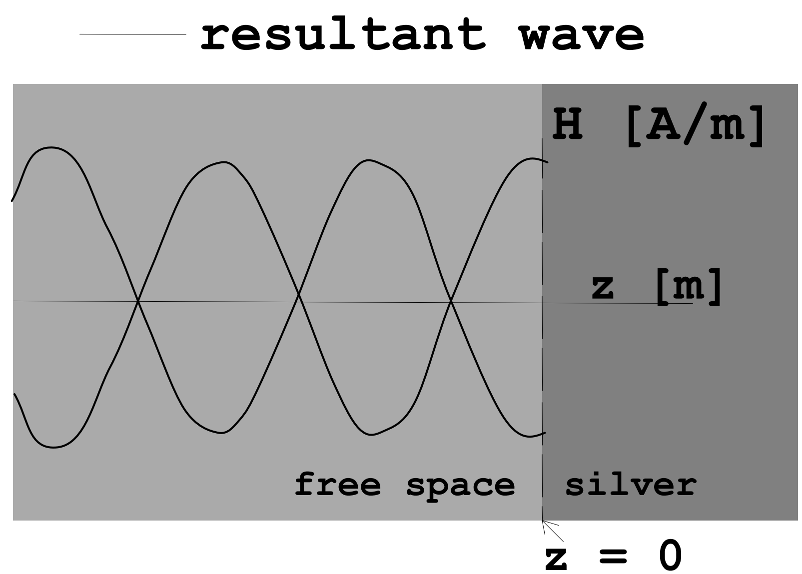
$$H_T(z,t) = \mathrm{Re}\{2H_i \cos\beta z \; (\cos\omega t + j\sin\omega t)\} = 2H_i \cos\beta z \; \cos\omega t$$

Reflection from a perfect conductor

$$H_T(z,t) = 2H_i \cos \beta z \cos \omega t$$

- Resultant magnetic field strength also has a standing-wave distribution
- In contrast to **E**, **H** has a maximum at the surface and zeros at $(2n+1)\lambda/4$ from the surface:





Reflection from a perfect conductor

$$E_T(z,t) = 2E_i \sin \beta z \sin \omega t$$

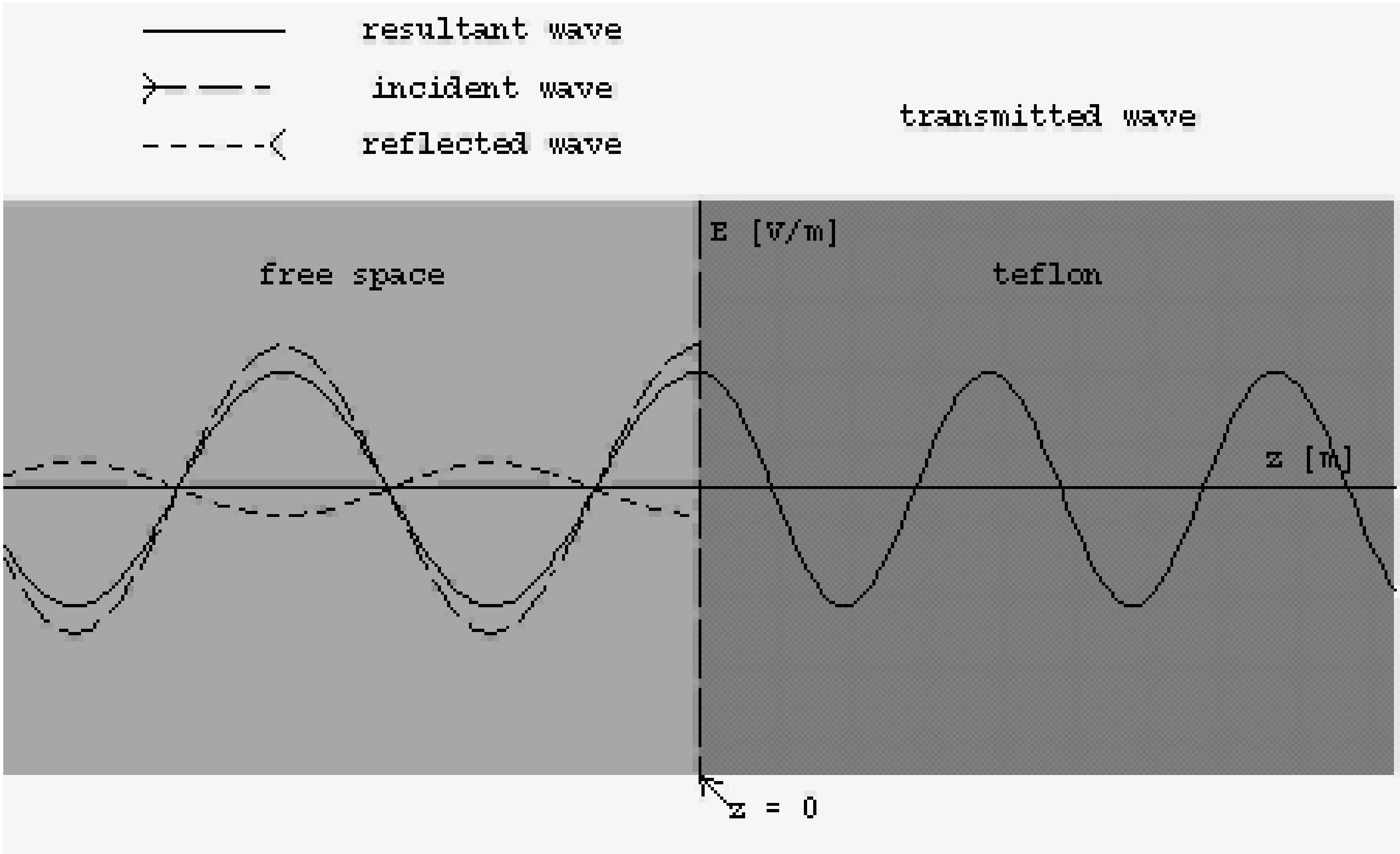
$$H_T(z,t) = 2H_i \cos \beta z \ \cos \omega t$$

- E_T and H_T are π/2 out of phase$(\sin \omega t = \cos(\omega t - \pi / 2))$
- No net power flow as expected
  - power flow in $+z$ direction is equal to power flow in - z direction

Reflection by a perfect dielectric

- Reflection by a perfect dielectric ($\mathbf{J}=\sigma\mathbf{E}=\mathbf{0}$)
  - no loss
- Wave is incident normally
  - **E** and **H** parallel to surface
- There are incident, reflected (in medium 1)and transmitted waves (in medium 2):

Reflection from a lossless dielectric



Reflection by a lossless dielectric

$$E_i = \eta_1 H_i$$
$$E_r = -\eta_1 H_r$$
$$E_t = \eta_2 H_t$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon_o\varepsilon_r}} = \sqrt{\frac{\mu}{\varepsilon}}$$

- Continuity of E and H at boundary requires:

$$E_i + E_r = E_t$$
$$H_i + H_r = H_t$$

Which can be combined to give

$$H_i + H_r = \frac{1}{\eta_1}\left(E_i - E_r\right) = H_t = \frac{1}{\eta_2}E_t = \frac{1}{\eta_2}\left(E_i + E_r\right)$$

$$\frac{1}{\eta_1}\left(E_i - E_r\right) = \frac{1}{\eta_2}\left(E_i + E_r\right) \quad \Rightarrow \quad \rho_E = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\Rightarrow \eta_2\left(E_i - E_r\right) = \eta_1\left(E_i + E_r\right)$$

$$\Rightarrow E_i\left(\eta_2 - \eta_1\right) = E_r\left(\eta_2 + \eta_1\right)$$

The reflection coefficient

Reflection by a lossless dielectric

$$E_i + E_r = E_t$$
$$H_i + H_r = H_t$$

- Similarly

$$\tau_E = \frac{E_t}{E_i} = \frac{E_r + E_i}{E_i} = \frac{E_r}{E_i} + 1 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} + \frac{\eta_2 + \eta_1}{\eta_2 + \eta_1} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$\tau_E = \frac{2\eta_2}{\eta_2 + \eta_1}$$

The transmission coefficient

Reflection by a lossless dielectric

- Furthermore:

$$\frac{H_r}{H_i} = -\frac{E_r}{E_i} = \rho_H$$

$$\frac{H_t}{H_i} = \frac{\eta_1 E_t}{\eta_2 E_i} = \frac{\eta_1}{\eta_2}\frac{2\eta_2}{\eta_2+\eta_1} = \frac{2\eta_1}{\eta_2+\eta_1}\tau_H$$

And because $\mu=\mu_o$ for all low-loss dielectrics

$$\rho_E = \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}} = \frac{n_1-n_2}{n_1+n_2} = -\rho_H$$

$$\tau_E = \frac{E_r}{E_i} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}} = \frac{2n_1}{n_1+n_2}$$

$$\tau_H = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}} = \frac{2n_2}{n_1+n_2}$$

Energy Transport - Poynting Vector

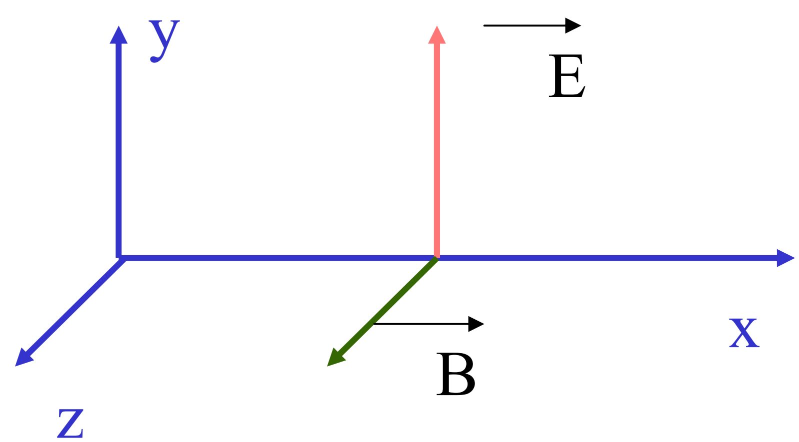
Electric and Magnetic Energy Density:

For an electromagnetic plane wave

$$\overline{E}_y(x,t) = \overline{E}_0 \, sin(kx - \omega t)$$

$$\overline{B}_z(x,t) = \overline{B}_0 \, sin(kx - \omega t)$$

$$where \; B_0 = E_0 / c$$



The electric energy density is given by

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 \overline{E}_0^2 \, sin^2(kx - \omega t)$$ *and the magnetic energy is*

$$u_B = \frac{1}{2\mu_0} B^2 = \frac{1}{2\mu_0 c} \overline{E}^2 = u_E$$

Note: I used $\overline{E} = c\overline{B}$

Energy Transport - Poynting Vector cont.

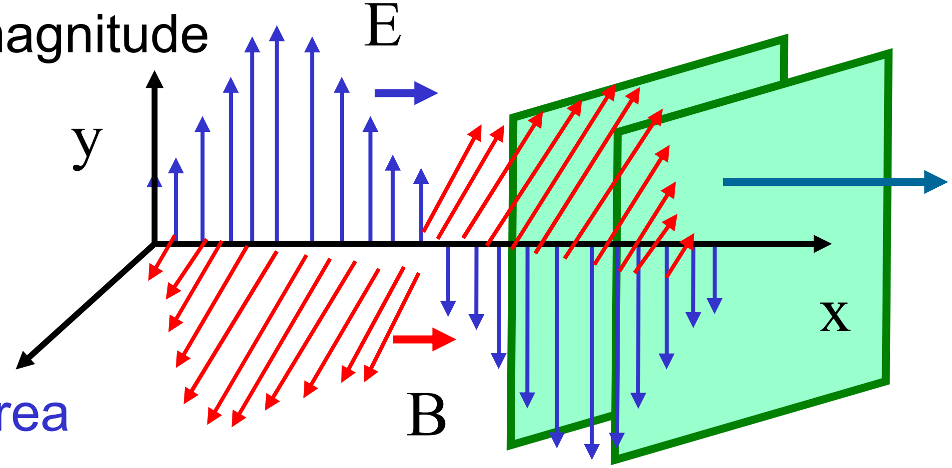
Thus, for light the electric and the magnetic field energy densities are equal and the total energy density is

$$u_{total} = u_E + u_B = \varepsilon_0 E^2 = \frac{1}{\mu_0} B^2 = \varepsilon_0 \overline{E}_0^2 \sin^2(kx - \omega t)$$

Poynting Vector $\left(\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \right)$:

The direction of the Poynting Vector is the direction of energy flow and the magnitude

$$\left(S = \frac{1}{\mu_0} EB = \frac{E^2}{\mu_0 c} = \frac{1}{A} \frac{dU}{dt} \right)$$



Is the energy per unit time per unit area (units of Watts/m^2).

Energy Transport - Poynting Vector cont.

Proof:

$$dU_{total} = u_{total}V = \varepsilon_0 E^2 Acdt \text{ so}$$

$$S = \frac{1}{A}\frac{dU}{dt} = \varepsilon_0 c E^2 = \frac{E^2}{\mu_0 c} = \frac{E_0{}^2}{\mu_0 c} sin^2(kx - \omega t)$$

Intensity of the Radiation (Watts/m^2):

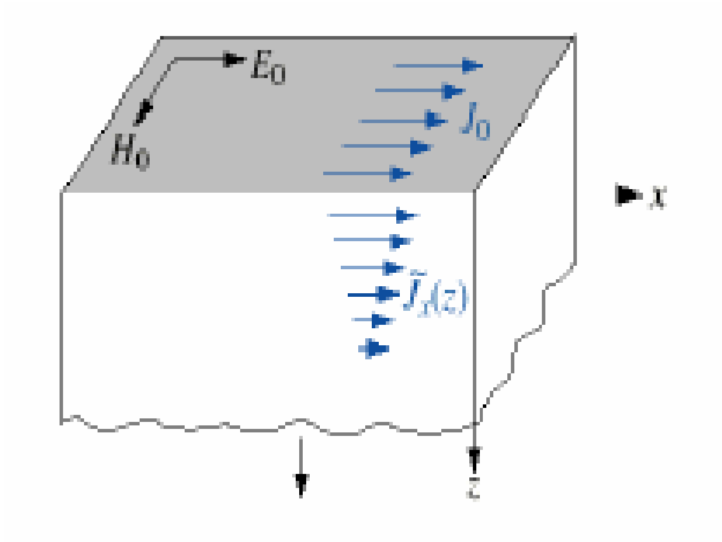
The intensity, I, is the average of S as follows:

$$I = \bar{S} = \frac{1}{A}\frac{d\overline{U}}{dt} = \frac{E_0{}^2}{\mu_0 c}\left\langle sin^2(kx - \omega t)\right\rangle = \frac{E^2}{2\mu_0 c}.$$

- **Ohm's law**

$$\bar{J} = \sigma \bar{E}$$

- **Skin depth**



Current density decays exponentially from the surface into the interior of the conductor

Phasors

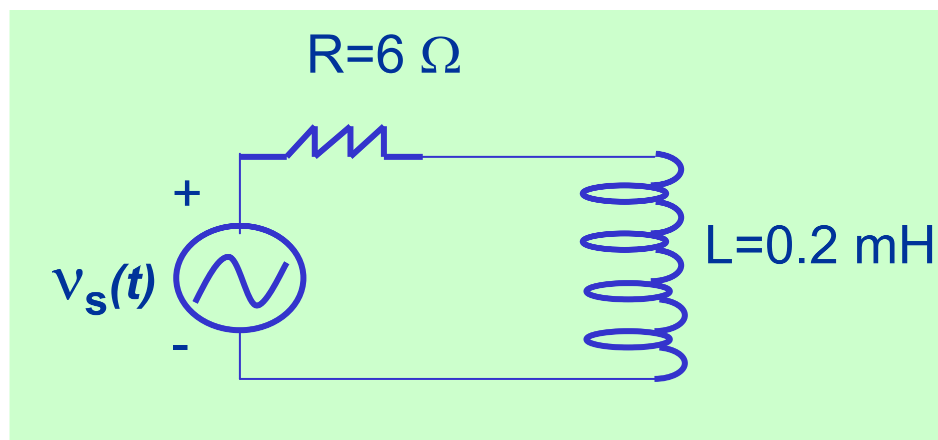
Fictitious way of dealing with AC circuits

$$i(t) = Re\left\{ I e^{j\omega t} \right\}$$

$$I = \frac{V_s}{R + j\omega L}$$

Measurable quantity

Phasor (not real)



Phasors cont.

- Phasors in lumped circuit analysis have no space components

- Phasors in distributed circuit analysis (RF) have a space component because they act as waves

$$v(x,t) = Re\left\{V_0 e^{\pm j\beta x}\ e^{j\omega t}\right\} = V_0 \cos\left(\omega t \pm \beta x\right)$$

Displacement Current

Observe that the vector field $\frac{1}{c}\frac{\partial E}{\partial t}$ appears to form a continuation of the conduction current distribution. Maxwell called it the displacement current, and the name has stuck although in no longer seem very appropriate.

We can define a displacement current density J_d , to be distinguished from the conduction current density J, by writing

$$curl\ B = \frac{4\pi}{c}\left(J + J_d\right)$$

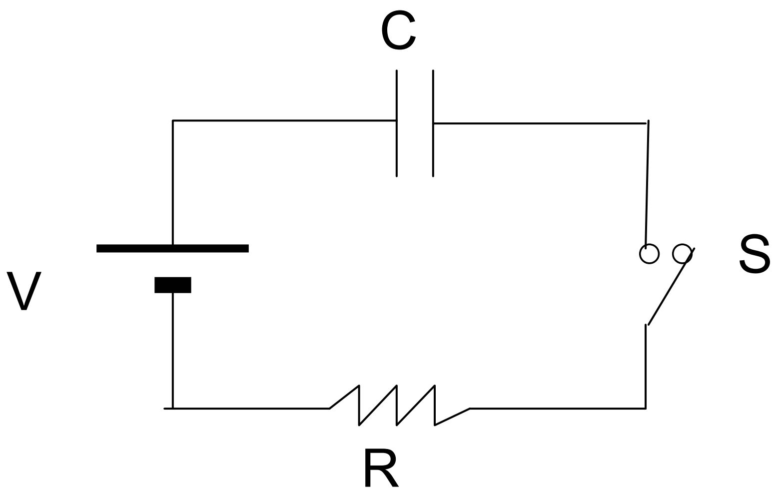
and define

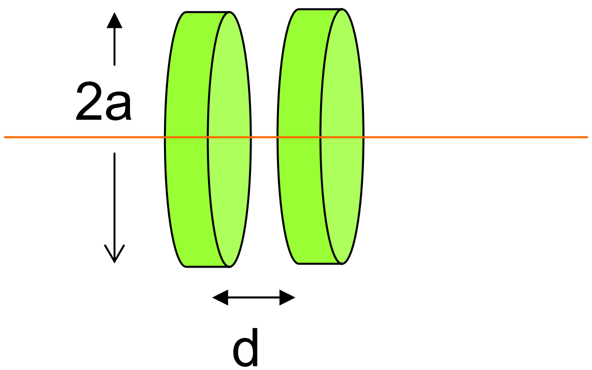
$$J_d = \frac{1}{4\pi}\frac{\partial E}{\partial t}$$

It turns out that physical displacement current lead to small magnetic fields that are difficult to detect. To see this effect, we need rapidly changing fields (Hertz experiment).

Displacement Current

Example: $I=I_d$ in a circuit branch having a capacitor





$$E(t)=\frac{V(t)}{d}=\frac{Q(t)}{Cd}$$

The displacement current density is given by

$$J_d=\frac{1}{4\pi}\frac{\partial E(t)}{\partial t}=\frac{1}{4\pi Cd}\frac{\partial Q(t)}{\partial t}=\frac{I(t)}{4\pi Cd}$$

Displacement Current

The direction of the displacement current is in the direction of the current. The total current of the displacement current is

$$I_d = A.J_d = \frac{A \cdot I}{4\pi C \cdot d} = I$$

Thus the current flowing in the wire and the displacement current flowing in the condenser are the same.

How about the magnetic field inside the capacitor? Since the is no real current in the capacitor,

$$curl\ B = \frac{1}{c}\frac{\partial E}{\partial t}$$

Integrating over a circular area of radius r,

$$\int_{S(r)} curl\ B \cdot da = \frac{1}{c}\int_{S(r)} \frac{\partial E}{\partial t} \cdot da$$

Displacement Current

$$l.h.s = \int_{S(r)} curlB \cdot da = \int_{C(r)} B \cdot ds = 2\pi B \cdot r$$

$$r.h.s = \frac{1}{c}\frac{\partial}{\partial t}\int_{S(r)} E \cdot da = \frac{\pi r^2}{c}\frac{\partial E}{\partial t}$$

$$= \frac{\pi r^2}{cd}\frac{\partial V}{\partial t} = \frac{\pi r^2}{cd}\frac{1}{C}\frac{\partial Q}{\partial t} = \frac{\pi r^2}{cd}\frac{I}{C} = \frac{4\pi I}{c}\frac{r^2}{a^2}$$

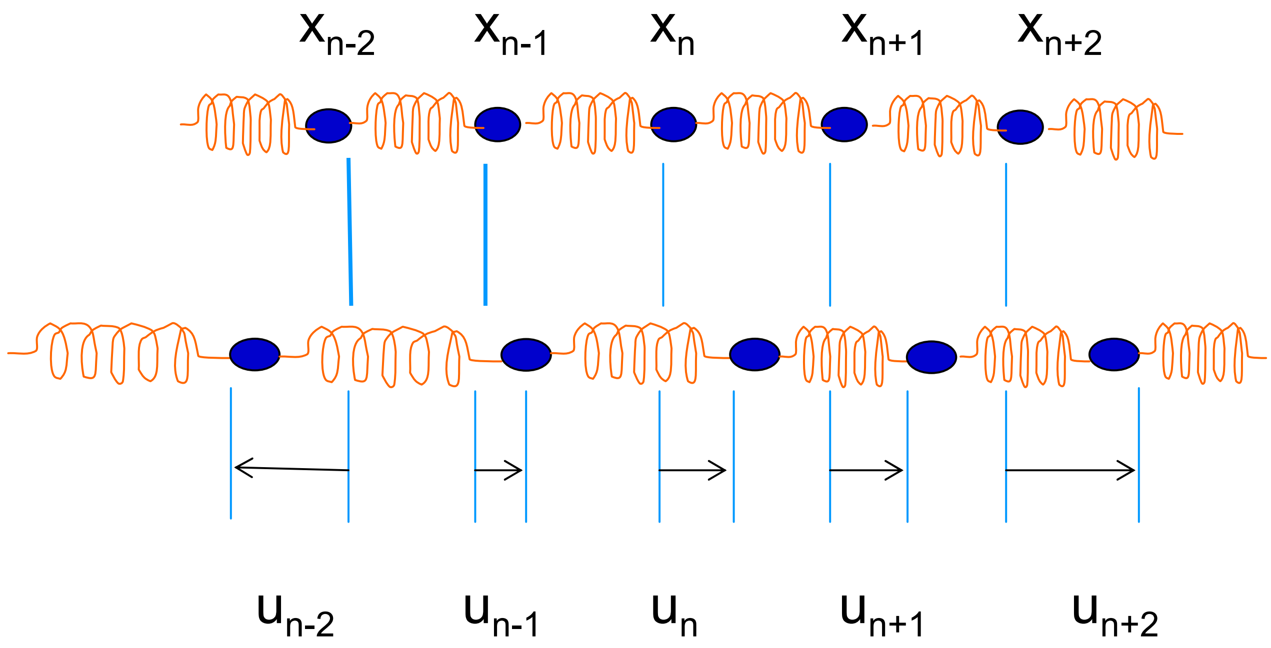
Thus the magnetic field in the capacitor is

$$2\pi B \cdot r = \frac{4\pi I}{c}\frac{r^2}{a^2} \to B(r) = \frac{2Ir}{ca^2}$$

$$2\pi B \cdot r = \frac{4\pi I}{c} \to B(r) = \frac{2I}{cr}$$ (at the edge of the capacitor)

This is the same as that produced by a current flowing in an infinitely long wire.

Wave in Elastic Medium



The equation of motion for n[th] mass is

$$m\frac{\partial^2 u_n}{\partial t^2} = -k\left(u_n - u_{n-1}\right) + k\left(u_{n+1} - u_n\right) = k\left(u_{n-1} - 2u_n + u_{n+1}\right)$$

By expanding the displacement $u_{n\pm1}(t)=u(x_{n\pm1},t)$ around x_n, we can convert the equation into a DE with variable x and t.

Wave in Elastic Medium

$$u_{n\pm1}(t)=u(x_n \pm \Delta x,t)=u(x_n,t)+\frac{\partial u(x_n,t)}{\partial x_n}(\pm\Delta x)+\frac{1}{2}\frac{\partial^2 u(x_n,t)}{\partial x_n^2}(\pm\Delta x)^2+\ldots$$

$$m\frac{\partial^2 u(x_n,t)}{\partial t^2}=k\Delta x^2\frac{\partial^2 u(x_n,t)}{\partial x_n^2}\rightarrow\frac{m}{\Delta x}\frac{\partial^2 u(x_n,t)}{\partial t^2}=k\Delta x\frac{\partial^2 u(x_n,t)}{\partial x_n^2}$$

Define $K \equiv k\,\Delta x$ as the elastic modulus of the medium and $\rho = m/\Delta x$ is the mass density. In continuous medium limit $\Delta x \longrightarrow 0$, we can take out n.

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2}=K\frac{\partial^2 u(x,t)}{\partial x^2}$$

We examine a wave equation in three dimensions. Consider a physical quantity that depends only on z and time t.

Wave along z-axis

$$\frac{\partial^2 \Psi(z,t)}{\partial t^2} = \nu^2 \frac{\partial^2 \Psi(z,t)}{\partial z^2}$$

We prove that the general solution of this DE is given by

$$\Psi(z,t) = f(z - vt) + g(z + vt)$$

f and *g* are arbitrary functions.

Insert a set of new variables,

$$\xi = z - vt \quad and \quad \eta = z + vt$$

Then

$$\frac{\partial}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial z}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -\nu\frac{\partial}{\partial \xi} + \nu\frac{\partial}{\partial \eta}$$

Wave along z-axis

$$\Rightarrow \quad \left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)^2 \Psi = \left(\frac{\partial}{\partial \xi}-\frac{\partial}{\partial \eta}\right)^2 \Psi$$

thus $$\frac{\partial^2}{\partial \eta \partial \xi}\Psi = 0$$

From this equation:

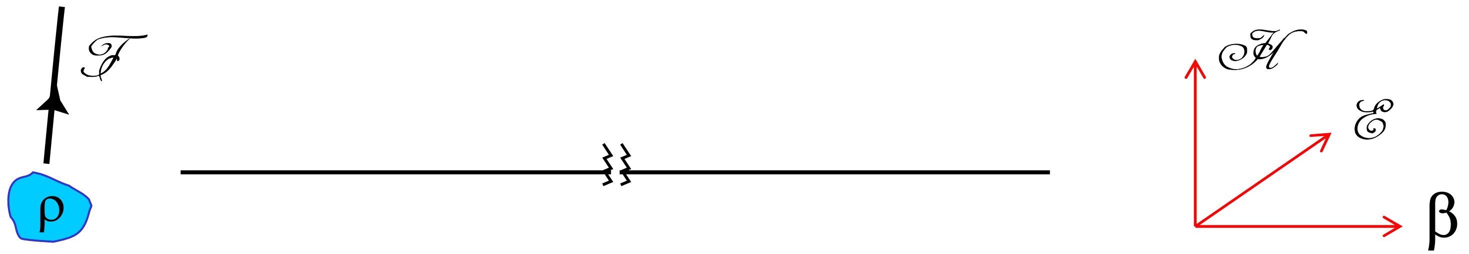
$$\frac{\partial}{\partial \eta}\frac{\partial \Psi}{\partial \xi} = 0 \rightarrow \frac{\partial \Psi}{\partial \xi} = F(\xi)$$

$$\frac{\partial \Psi}{\partial \xi} = F(\xi) \rightarrow \Psi = \int F(\xi) d\xi + g(\eta) \equiv f(\xi) + g(\eta)$$

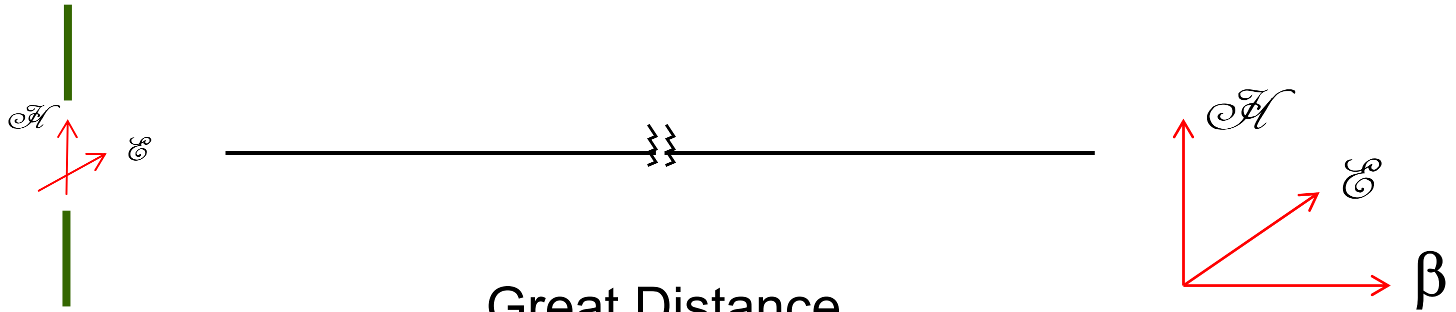
Thus

$$\Psi(z,t) = f(z - vt) + g(z + vt)$$

Radiation



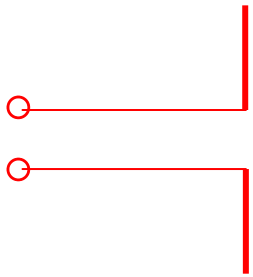
Charges and currents | Great Distance | Approximate plane waves



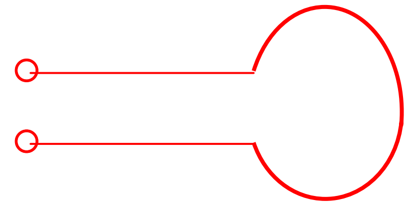
Aperture fields | Great Distance | Approximate plane waves

Radiation

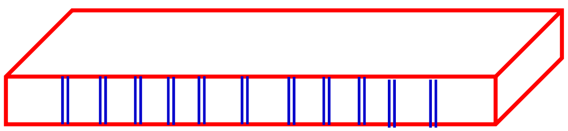
Antennas



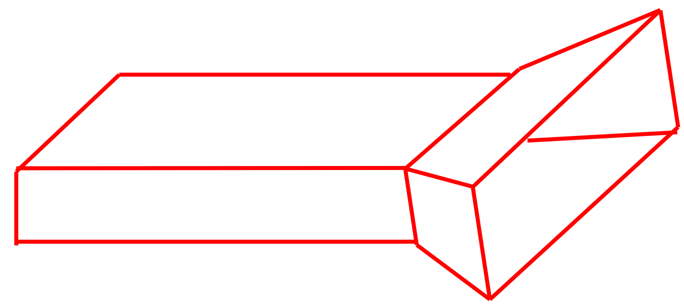
Transmission line fed dipole



Transmission line fed current loop



Slots in waveguide



Waveguide fed horn

Radiation

In the time domain the electric scalar potential $\phi(r_2,t)$ and the magnetic vector potential $A(r_2,t)$ produced at time t at a point r_2 by charge and current distribution $\rho(r_1)$ and $J(r_1)$ are given by

$$\phi(r_2,t)=\frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(r_1,t-r_{12}/c)}{r_{12}}dv$$

and

$$A(r_2,t)=\frac{\mu_0}{4\pi}\int_V \frac{J(r_1,t-r_{12}/c)}{r_{12}}dv$$

Sinusoidal steady state

$$\phi(r_2)=\frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(r_1)e^{-j\beta r_{12}}}{r_{12}}dv$$

$e^{-j\beta r_{12}}$ ***is the phase retardation factor***

$$A(r_2)=\frac{\mu_0}{4\pi}\int_V \frac{J(r_1)e^{-j\beta r_{12}}}{r_{12}}dv$$

We start with

$$B = curl\ A \quad \textbf{and} \quad E = -grad\ \phi - j\omega A$$

Charge conservation:

$$div\ J + \frac{\partial \rho}{\partial t} = 0 \xrightarrow{\text{Sinusoidal steady state}} div\ J + j\omega\rho = 0$$

Because ρ and J are related by the charge conservation equation, ϕ and A are also related. In the time domain,

$$div\ A + \mu_0\varepsilon_0 \frac{\partial \phi}{\partial t} = 0 \xrightarrow{\text{Sinusoidal steady state}} div\ A + j\omega\mu_0\varepsilon_0\phi = 0$$

With $\omega \neq 0$

$$\phi = -\frac{div\ A}{j\omega\mu_0\varepsilon_0}$$

Substituting for ϕ:

$$H = \frac{1}{\mu_0} curl\ A$$

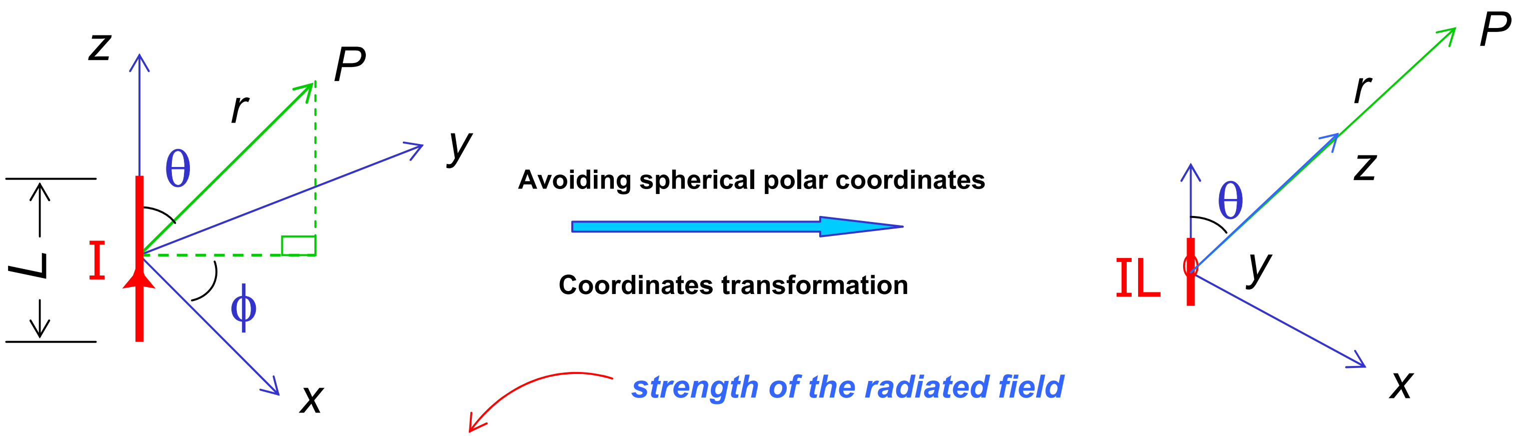
$$E = \frac{1}{j\omega\mu_0\varepsilon_0} grad\ div\ A - j\omega A$$

$$= -\frac{j\omega}{\beta^2} grad\ div\ A - j\omega A$$

$$\boxed{c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \quad \omega = c\beta}$$

Near and far fields

We consider the transmission characteristics of a particular antenna in the form of a straight wire, carrying an oscillatory current whose length is much less than the electromagnetic wavelength at the operating frequency. Such antenna is called a *short electric dipole.*



$$j\omega P = IL$$

The components of the dipole vector in these coordinates are

$$P = \begin{bmatrix} p_x \\ 0 \\ p_z \end{bmatrix} = \begin{bmatrix} -p\sin\theta \\ 0 \\ p\cos\theta \end{bmatrix}$$

Dipole radiation

The retarded vector potential is then

$$A = \frac{\mu_0}{4\pi}\int_V \frac{Je^{-j\beta z}}{z}dv$$

Where we used $\beta = \dfrac{\omega}{c}$. We also replace $\int_V Jdv$ by $IL = j\omega P$ and obtain

$$A = \frac{\mu_0}{4\pi}\left(j\omega P\right)\frac{e^{-j\beta z}}{z}$$

$$curl\ A \approx \frac{j\omega\mu_0}{4\pi z}\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P_x e^{-j\beta z} & 0 & P_z e^{-j\beta z}\end{vmatrix} = \frac{j\omega\mu_0}{4\pi z}\begin{bmatrix} 0 \\ j\beta P_x e^{-j\beta z} \\ 0 \end{bmatrix}$$

Thus the radiation component of the magnetic field has a y component only given by

$$H_y = -j\beta j\omega \frac{P_x e^{-j\beta z}}{4\pi z}$$

Dipole radiation

Electric field:

We start with
$$divA \approx \frac{\partial A_z}{\partial z} = \frac{j\omega\mu_0 P_z(-j\beta)e^{-j\beta z}}{4\pi z}$$

then
$$grad\ divA = \frac{\mu_0 j\omega P_z(-j\beta)}{4\pi z}\begin{bmatrix} 0 \\ 0 \\ (-j\beta)e^{-j\beta z} \end{bmatrix}$$

The first term we require for the electric field is simply

$$\frac{-j\omega}{\beta^2} grad\ divA = \frac{-\omega^2\mu_0 e^{-j\beta z}}{4\pi z}\begin{bmatrix} 0 \\ 0 \\ P_z \end{bmatrix}$$

The second term we require for the electric field is

$$-j\omega A = \frac{-\omega^2\mu_0 e^{-j\beta z}}{4\pi z}\begin{bmatrix} -P_x \\ 0 \\ -P_z \end{bmatrix}$$

Dipole radiation

Electric field:

The electric field is the sum of these two terms. It may be seen that the z components cancel, and we are left with only x component of field given by

$$E_x = \frac{\omega^2 \mu_0 M_x e^{-j\beta z}}{4\pi z}$$

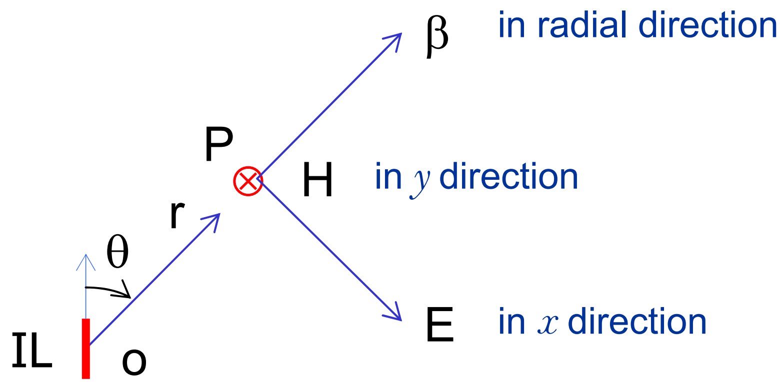
Note that this expression also fits our expectation of an approximately uniform plane wave. The ratio of electric to magnetic field amplitudes is

$$\frac{E_x}{H_y} = \frac{\mu_0 \omega^2}{\beta\omega} = \mu_0 \frac{\omega}{\beta} = \mu_0 c = \mu_0 \sqrt{\frac{1}{\mu_0 \varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta$$

as expected for a uniform plane wave.

Dipole radiation

We will now translate the field components into the spherical polar coordinates.



since $P_x = -P \sin\theta$ we have

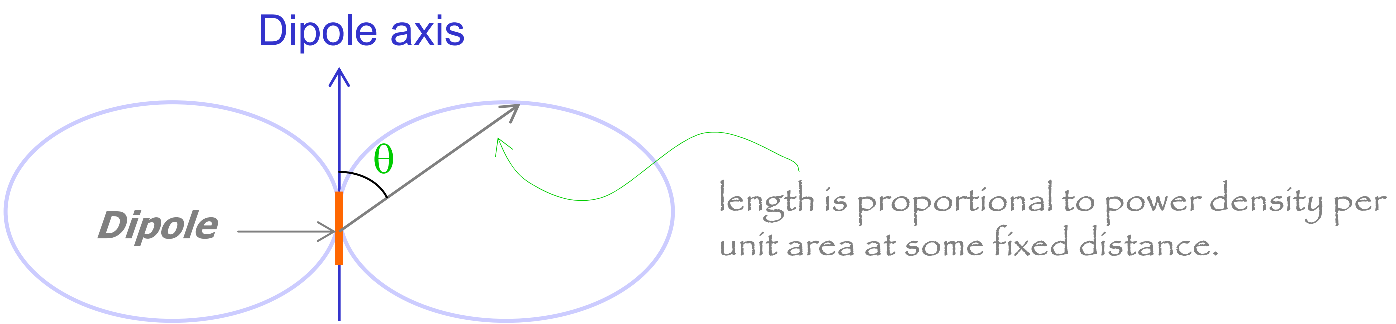
$$E_\theta = E_x = \frac{\omega^2 \mu_0 P \sin\theta e^{-j\beta r}}{4\pi r} \quad \text{and} \quad H_\phi = H_y = \frac{-\omega\beta P \sin\theta e^{-j\beta r}}{4\pi r}$$

The Poynting vector $\frac{1}{2}\left(\overline{E} \times \overline{H}^*\right)$ is in *r* direction and has the value

$$\boxed{S_r = S_z = \frac{\mu_0 \omega^3 \beta |P|^2 \sin^2\theta}{2(4\pi r)^2}}$$

This vector (real) gives the real power per unit area flowing across an element of area ⊥ to *r* at a great distance.

Radiation pattern



Note: *No radiation takes place along the dipole axis, and the radiation pattern has axial symmetry, with maximum radiation being in the equatorial plane.*

Because of the non-uniform nature of the pattern we have the concept of antenna gain, which for a lossless antenna is the power flow per unit area for the antenna in the most efficient direction over the power flow per unit area we would obtain if the energy were uniformly radiated in all directions. The total radiated power is

$$W = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \Re e\{S_r\}\left(r^2 \sin\theta \, d\theta \, d\phi\right) = \frac{\mu_0 \omega^3 \beta |P|^2}{32\pi^2} \int_{\theta=0}^{\pi} \sin^3\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{\mu_0 \omega^3 \beta |P|^2}{12\pi^2}$$

The average radiated power per unit area is

$$\frac{W}{4\pi r^2} = \frac{\mu_0 \omega^3 \beta |P|^2}{48\pi^2 r^2}$$

Hence the antenna gain, *g* defined by

$$g = \frac{\text{radiated power/unit area in the most efficient direction}}{\text{average radiated power/unit area over a large sphere}}$$

becomes

$$\boxed{g = \frac{\omega^3 \beta |P|^2}{32\pi^2 r^2} \frac{48\pi^2 r^2}{\omega^3 \beta |P|^2} = \frac{3}{2}}$$

This result is the gain of a small dipole.

Radiation resistance

Recall

$$W = \frac{\mu_0 \omega^3 \beta |P|^2}{12\pi} = \frac{\mu_0 \omega \beta |I|^2 L^2}{12\pi}$$

The radiation resistance R_r is defined as the equivalent resistance which would absorb the same power W from the same current I, i.e.

$$W = \frac{R_r |I|^2}{2}$$

Combining these results we obtain

$$R_r = \frac{\mu_0 \omega \beta L^2}{6\pi}$$

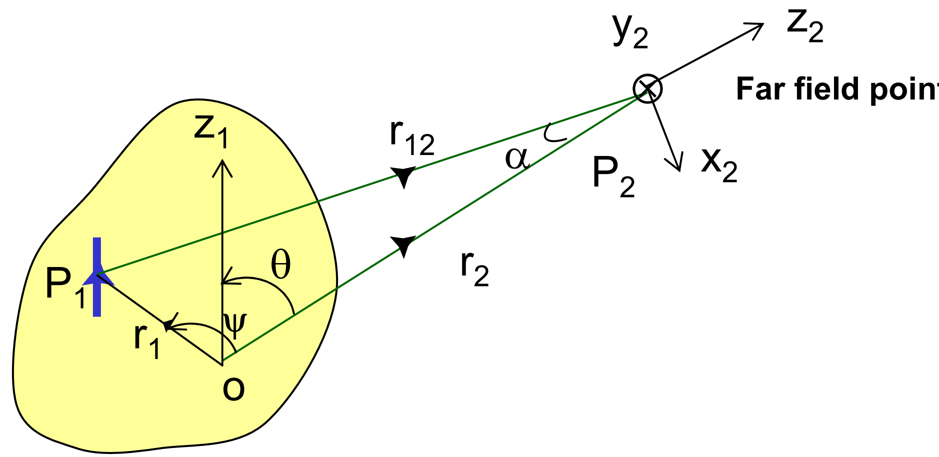
Using $\omega = c\beta,\ \beta = 2\pi/\lambda,\ \ c = 1/\sqrt{\mu_0 \varepsilon_0}$ and $\eta = \sqrt{\mu_0/\varepsilon_0}$, we find

$$R_r = \frac{\eta}{6\pi}(\beta L)^2 = \left(\frac{2\pi}{3}\right)\eta\left(\frac{L}{\lambda}\right)^2 \Longrightarrow R_r \approx 20(\beta L)^2 \Omega \quad (\eta \approx 120\pi\Omega)$$

Consider an arbitrary system of radiating currents

We start with the vector potential

$$A(r_2) = \frac{\mu_0}{4\pi} \int_V \frac{J(r_1) e^{-j\beta r_{12}}}{r_{12}} dv$$



We will regard r_{12} fixed. For P_2 a distance point, we replace r_{12} with r_2

So $$A(r_2) = \frac{\mu_0}{4\pi r_2} \int_V J(r_1) e^{-j\beta r_{12}} dv$$

Approximations for r_{12} in $e^{-j\beta r_{12}}$ require more care, sine phase differences in radiation effects are crucial. We use the following approximation

$$r_2 = r_1 + r_{12} \qquad r_2 \approx r_1 \cos\psi + r_{12} \qquad r_{12} \approx r_2 - r_1 \cos\psi$$

$$\Longrightarrow \quad A(r_2) = \frac{\mu_0 e^{-j\beta r_2}}{4\pi r_2} \int_V J(r_1) e^{+j\beta r_1 \cos\psi} dv$$

factor $e^{+j\beta r_1 \cos\psi}$ ***expresses the phase advance of the radiation from the element at P_1 relative to the phase at the origin.***

We have

$$A(r_2) = \frac{\mu_0 e^{-j\beta r_2}}{4\pi r_2}\Re$$

where

$$\Re = \int_V J(r_1) e^{j\beta r_1 \cos\psi} dv$$

$\Re$ is called the *radiation vector.* It depends on the *internal geometrical distribution* of the currents and on *the direction of P_2 from the origin O,* but not on the *distance.*

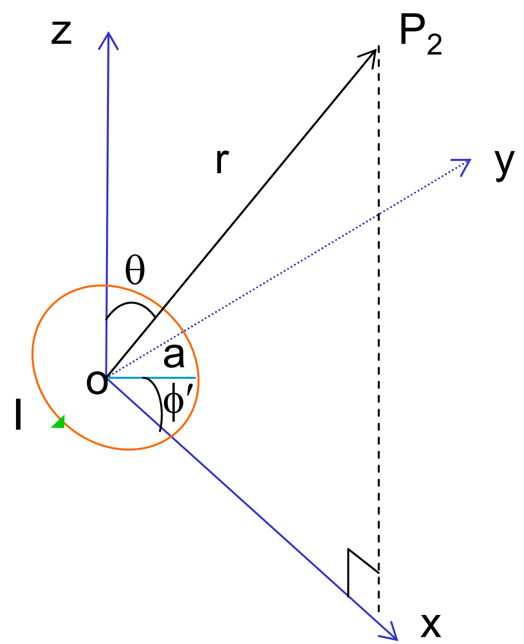
The factor $\frac{\mu_0 e^{-j\beta r_2}}{4\pi r_2}$ depends only on the distance from the origin O to the field point P_2 but not on the internal distribution of the currents in the antenna.

The radiation vector $\Re$ can be regarded as an *effective dipole* equal to the sum of the individual dipole elements J*dv*, each weighted by phase factor $e^{j\beta r_1 \cos\psi}$, which depends on the *phase advance* $\beta r_1 \cos\psi$ of the element in relation to the origin, and direction OP_2.

$$H_\theta = j\beta \frac{e^{-j\beta r}}{4\pi r}\Re_\phi \quad \text{and} \quad H_\phi = -j\beta \frac{e^{-j\beta r}}{4\pi r}\Re_\theta$$

$$E_\theta = \eta H_\phi \quad \text{and} \quad E_\phi = -\eta H_\theta$$

Small circular loop



Calculate the radiated fields and power at large distance.

Using the symmetry the results will be independent of the azimuth coordinate ϕ.

The spherical polar coordinates of a point P_1 at a general position on the loop are (a, $\pi/2$, ϕ').

ψ being the angle between OP_1 and OP_2 with a unit vector in the direction of OP_1 $(\cos\phi', \sin\phi', 0)$ and a unit vector in the direction of OP_2 $(\sin\theta, 0, \cos\theta)$:

We have $$\cos\psi = \sin\theta\cos\phi'$$

The radiation vector is then given by

$$\Re(\theta,0) = \int J(r_1)\cdot\hat{u}_\phi e^{j\beta a \sin\theta\cos\phi'} dv \xrightarrow{\text{filamentary current}} \Re(\theta,0) = \int I dr_1 \cdot \hat{u}_\phi e^{j\beta a \sin\theta\cos\phi'}$$

$$\Re(\theta,0) = \int_0^{2\pi} I a e^{j\beta a \sin\theta\cos\phi'} \cos\phi' d\phi' \longrightarrow \Re(\theta,0) \approx Ia\int_0^{2\pi} (1 + j\beta a \sin\theta\cos\phi')\cos\phi' d\phi'$$

$$\Rightarrow \boxed{\Re_\varphi(\theta,0) = j\beta\pi I a^2 \sin\theta}$$

Electric and magnetic fields

$$H_\theta = \frac{j\beta e^{-j\beta r}}{4\pi r}\Re_\varphi = \frac{-(\beta a)^2 I \sin\theta}{4r} e^{-j\beta r}$$ and $$E_\phi = -\eta H_\theta = \frac{(\beta a)^2 \eta I \sin\theta}{4r} e^{-j\beta r}$$

Poynting vector

$$S_r = -\frac{1}{2} E_\phi H_\theta^* = \frac{(\beta a)^4 \eta I^2 \sin^2\theta}{32 r^2}$$

Total power radiated

$$W = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} S_r r \sin\theta \, d\phi \, r d\theta$$

Substituting for $\boldsymbol{S_r}$ *and using* $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

$$\boxed{W = \frac{\pi\eta |I|^2 (\beta a)^4}{12}}$$

Radiation resistance

$$W = \frac{1}{2}\Re_r |I|^2 \longrightarrow \Re_r = \frac{\pi\eta}{6}(\beta a)^4 \longrightarrow \Re_r = 20\pi^2(\beta a)^4 \Omega \quad (\eta = 120\pi\Omega)$$