# Lecture 2

- Maxwell's equations
- Grad, div and curl
- Wave equations
- Plane Waves
- Boundary conditions

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# Maxwell's Equations

The general form of the time-varying Maxwell's equations can be written in differential form as:

$$\boxed{\begin{aligned}
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}}$$

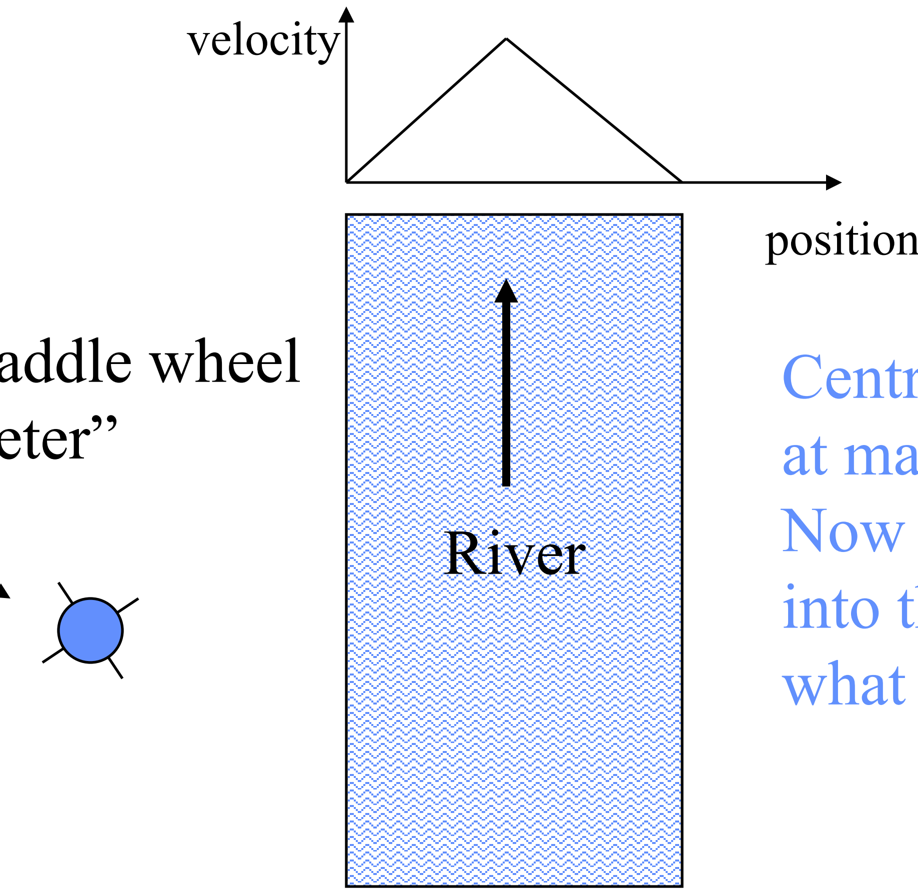
# What is Maxwell’s theory?

**“I cannot give any clearer or briefer answer than the following: Maxwell’s theory is the system of Maxwell’s equations”**

**-Heinrich Rudolph Hertz**
**(1857-94)**

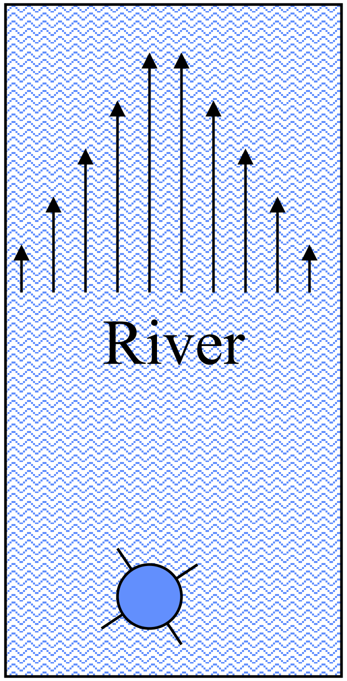
# Physical interpretation of curl

Rotating paddle wheel
or "curl-meter"

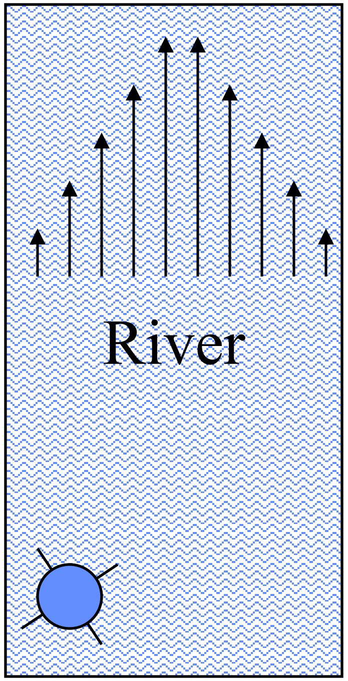



Centre of river flows
at maximum speed.
Now put the "curl meter"
into the river and see
what happens!

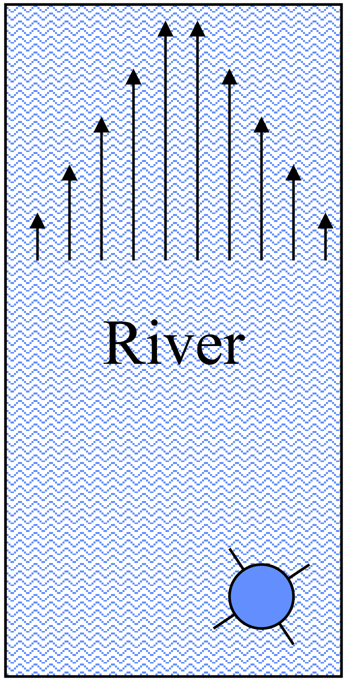
# Three different positions (centre left and right)




No rotation!




Anti-clockwise rotation.




Clockwise rotation.

# Conclusions

- Curl-meter only indicates rotation if there is non-uniformity in the vector field.
- The amount of rotation is proportional to the degree of non-uniformity.
- The rotation cannot be described just with a scalar. Direction should also be given. Rotation is therefore a vector quantity.

Curl therefore describes the variation across the field. The above physical interpretation is concerned with only one dimension or one component. For electromagnetic fields we must imagine the concept applying to three dimensions.

# A few other fundamental relationships

$$\boxed{\begin{aligned} &\mathbf{J} = \sigma \mathbf{E} \\ &\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \\ &\left.\begin{aligned}\mathbf{D} &= \varepsilon\ \mathbf{E} \\ \mathbf{B} &= \mu\, \mathbf{H}\end{aligned}\right\} \end{aligned}}$$

"Ohm's law"

"continuity equation"

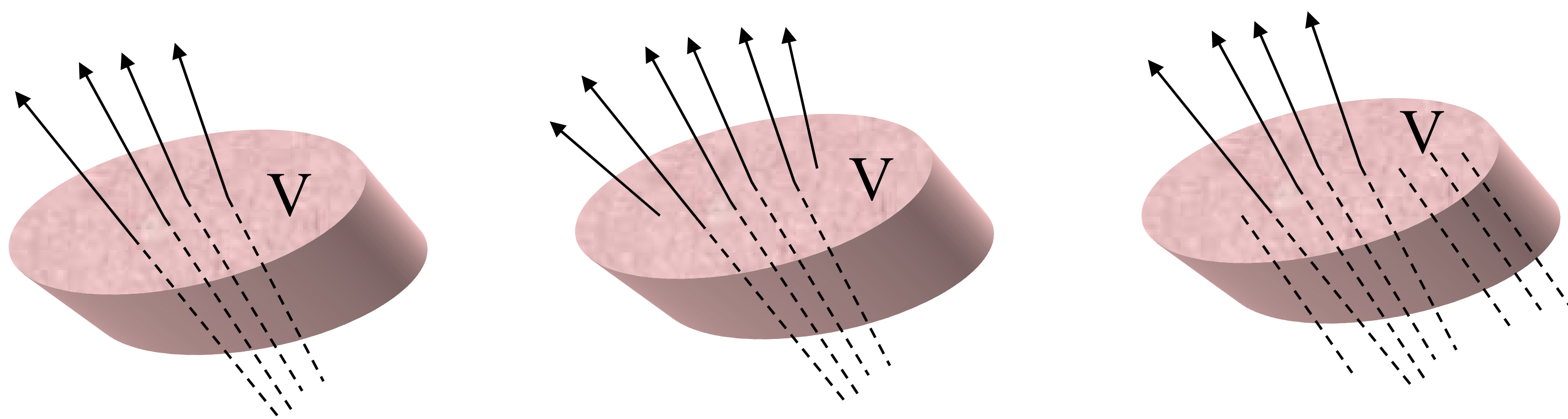
"constitutive relationships"

Note this well!!!

here $\varepsilon = \varepsilon_0 \varepsilon_r$ (permittivity) and $\mu = \mu_0 \mu_r$ (permeability)

with $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m, $\mu_0 = 4\pi \times 10^{-7}$ H/m

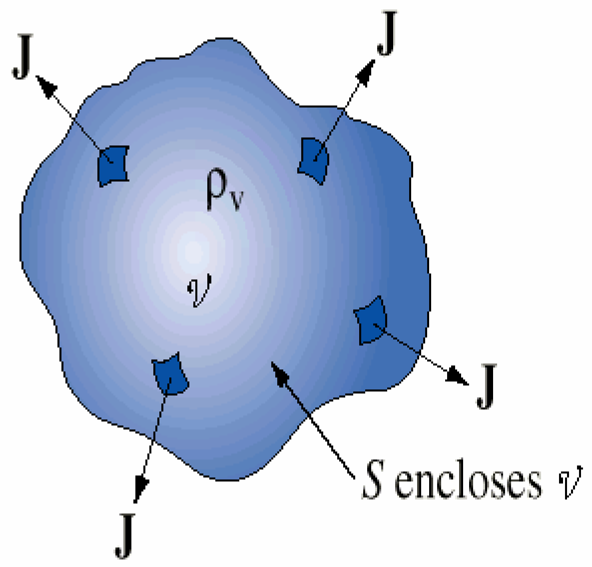
# Physical interpretation of divergence




| | | |
|---|---|---|
| Flux in = flux out so no sources or sinks inside V. | Flux out > flux in Positive divergence. Must be a source inside V. | Flux out < flux in Negative divergence. Must be a sink or drain inside V. |

# Example

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_V}{\partial t}$$




The current density flux flowing out of the closed surface equals the rate of decrease of the positive charge density.

# Maxwell's Equations

– All (macroscopic) electromagnetic phenomena can be explained in terms of Maxwell's equations, the continuity equation and the Lorentz force equation.

– We have only listed the differential form of Maxwell's equations. There are corresponding integral forms of these equations as well.

–We also note that Maxwell's equations are not independent. In fact, the two Divergence equations can be derived from the two Curl equations.

# Maxwell's Equations

- In vacuum,

SI

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

We use SI

CGS

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

- Try eliminating **B**

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \qquad \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Do you know the "BAC-CAB" rule?

# Wave Equations

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Very useful rule
Prove it!

- Using this rule,

$$\nabla\times(\nabla\times\mathbf{E})=\nabla\underbrace{(\nabla\cdot\mathbf{E})}_{=0}-(\nabla\cdot\nabla)\mathbf{E}=-\nabla^2\mathbf{E}$$

- We get

$$\frac{\partial^2\mathbf{E}}{\partial t^2}=c^2\nabla^2\mathbf{E}$$

- Similarly, we can derive

$$\frac{\partial^2\mathbf{B}}{\partial t^2}=c^2\nabla^2\mathbf{B}$$

Wave equations for the EM waves in free space

- They look just like the 3-D wave equation from 2 weeks ago
  - We know the solutions

# Plane Waves

- Solutions must be plane waves

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \qquad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \qquad \omega = ck$$

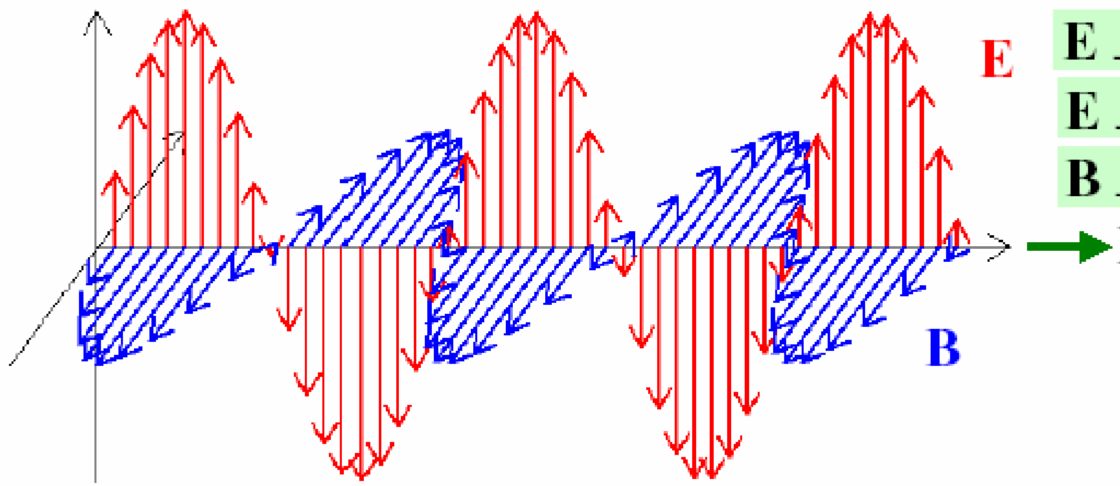
  - $\mathbf{E}_0$ and $\mathbf{B}_0$ are not completely free
  - Must satisfy all of Maxwell's equations
    - $\nabla \cdot \mathbf{E} = 0 \Rightarrow \mathbf{k} \cdot \mathbf{E} = 0$
    - $\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{k} \cdot \mathbf{B} = 0$
    - $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \Rightarrow \mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$
    - $\nabla \times \mathbf{B} = \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} \Rightarrow \mathbf{k} \times \mathbf{B} = -\dfrac{\omega}{c^2}\mathbf{E}$

**E** and **B** are perpendicular to **k**

**E** and **B** are perpendicular to each other

# Transverse Waves

- EM waves in free space is transverse




$\mathbf{E} \perp \mathbf{B}$

$\mathbf{E} \perp \mathbf{k}$

$\mathbf{B} \perp \mathbf{k}$

- From $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$ and $\omega = ck$ ⇒ $|\mathbf{E}| = c|\mathbf{B}|$
- If you want **H**,

$$|\mathbf{H}| = \frac{|\mathbf{B}|}{\mu_0} = \frac{|\mathbf{E}|}{c\mu_0} = \frac{|\mathbf{E}|}{Z_0}$$

Vacuum impedance (377Ω)

$|\mathbf{E}| = |\mathbf{B}|$ if we were using CGS

# Maxwell's Equations

- Now we go back to Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

  - Movement of the charges in matter → Current $\mathbf{J} = q n_0 \mathbf{v}$
  - We assumed $\mathbf{x} = \frac{q\mathbf{E}}{k_s} \Rightarrow \mathbf{v} = \frac{q}{k_s}\frac{\partial \mathbf{E}}{\partial t}$
  - Usual trick with BAC-CAB rule gives us

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\mu_0 q^2 n_0}{k_s}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Simple wave equation

# Plane Wave Solution

$$\nabla^2\mathbf{E} = \left(\frac{1}{c^2} + \frac{\mu_0 q^2 n_0}{k_s}\right)\frac{\partial^2\mathbf{E}}{\partial t^2}$$

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

- Wave equation reduces to $-k^2\mathbf{E} = -\omega^2\left(\frac{1}{c^2} + \frac{\mu_0 q^2 n_0}{k_s}\right)\mathbf{E}$
- Dispersion relation is

$$k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{q^2 n_0}{\varepsilon_0 k_s}\right) \Rightarrow c_p = \frac{\omega}{k} = \frac{c}{\sqrt{1+\frac{q^2 n_0}{\varepsilon_0 k_s}}}$$

$$n = \sqrt{1+\frac{q^2 n_0}{\varepsilon_0 k_s}}$$

Same $n_{\text{true}}$

- We found the same solution
  - We used the short-cut by trusting Maxwell's **J** term
  - It can be made even simpler…

# Maxwell's Equation

- Take the equation $\nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$
  - We are assuming

$$\mathbf{J} = q n_0 \mathbf{v} \qquad \mathbf{v} = \frac{q}{k_s} \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\mu_0 q^2 n_0}{k_s} \frac{\partial \mathbf{E}}{\partial t} = \left( \varepsilon_0 + \frac{q^2 n_0}{k_s} \right) \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

  - We could define

$$\varepsilon = \varepsilon_0 + \frac{q^2 n_0}{k_s} \qquad \nabla \times \mathbf{B} = \varepsilon \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

  - We absorbed the **J** term into the matter's permittivity $\varepsilon$
  - Now it's easy to get $n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{1 + \frac{q^2 n_0}{\varepsilon_0 k_s}}$

# Integral form of the equations

$$\nabla \cdot \vec{D} = \rho \quad \Rightarrow \quad \oint_S \vec{D} \cdot d\vec{s} = Q$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \oint_S \vec{B} \cdot d\vec{s} = 0$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \Rightarrow \quad \oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s}$$

# Wave Equations

In any problem with unknown **E, D, B, H** we have 12 unknowns. To solve for these we need 12 scalar equations. Maxwell's equations provide 3 each for the two curl equations. and 3 each for both constitutive relations (difficult task).

Instead we anticipate that electromagnetic fields propagate as waves. Thus if we can find a wave equation, we could solve it to find out the fields directly.

# Maxwell's equations not independent!

E.g.

$$\nabla\times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

$$\nabla\cdot\nabla\times \mathrm{E} = 0 = -\frac{\partial}{\partial t}[\nabla\cdot \mathrm{B}]$$

This implies

$$\nabla\cdot\mathbf{B} = 0$$

Use the identity

$$\nabla\cdot\nabla\times\vec{A} \equiv 0$$

$$\nabla\cdot\nabla\times\mathbf{H} = \nabla\cdot\mathbf{J} + \frac{\partial}{\partial t}[\nabla\cdot\mathbf{D}]$$

$$0 = -\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial t}$$

This implies

$$\triangle\cdot\mathbf{D}=\rho$$

# Wave equations

Take the curl of the first Maxwell:

$$\begin{aligned}\nabla \times \nabla \times \mathbf{H} &= \nabla \times \mathbf{J} + \nabla \times \frac{\partial}{\partial t}\left(\varepsilon\, \mathbf{E}\right) \\ &= \nabla \times \mathbf{J} + \varepsilon \frac{\partial}{\partial t}\left(\nabla \times \mathbf{E}\right) \\ &= \nabla \times \mathbf{J} + \varepsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \mathbf{H}}{\partial t}\right) \\ &= \nabla \times \mathbf{J} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}\end{aligned}$$

Now use $\nabla \times \nabla \times \mathbf{H} \equiv \nabla\left(\underbrace{\nabla \cdot \mathbf{H}}_{0}\right) - \nabla^2 \mathbf{H}$ on the LHS

# Wave Equations

The result is:

$$\boxed{\nabla^2 \vec{\mathbf{H}} - \mu\varepsilon \frac{\partial^2 \vec{\mathbf{H}}}{\partial t^2} = -\nabla \times \vec{\mathbf{J}}}$$

Similarly, the same process for the second Maxwell produces

$$\boxed{\nabla^2 \vec{\mathbf{E}} - \mu\varepsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = \mu \frac{\partial \vec{\mathbf{J}}}{\partial t} + \nabla \frac{\rho}{\varepsilon}}$$

Note how in both case we have a wave equation (2nd order PDE) for both **E** and **H** with fields to the left of the = sign and sources to the right. These two wave equations are completely equivalent to the Maxwell equations.

# Solutions to the wave equations

Consider a region of free space ($\sigma = 0$) where there are no sources ($\mathbf{J} = 0$). The wave equations become homogeneous:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

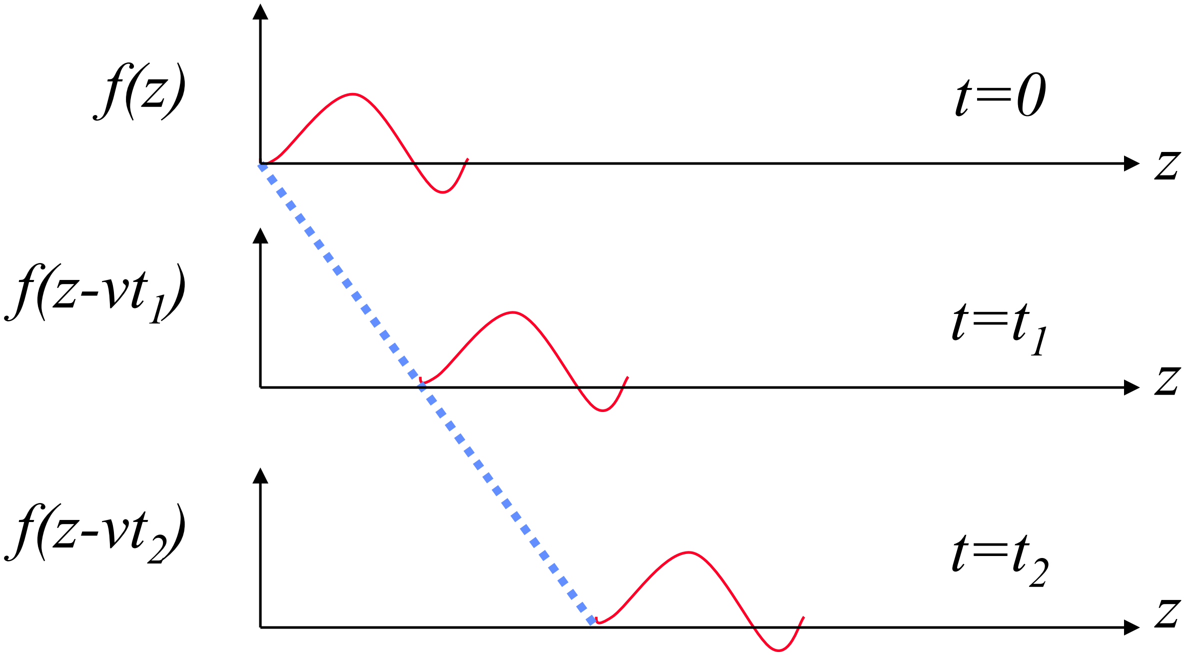
Actually there are 6 equations; we will only consider one component:

e.g. $E_x(z,t)$

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 E_x}{\partial t^2} = 0 \qquad \text{where} \quad v^2 = \frac{1}{\mu_0 \varepsilon_0} = c^2$$

# Solutions to the wave equation

Try a solution of the form $f(z-vt)$ *e.g.* $\sin[\beta(z-vt)]$. By differentiating twice and substituting back into the scalar wave equation, we find that it satisfies!



# Plane Waves

- First treat plane waves in free space.
- Then interaction of plane waves with media.
- We assume time harmonic case, and source free situation.

We require solutions for E and H (which are solutions to the following PDE) in free space

$$\nabla^2 \vec{E} + k_0^2 \vec{E} = 0$$

No potentials here!
(no sources)

Note that this is actually three equations:

$$\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} + k_0^2 E_i = 0 \qquad i = x, y, z$$

# How do we find a solution?

Usual procedure is to use Separation of Variables (SOV). Take one component for example $E_x$.

$$E_x = f(x)g(y)h(z)$$

$$ghf'' + fhg'' + fgh'' + k_0^2 fgh = 0$$

$$\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} + k_0^2 = 0$$

Functions of a single variable $\Rightarrow$ sum = constant = $-k_0^2$

$$\frac{f''}{f} = -k_x^2; \quad \frac{g''}{g} = -k_y^2; \quad \frac{h''}{h} = -k_z^2$$

and so $$\boxed{k_x^2 + k_y^2 + k_z^2 = k_0^2 \quad \text{with } k_0 = \frac{2\pi}{\lambda} = \frac{\omega}{c}}$$

# Mathematical Solution

We note we have 3 ODEs now.

$$\frac{d^2 f}{dx^2} + k_x^2 f = 0 \qquad \text{solution is} \qquad f = e^{\pm j k_x x}$$

$$\frac{d^2 g}{dy^2} + k_y^2 g = 0 \qquad \text{solution is} \qquad g = e^{\pm j k_y y}$$

$$\frac{d^2 h}{dz^2} + k_z^2 h = 0 \qquad \text{solution is} \qquad h = e^{\pm j k_z z}$$

$$\boxed{E_x = A\, e^{\pm j\left(k_x x + k_y y + k_z z\right)}}$$

# But, what does it mean physically?

$$\boxed{E_x = A\, e^{\pm j\left(k_x x + k_y y + k_z z\right)}}$$

This represents the x-component of the travelling wave E-field (like on a transmission line) which is travelling in the direction of the propagation vector, with Amplitude A. The direction of propagation is given by

$$\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$$

# Physical interpretation

The solution represents a wave travelling in the $+z$ direction with velocity $c$. Similarly, $f(z+vt)$ is a solution as well. This latter solution represents a wave travelling in the $-z$ direction.
So generally,

$$E_x(z,t) = f\left[(x \pm vt)(y \pm vt)(z \pm vt)\right]$$

In practice, we solve for either E or H and then obtain the other field using the appropriate curl equation.

What about when sources are present? Looks difficult!

# Generalize for all components

If we define the normal 3D position vector as:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

then $\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z$

so $E_x = Ae^{-j\vec{k}\cdot\vec{r}}$

similarly $E_y = Be^{-j\vec{k}\cdot\vec{r}}$

$E_z = Ce^{-j\vec{k}\cdot\vec{r}}$

+sign dropped here

General expression for a plane wave

$$\vec{E} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} \quad \text{where} \quad \vec{E}_0 = A\hat{x} + B\hat{y} + C\hat{z}$$

# Properties of plane waves

For source free propagation we must have $\nabla \cdot \mathbf{E} = 0$. If we satisfy this requirement we must have $\mathbf{k} \cdot \mathbf{E_0} = 0$. This means that $\mathbf{E_0}$ is perpendicular to **k**.

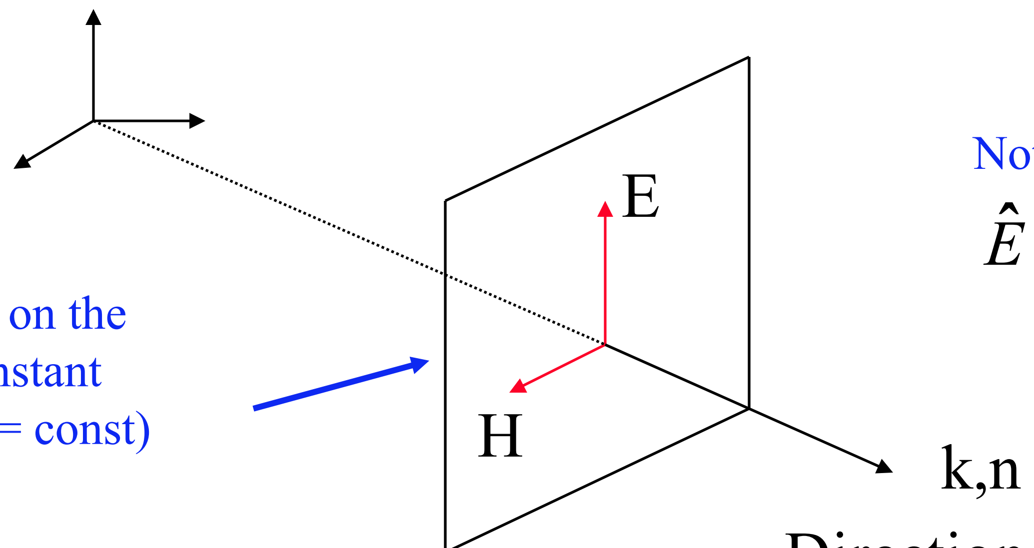
The corresponding expression for **H** can be found by substitution of the solution for **E** into the $\nabla \times \mathbf{E}$ equation. The result is:

$$\vec{H} = \frac{k_0}{\omega \mu_0} \hat{n} \times \vec{E}$$

Where **n** is a unit vector in the **k** direction.

# Transverse Electromagnetic (TEM) wave

Note that **H** is also perpendicular to **k** and also perpendicular to **E**. This can be established from the expression for **H**.




Direction of propagation

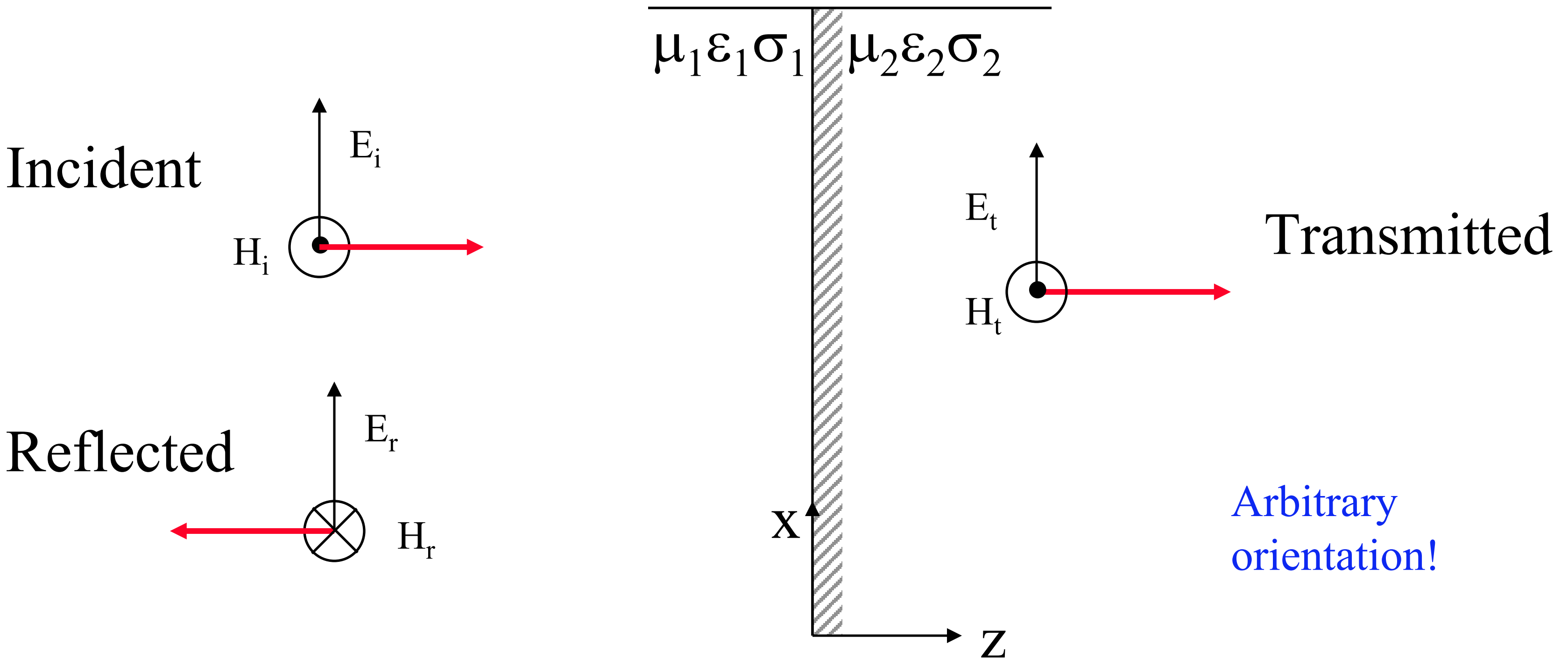
Note that:

$$\hat{E} \times \hat{H} = \hat{n} \ \text{ or } \ \hat{k}$$

**E** and **H** lie on the plane of constant phase (**k**·**r** = const)

# Plane waves at interfaces (normal incidence)

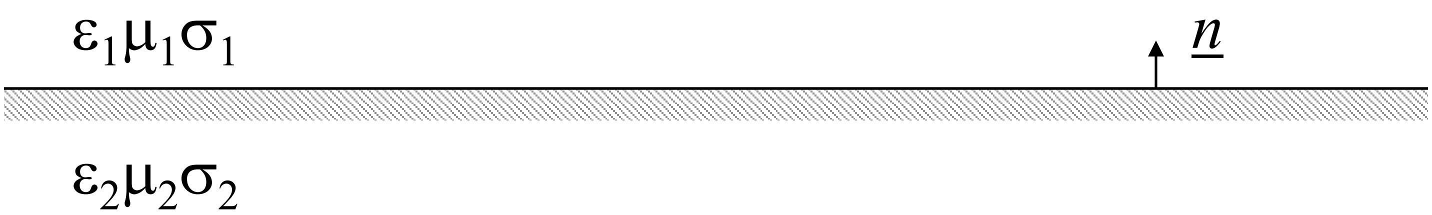
Consider a linearly polarized (in x-direction) wave travelling in the +z direction with magnitude $E_i$



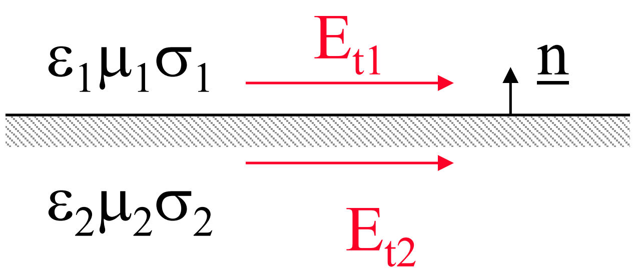
# Boundary conditions

Maxwell's equations in differential form require known boundary values in order to have a complete and unique solution. The so called boundary conditions (B/C) can be derived by considering the integral form of Maxwell's equations.

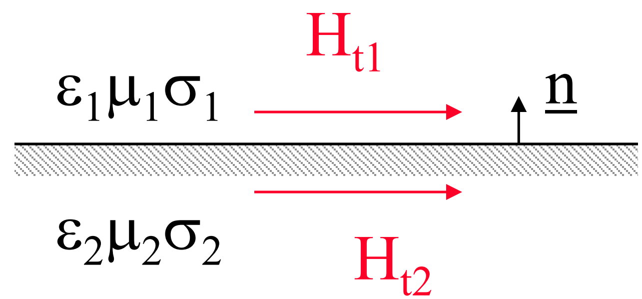
We deal with a general dielectric interface and two special cases. First the general case. For convenience we consider the boundary to be planar.



# General case

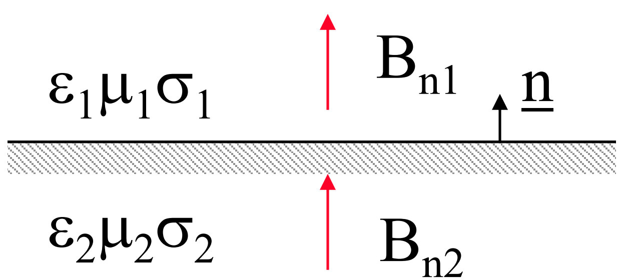



Tangential E continuous

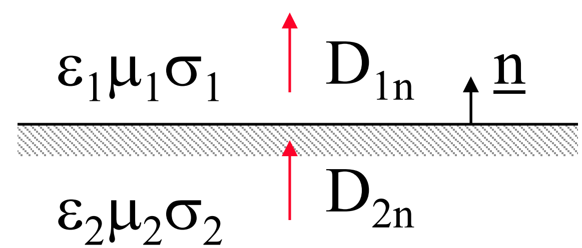



$\underline{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$

Equivalent




Normal B continuous




$\underline{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$

# Special case (a) Lossless dielectric

$\varepsilon_1 \mu_1 \sigma_1 = 0$ $\underline{n}$

$\varepsilon_2 \mu_2 \sigma_2 = 0$

$E_{t1} = E_{t2} \longrightarrow$ tangential E fields continuous)

$H_{t1} = H_{t2} \longrightarrow$ tangential H fields continuous (no current)

$D_{n1} = D_{n2} \longrightarrow$ normal D fields continuous (no charge)

$B_{n1} = B_{n2} \longrightarrow$ normal B fields continuous

# Special case (b) Perfect Conductor

$\varepsilon_1 \mu_1 \sigma_1 = 0$


$\sigma_2 \rightarrow \infty$ Perfect Electric Conductor $E_{t2}=H_{t2}=0$

$\mathbf{E}_{t1}= 0$ ⟶ Tangential Electric field on conductor is zero.

$\underline{n} \times \mathbf{H}_1=\mathbf{J}_s$ ⟶ H field is discontinuous by the surface current

$\underline{n} \,.\, \mathbf{D}_1= \rho$ ⟶ Normal D(E) field is discontinuous by surface charge

$\mathbf{B}_{n1}= 0$ ⟶ Normal B(H) field is zero on conductor.

# Boundary conditions

Continuity at the boundary for the tangential fields requires:

$$E_i + E_r = E_t \qquad (1)$$

$$H_i + H_r = H_t \qquad (2)$$

Fix signs when defining impedance!

Now define: $\dfrac{E_i}{H_i} = Z_1 \qquad \dfrac{E_r}{H_r} = -Z_1 \qquad \dfrac{E_t}{H_t} = Z_2$

Substituting into (1) and (2) and eliminating $E_r$ gives

Transmission coefficient $\boxed{\tau = \dfrac{E_t}{E_i} = \dfrac{2Z_2}{Z_1 + Z_2}}$

# Plane Wave in Dispersive Media

- Recall the Maxwell's equations:

$$\vec{\nabla}\times\vec{E} = -j\omega\vec{B}$$

$$\vec{\nabla}\times\vec{B} = j\omega\vec{D} + \vec{J}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\cdot\vec{D} = \rho_v$$

$$\vec{E}(x,y,x;t) = \vec{E}(x,y,z)e^{j\omega t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$$

$$\int \nabla\times\vec{E}(x,y,z)e^{j\omega t} = -\int\frac{\partial\vec{B}}{\partial t}$$

$$\frac{1}{j\omega}\left(\vec{\nabla}\times\vec{E}\right) = \vec{B} \Rightarrow \vec{\nabla}\times\vec{E} = -j\omega\vec{B}$$

# Plane Wave in Dispersive Media

- So far, for lossless media, we considered J=0, and $\rho_v$=0 but, there are actually two types of current and one of them should not be ignored.
- Total current is a sum of the Source current and Conduction current.

$$\vec{J} = \vec{J}_c + \vec{J}_o$$

set $\vec{J}_c = \sigma \vec{E}$

$$\vec{\nabla} \times \vec{H} = j\omega \vec{D} + \vec{J}$$

$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon \vec{E} + \sigma \vec{E} + \vec{J}_o = j\omega \left( \varepsilon - j\frac{\sigma}{\omega} \right) \vec{E} + \vec{J}_o$$

# Plane Wave in Dispersive Media

Defining complex permittivity

$$\underline{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega}$$

Maxwell's equations in a conducting media (source free) can be written as

$$\vec{\nabla} \times \vec{E} = -j\omega\mu\vec{H}$$

$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon\vec{E}$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

# Plane Wave in Dispersive Media

We have considered so far:

| Plane Waves in Free space | Plane Waves in Isotropic Dielectric | Plane Waves in anisotropic Dielectric | Plane Waves in Dissipative Media |
|---|---|---|---|
| $\vec{\nabla}\times\vec{E} = -j\omega\mu_0\vec{H}$ | $\vec{\nabla}\times\vec{E} = -j\omega\mu\vec{H}$ | $\vec{\nabla}\times\vec{E} = -j\omega\mu\vec{H}$ | $\vec{\nabla}\times\vec{E} = -j\omega\mu\vec{H}$ |
| $\vec{\nabla}\times\vec{H} = j\omega\varepsilon_0\vec{E}$ | $\vec{\nabla}\times\vec{H} = j\omega\varepsilon_0\vec{E}$ | $\vec{\nabla}\times\vec{H} = j\omega\varepsilon\vec{E}$ | $\vec{\nabla}\times\vec{H} = j\omega\varepsilon\vec{E}$ |
| $\vec{\nabla}\cdot\vec{H} = 0$ | $\vec{\nabla}\cdot\vec{E} = 0$ | $\vec{\nabla}\cdot\vec{D} = 0$ | $\vec{\nabla}\cdot\vec{E} = 0$ |
| $\vec{\nabla}\cdot\vec{E} = 0$ | $\vec{\nabla}\cdot\vec{H} = 0$ | $\vec{\nabla}\cdot\vec{B} = 0$ | $\vec{\nabla}\cdot\vec{H} = 0$ |

# Plane Wave in Dispersive Media

Wave equation for dissipative media becomes:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -j\omega\mu\vec{\nabla} \times \vec{H}$$

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \vec{\nabla}^2 \vec{E} = -j\omega\mu\left(j\omega\varepsilon\vec{E}\right)$$

$$\vec{\nabla}^2 \vec{E} = -\omega^2\mu\varepsilon\vec{E}$$

$$\vec{\nabla}^2 \vec{H} = -\omega^2\mu\varepsilon\vec{H}$$

The set of plane-wave solutions are:

$$\vec{E} = \hat{x}E_0 e^{-j\kappa z}$$

$$\vec{H} = \hat{y}\left(\frac{E_0}{\eta}\right)e^{-j\kappa z}$$

# Plane Wave in Dispersive Media

Substituting into $\vec{\nabla}^2 \vec{E} = -\omega^2 \mu \varepsilon \vec{E}$ and $\vec{\nabla}^2 \vec{H} = -\omega^2 \mu \varepsilon \vec{H}$

yields the dispersion relation

$$\kappa^2 = \omega^2 \mu \varepsilon$$

and

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Is the complex intrinsic impedance of the isotropic media.

# Plane Wave in Dispersive Media

Denoting the complex values:

$$\kappa = \kappa_R - j\kappa_I$$

$$\eta = |\eta| e^{j\phi}$$

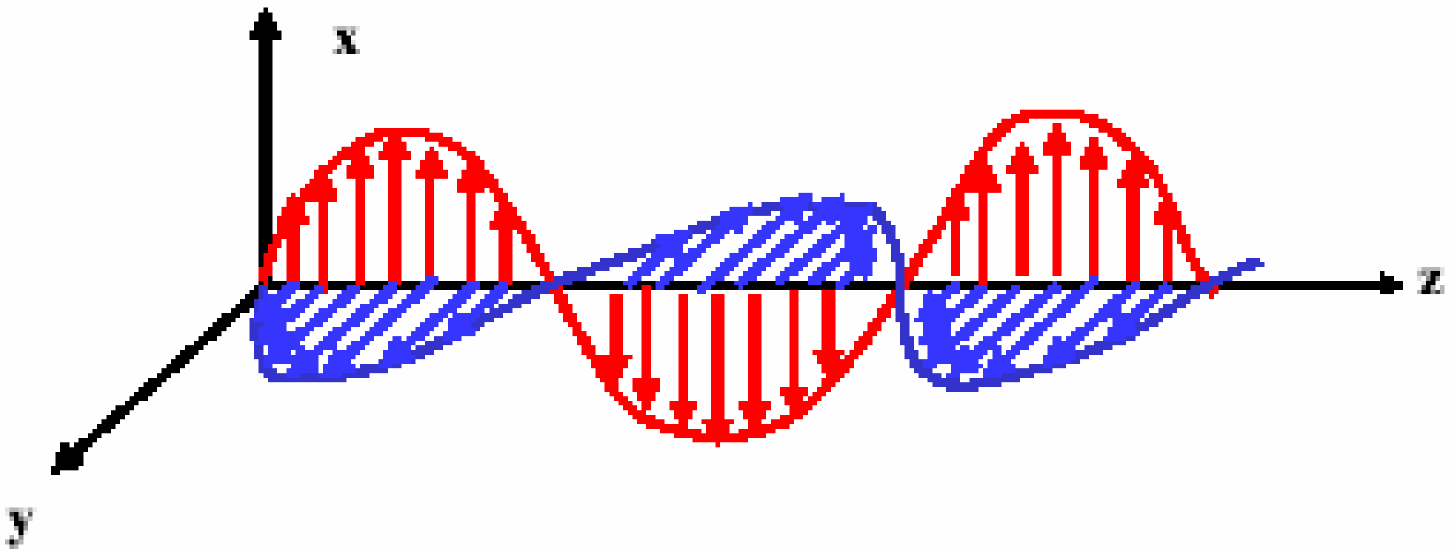
then,

$$\vec{E} = \hat{x} E_0 e^{-j\kappa z} = \hat{x} E_0 e^{-j(\kappa_R - j\kappa_I)z} = \hat{x} E_0 e^{-j\kappa_R z} e^{-\kappa_I z} = \hat{x} \underline{E_x}$$

$$\vec{H} = \hat{y}\left(\frac{E_0}{\eta}\right) e^{-j(\kappa_R - j\kappa_I)z} = \hat{y} \frac{E_0}{|\eta|} e^{-j(\kappa_R - j\kappa_I)z} e^{-j\phi}$$

# Plane Wave in Dispersive Media

- Picture it: Lossless Media



# Plane Wave in Dispersive Media

Loss tangent is defined from

$$\kappa = \kappa_R - j\kappa_I = \omega\sqrt{\mu\underline{\varepsilon}} = \omega\sqrt{\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)}$$

$$= \omega\sqrt{\mu\varepsilon}\sqrt{\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}$$

$\dfrac{\sigma}{\omega\varepsilon}$ is defined as loss tangent

$$\underline{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right) = \varepsilon' - j\varepsilon''$$

$$tan\,\delta = \frac{\varepsilon''}{\varepsilon'}$$

# Plane Wave in Dispersive Media

Slightly lossy case: $\dfrac{\sigma}{\omega\varepsilon} << 1$

$$\kappa = \omega\sqrt{\mu\varepsilon}\sqrt{\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} = \omega\sqrt{\mu\varepsilon}\left(1 - j\frac{\sigma}{2\omega\varepsilon}\right)$$

$$\kappa_R = \omega\sqrt{\mu\varepsilon}$$

$$\kappa_I = \omega\sqrt{\mu\varepsilon}\frac{\sigma}{2\omega\varepsilon} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$d_p = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

# Plane Wave in Dispersive Media

Highly lossy case: $\dfrac{\sigma}{\omega\varepsilon} >> 1$

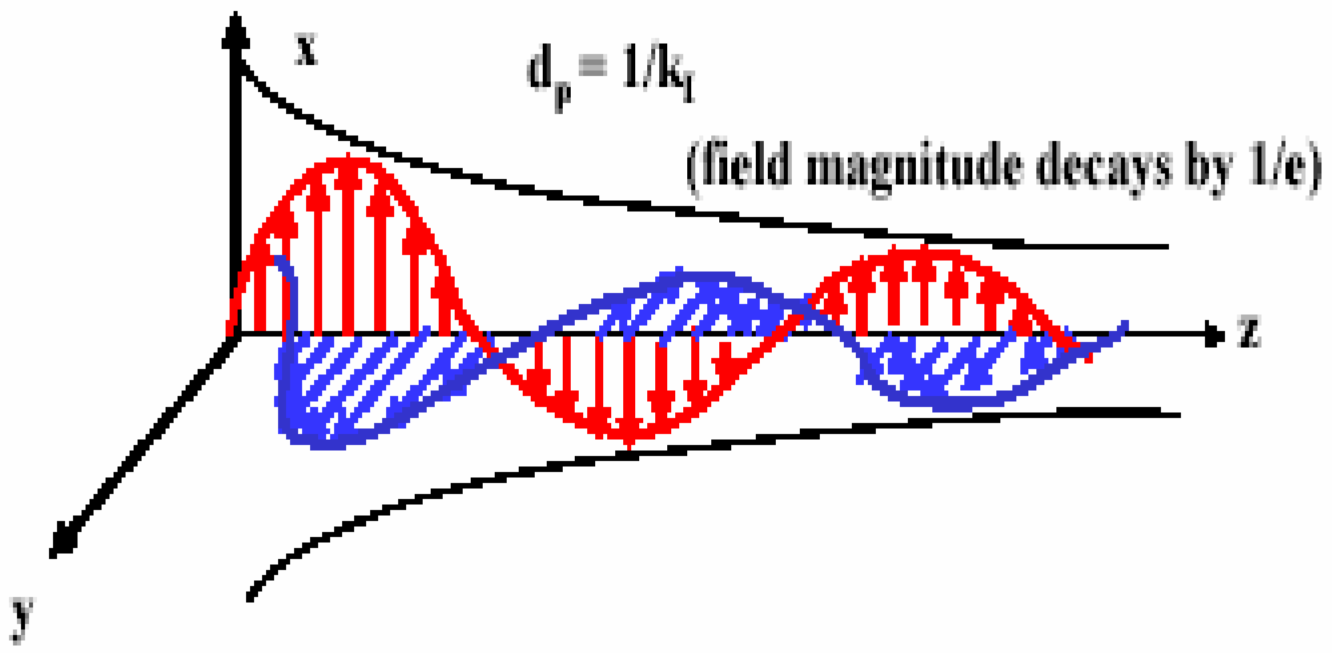
$$\kappa = \omega\sqrt{\mu\varepsilon}\sqrt{\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} = \omega\sqrt{\mu\varepsilon}\left(\sqrt{-j\frac{\sigma}{2\omega\varepsilon}}\right)$$

$$= \sqrt{\omega\mu\frac{\sigma}{2}}(1 - j)$$

$$d_p = \sqrt{\frac{2}{\omega\mu\sigma}} \approx \delta$$ Skin depth

# Plane Wave in Dispersive Media

- Picture it: Lossy Media



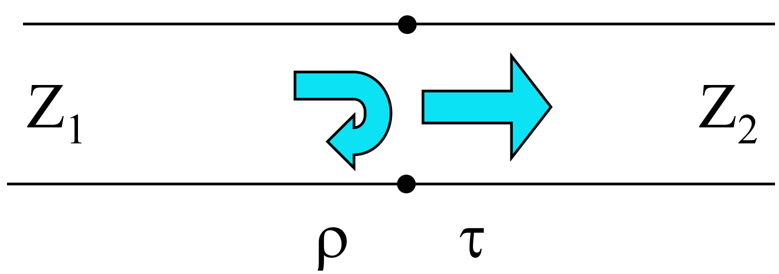
# Reflection & Transmission

Similarly, substituting into (1) and (2) and eliminating $E_t$

Reflection coefficient $$\boxed{\rho = \frac{E_r}{E_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1}}$$

**<u>Not</u> 1-ρ**

We note that $\tau = 1 + \rho$, and that the values of the reflection and transmission are the same as occur in a transmission line discontinuity.



# Special case (1)

(1) Medium 1: air; Medium 2: conductor

$$Z_1 = 377\Omega \quad >> \quad Z_2 = Z_m = \frac{1+j}{\sigma\delta}$$

So $\quad E_t = \tau E_i \approx \dfrac{2Z_2}{Z_1} E_i$

then use $\quad H_t = \dfrac{E_t}{Z_2} \Rightarrow H_t = \dfrac{2}{Z_1} E_i \approx 2H_i$

This says that the transmitted magnetic field is almost doubled at the boundary before it decays according to the skin depth. On the reflection side $H_i \approx H_r$ implying that almost all the H-field is reflected forming a standing wave.

# Special case (2)

(2) Medium 1: conductor; Medium 2: air

Reversing the situation, now where the wave is incident from the conducting side, we can show that the wave is almost totally reflected within the conductor, but that the standing wave is attenuated due to the conductivity.

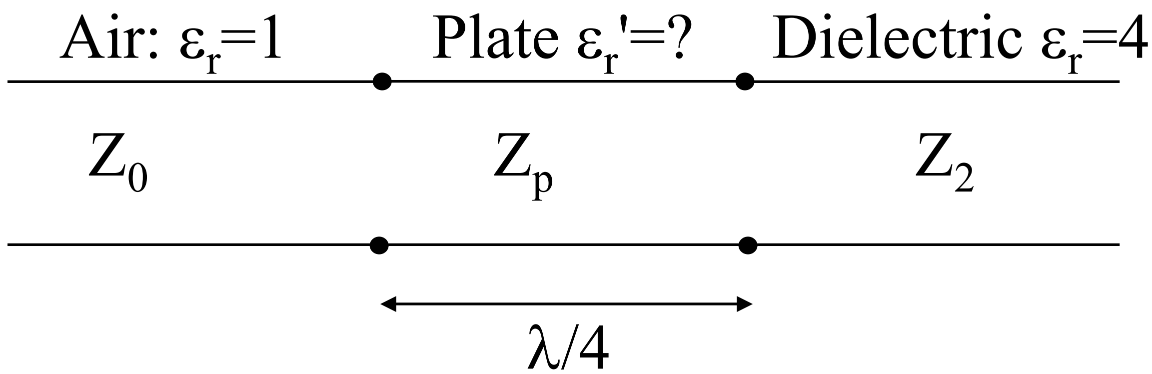
# Special Case (3)

(2) Medium1: dielectric; Medium2: dielectric

$$Z_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}}, \qquad Z_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}} \qquad \Rightarrow \qquad \rho = \frac{\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} - 1}{\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} + 1}$$

This result says that the reflection can be controlled by varying the ratio of the dielectric constants. The transmission analogy can thus be used for a quarter-wave matching device.

# λ/4 Matching Plate




Transmission line theory tells us that for a match

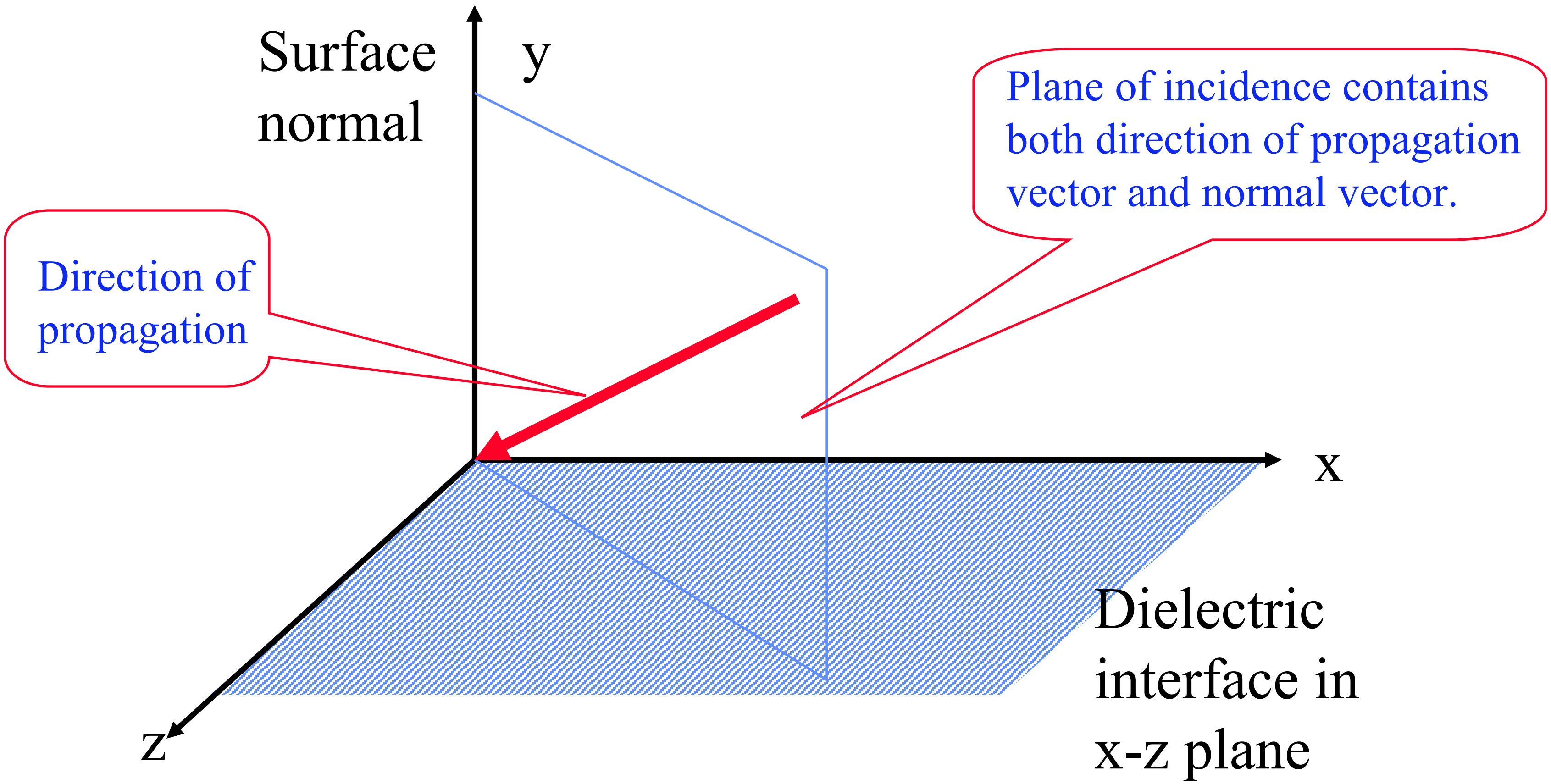
$$Z_p = \sqrt{Z_0 Z_2}$$

We will see TL lectures later

$$Z_0 = 376.7\Omega, \quad Z_2 = \frac{Z_0}{\sqrt{\varepsilon_r}} = \frac{376.7}{2} = 188\Omega$$

So $\quad Z_p = 266\Omega \quad$ and $\quad \varepsilon_r' = \frac{Z_0}{Z_2} = 2$

# Plane of Incidence



# Applications

The principle of $\lambda/4$ matching is not only confined to transmission line problems! In fact, the same principle is used to eliminate reflections in many optical devices using a $\lambda/4$ coating layer on lenses & prisms to improve light transmission efficiency.

Similarly, a half-wave section can be used as a dielectric window. Ie. Full transparency. In this case $Z_2=Z_0$ and the matching section is $\lambda/2$. Such devices are used to protect antennas from weather, ice snow, etc and are called radomes.

Note that both applications are frequency sensitive and that the matching section is only $\lambda/4$ or $\lambda/2$ at one frequency.

# Oblique Incidence

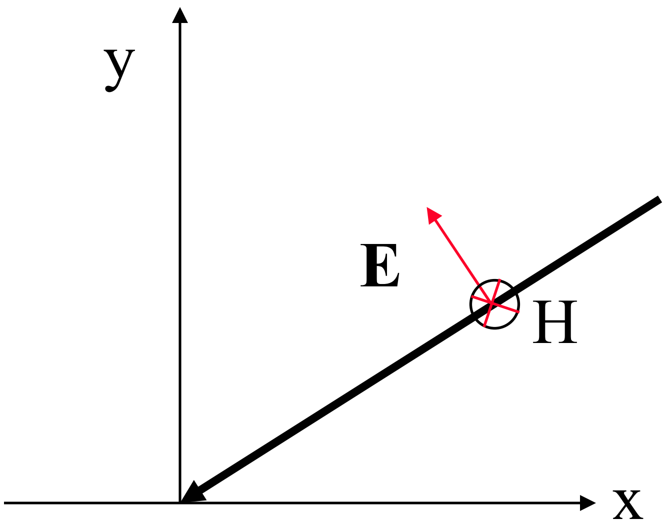
The transmission line analogy only works for normal incidence. When we have oblique incidence of plane waves on a dielectric interface the reflection and transmission characteristics become polarization and angle of incidence dependent.

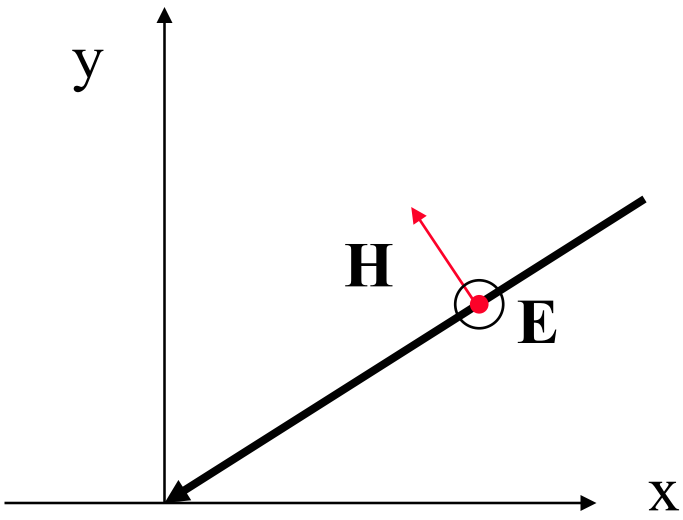
We need to distinguish between the two different polarizations. We do this by first, explaining what a plane of incidence is, then we will point out the distinguishing features of each polarization. We are aiming for expressions for reflection coefficients.

We note again that we are only dealing with plane waves

# Parallel & Perpendicular Incidence

Plane of incidence is the x-y plane

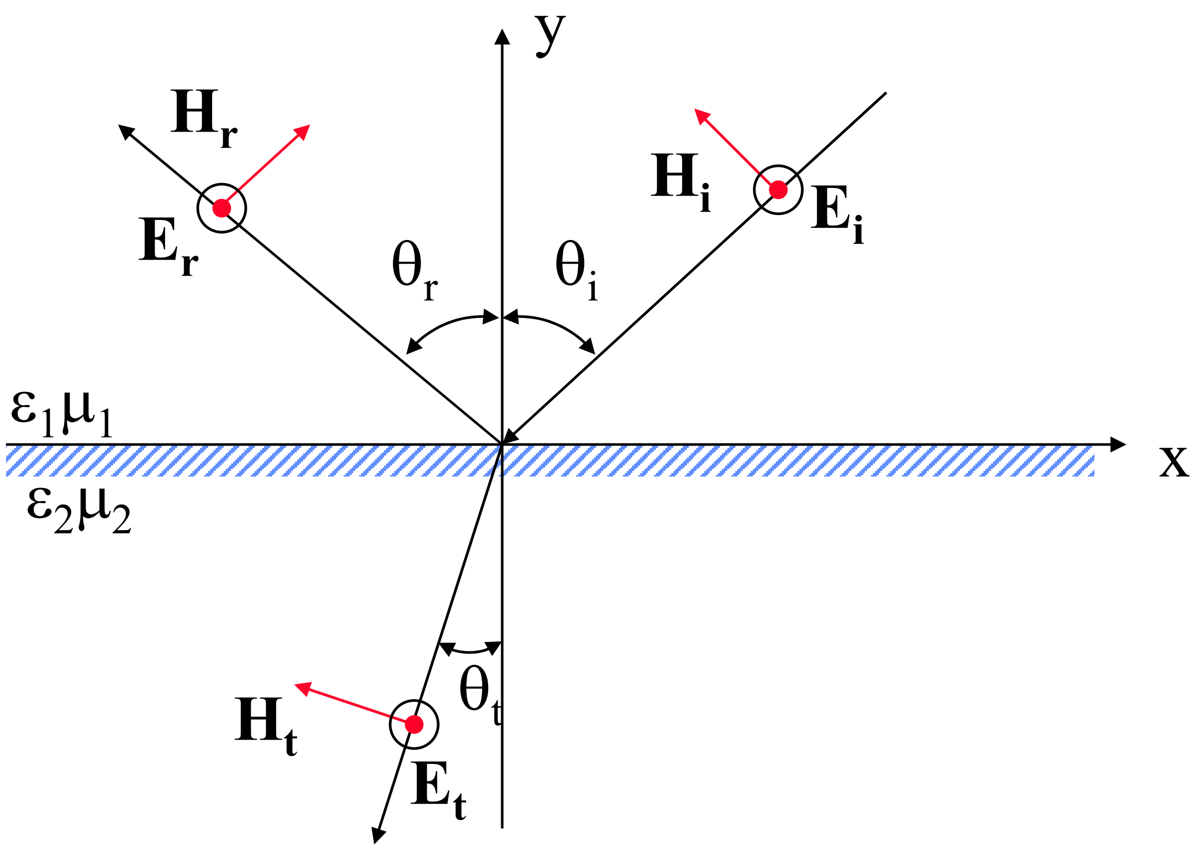



E is Parallel to the plane of incidence

E is Perpendicular to the plane of incidence

# Perpendicular incidence



# Write math expression for fields!

$$E_i = \hat{z} E_0 \exp\left[j\beta_1\left(x\sin\theta_i + y\cos\theta_i\right)\right]$$

$$H_i = \left(-\hat{x}\cos\theta_i + \hat{y}\sin\theta_i\right)\frac{E_0}{Z_1}\exp\left[j\beta_1\left(x\sin\theta_i + y\cos\theta_i\right)\right]$$

---

$$E_r = \hat{z}\rho_\perp E_0 \exp\left[j\beta_1\left(x\sin\theta_r - y\cos\theta_r\right)\right]$$

$$H_r = \left(\hat{x}\cos\theta_r + \hat{y}\sin\theta_r\right)\frac{\rho_\perp E_0}{Z_1}\exp\left[j\beta_1\left(x\sin\theta_r - y\cos\theta_r\right)\right]$$

---

$$E_t = \hat{z}\tau_\perp E_0 \exp\left[j\beta_2\left(x\sin\theta_t + y\cos\theta_t\right)\right]$$

$$H_t = \left(-\hat{x}\cos\theta_t + \hat{y}\sin\theta_t\right)\frac{\tau_\perp E_0}{Z_2}\exp\left[j\beta_2\left(x\sin\theta_t + y\cos\theta_t\right)\right]$$

# Say What!!! How did you get that?

Within the exponential: This tells the direction of propagation
Of the wave. E.g. for both the incident $E_i$ and $H_i$

$$j\beta_1\left(x\sin\theta_i + y\cos\theta_i\right)$$

Propagating
In medium 1

A component in the – x direction

Another component in the –y direction

Outside the exponential tells what vector components of the field
Are present. E.g. for $H_r$

$$\left(\hat{x}\cos\theta_r + \hat{y}\sin\theta_r\right)\frac{\rho_\perp E_0}{Z_1}$$

Perpendicular reflection coefficient

+x and +y components of $H_r$

$E_0/Z_1$ converts E to H

# Apply boundary conditions

Tangential E fields ($E_z$) matches at y=0
Tangential H fields ($H_x$) matches at y=0

$$\mathbf{exp}\left(j\beta_1 x\,\mathbf{sin}\,\theta_i\right)+\rho_\perp\,\mathbf{exp}\left(j\beta_1 x\,\mathbf{sin}\,\theta_r\right)=\tau_\perp\,\mathbf{exp}\left(j\beta_2 x\,\mathbf{sin}\,\theta_t\right)$$

We know that τ =1+ ρ, so then the arguments of the exponents must be equal. Sometimes called Phase matching in optical context. It is the same as applying the boundary conditions.

$$\Rightarrow \quad j\beta_1\,\mathbf{sin}\,\theta_i = j\beta_1\,\mathbf{sin}\,\theta_r = j\beta_2\,\mathbf{sin}\,\theta_t$$

# Snell's laws and Fresnel coefficients

The first equation gives

$$\theta_r = \theta_i$$

and from the second using $\beta = \frac{2\pi}{\lambda}$

$$\sin\theta_t = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\sin\theta_i$$

By matching the $H_x$ components and utilizing Snell, we can obtain the Fresnel reflection coefficient for perpendicular incidence.

$$\boxed{\rho_\perp = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}}$$

# Alternative form

Alternatively, we can use Snell to remove the $\theta_t$ and write it in terms of the incidence angle, at the same time assuming non-magnetic media ($\mu = \mu_0$ for both media).

$$\boxed{\rho_\perp = \frac{\cos\theta_i - \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}}}$$

Note how both forms reduce to the transmission line form when $\theta_i=0$

This latter form is the one that is most often quoted in texts, the previous version is more general

# Some interesting observations

- If $\varepsilon_2 > \varepsilon_1$ Then the square root is positive, $\rho_\perp$ Is real
- If $\varepsilon_1 > \varepsilon_2$ i.e. the wave is incident from more dense to less dense

AND

$$\sin^2\theta_i \geq \frac{\varepsilon_2}{\varepsilon_1}$$

Then $\rho_\perp$ is complex and $\left|\rho_\perp\right| = 1$

This implies that the incident wave is totally internally reflected (TIR) into the more dense medium

# Critical angle

When the equality is satisfied we have the so-called critical angle. In other words, if the incident angle is greater than or equal to the critical angle AND the incidence is from more dense to less dense, we have TIR.

$$\boxed{\theta_{ic} = \mathbf{sin}^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$

For $\theta_i > \theta_{ic}$ Then $\left|\rho_\perp\right| = 1$ as noted previously.

# Strange results

Now $\quad \sin\theta_t = \sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i$ so since $\varepsilon_1 > \varepsilon_2 \quad \Rightarrow \quad \sin\theta_t > 1$ !

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = jA$$

$\cos\theta_t$ is imaginary!

where $\mathrm{A} = \sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}$

What is the physical interpretation of these results? To see what is happening we go back to the expression for the transmitted field and substitute the above results.

# Transmitted field

previously $$E_t = \hat{z}\tau_\perp E_0 \,\mathbf{exp}\left[j\beta_2\left(x\,\mathbf{sin}\,\theta_t + y\,\mathbf{cos}\,\theta_t\right)\right]$$

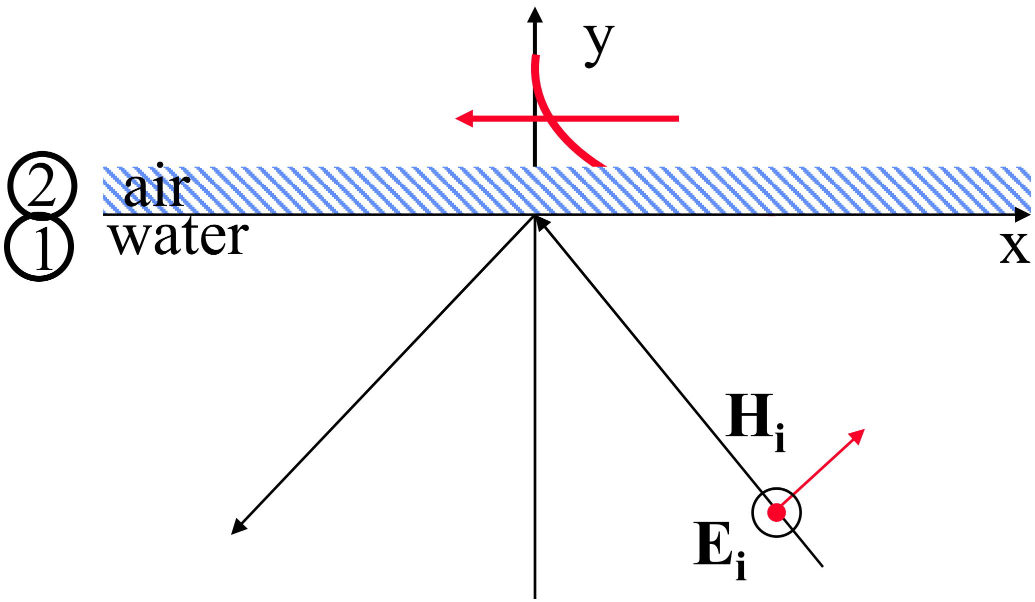
cos $\theta_t$=jA

$$= \hat{z}\tau_\perp E_0 \,\mathbf{exp}\left[j\beta_2 x\,\mathbf{sin}\,\theta_t\right]\mathbf{exp}\left[-\alpha y\right]$$

where $\alpha = \beta_2 A = \omega\sqrt{\mu_2\varepsilon_2}\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\mathbf{sin}^2\theta_i - 1}$

Physically, it is apparent that the transmitted field propagates along the surface (-x direction) but attenuates in the +y direction
This type of wave is a surface wave field

# Example




Assume:

$\varepsilon_r = 81$

$\sigma = 0$

$\mu_r = 1$

Let $\theta_i = 45°$

evaluate $\theta_{ic} = \mathbf{sin}^{-1}\sqrt{\frac{1}{81}} = 6.38^\circ$ so $\theta_i > \theta_{ic} \Rightarrow TIR$

# Example (ctd)

Using Snell $\sin\theta_t = \sqrt{\frac{81}{1}}\sin 45^\circ = 6.38$

$$\cos\theta_t = \pm j\sqrt{81\sin^2 45^\circ - 1} = +j6.28$$

Choose + sign to allow for attenuation in +y direction

$$\alpha = \beta_2 A = \frac{2\pi}{\lambda_0}6.28 = \frac{39.5}{\lambda_0} Nep/m$$

$$\tau = 1 + \rho_\perp$$

$$= 1 + \frac{0.707 - \sqrt{\frac{1}{81} - 0.5}}{0.707 + \sqrt{\frac{1}{81} - 0.5}}$$

$$= 1.42\angle -44.6^\circ$$

This means that if the field strength on the surface is1Vm-1, then

$$|E_t| = |\tau||E_i| = 1.42\text{Vm}^{-1}$$

# Evaluate the field just above the surface

Lets evaluate the transmitted E field at λ/4 above the surface.

$$E_t = 1.42\,\mathbf{exp}\left[\frac{-39.49}{\lambda_0}\frac{\lambda_0}{4}\right] = 73.2\mu Vm^{-1}$$

$$= 20\,\mathbf{log}\left(\frac{73.2\times 10^{-6}}{1.42}\right) = -85.8dB$$

This means that the surface wave is very tightly bound to the surface and the power flow in the direction normal to the surface is zero.

# What about the factor $\dfrac{k_0}{\omega\mu_0}$ ?

$$\frac{k_0}{\omega\mu_0} = \frac{2\pi}{\lambda_0\omega\mu_0} = \frac{2\pi f}{c\omega\mu_0} = \frac{1}{c\mu_0} = \sqrt{\frac{\varepsilon_0}{\mu_0}}$$

This term has the dimensions of admittance, in fact

$$\boxed{Y_0 = \frac{1}{Z_0} = \frac{1}{\eta_0} = \sqrt{\frac{\varepsilon_0}{\mu_0}}}$$

where $\mathrm{Z_0} = \text{impedance of free space} \approx 377\Omega$

And now $$\boxed{\vec{H} = \frac{1}{\eta_0}\hat{n}\times\vec{E}}$$

# Propagation in conducting media

We have considered propagation in free space (perfect dielectric with $\sigma = 0$). Now consider propagation in conducting media where $\sigma$ can vary from a finite value to $\infty$.

Start with $$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu \frac{\partial \mathbf{J}}{\partial t} + \nabla \frac{\rho}{\varepsilon}$$

Assuming no free charge and the time harmonic form, gives

$$\nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = j\omega\mu\sigma\vec{E}$$

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$$

where $$\boxed{\gamma^2 = j\omega\mu\sigma - \mu\varepsilon\omega^2}$$

Complex propagation coefficient due to finite conductivity

# Conduction current and displacement current

In metals, the conduction current ($\sigma\mathbf{E}$) is much larger than the displacement current ($j\omega\varepsilon_0\mathbf{E}$). Only as frequencies increase to the optical region do the two become comparable.

E.g. $\sigma = 5.8\text{x}10^{7}$ for copper

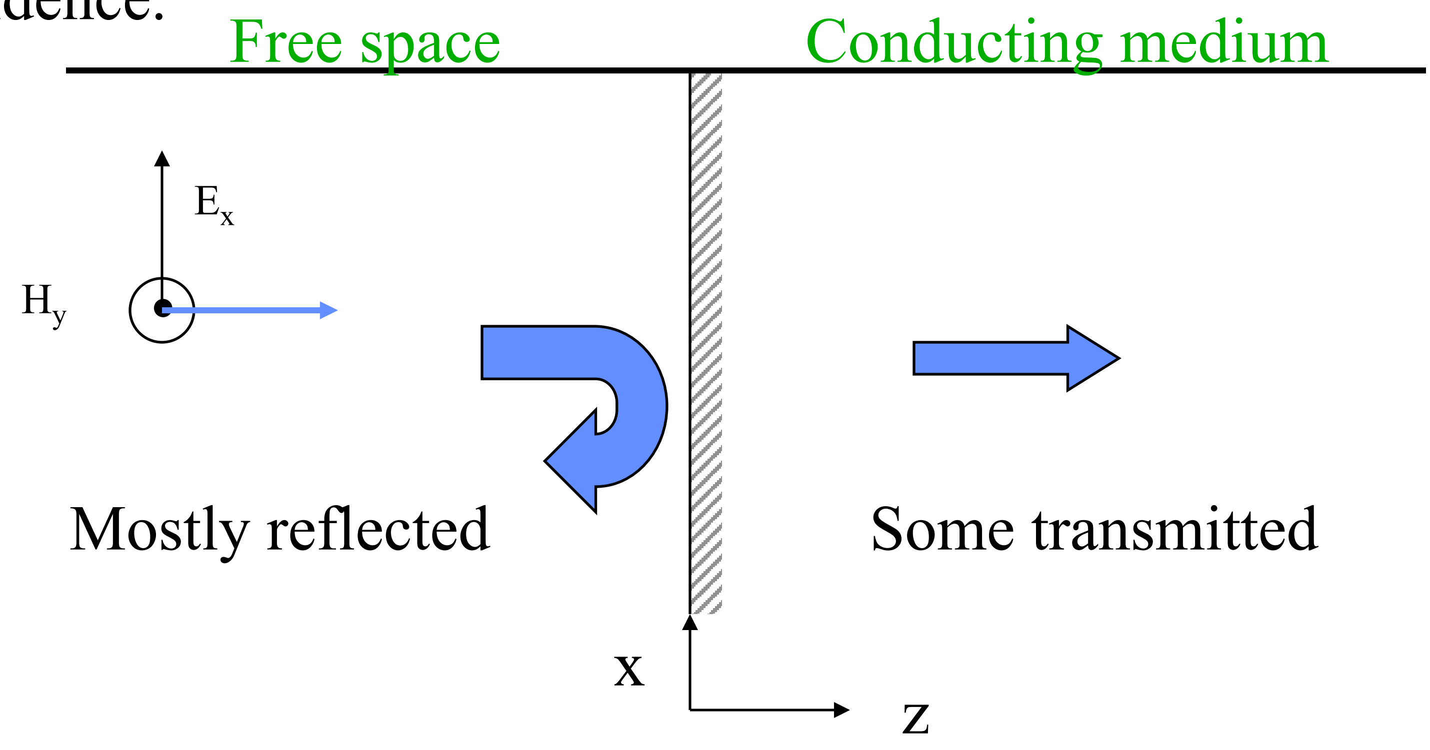
$\omega\varepsilon_0 = 2\pi\text{x}10^{10}\text{x }8.854\text{x}10^{-12} = 0.556$

So retain only the $j\omega\mu\sigma$ term when considering highly conductive material at frequencies below light. The PDE becomes:

$$\nabla^2\vec{E} - j\omega\mu_0\sigma\vec{E} = 0$$

# Plane wave incident on a conductor

Consider a plane wave entering a conductive medium at normal incidence.



# Mathematical solution

The equation for this is:

$$\frac{\partial^2 E_x}{\partial z^2} - j\omega\mu_0\sigma E_x = 0$$

The solution is:

$$E_x = E_0 e^{-\sqrt{j\omega\mu_0\sigma}\, z}$$

We can simplify the exponent:

$$\gamma = \sqrt{j\omega\mu_0\sigma} = (1+j)\sqrt{\frac{\omega\mu_0\sigma}{2}}$$

So now $\gamma$ has equal real and imaginary parts.

$$E_x = E_0 e^{-\alpha z} e^{-\beta z} \quad \text{with} \quad \alpha = \beta = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

Alternatively write

$$E_x = E_0 e^{-z/\delta} e^{-jz/\delta}$$

# Skin Depth

The last equation

$$E_x = E_0 e^{-z/\delta} e^{-jz/\delta}$$

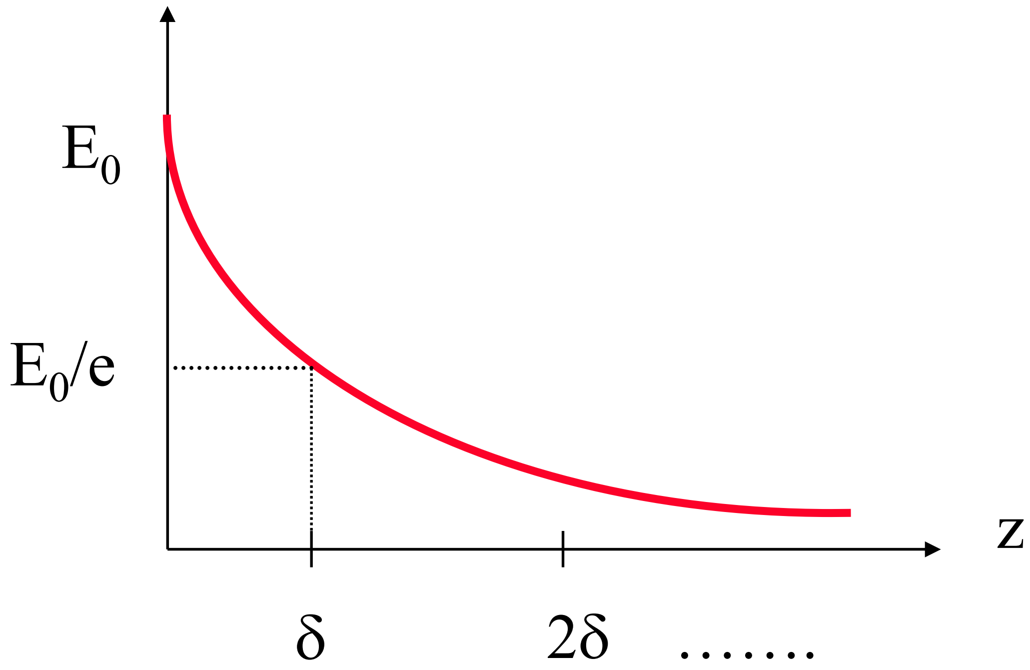
gives us the notion of skin depth:

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} = \frac{1}{\alpha} = \frac{1}{\beta}}$$

On the surface at z=0 we have $E_x=E_0$
at one skin depth $z=\delta$ we have $E_x=E_0/e$

field has decayed to 1/e or 36.8% of value on the surface.

# Plot of field into conductor



# Examples of skin depth

Copper

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} = \frac{6.61\times10^{-2}}{\sqrt{f}}$$

σ = 5.8x10$^{7}$ S/m

| | | |
|---|---|---|
| at | 60Hz | δ=8.5x10$^{-3}$ m |
| at | 1MHz | δ=6.6x10$^{-5}$ m |
| at | 30GHz | δ=3.8x10$^{-7}$ m |

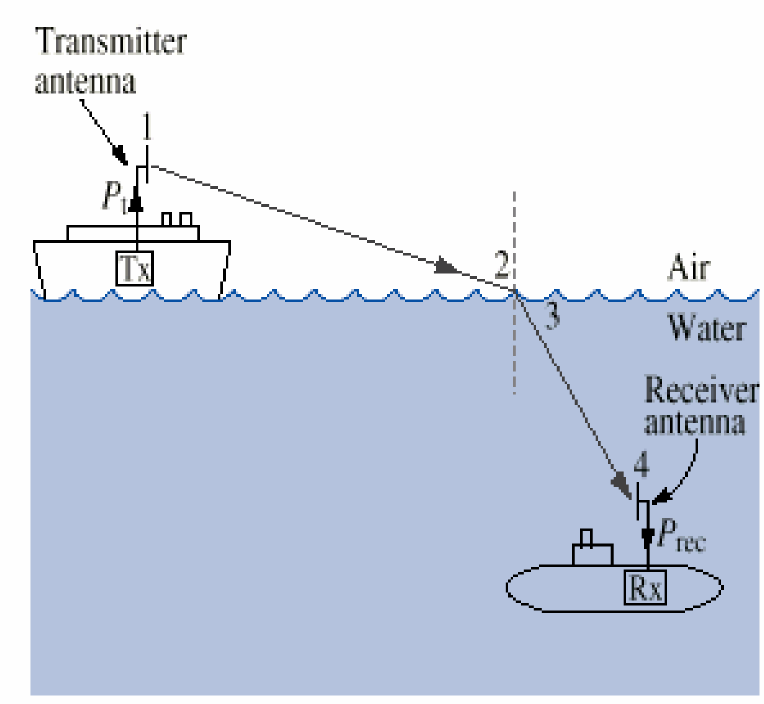
Seawater

$$\delta = \frac{2.52\times10^{2}}{\sqrt{f}}$$

σ = 4 S/m

at 1 kHz δ=7.96m Submarine comms!

# Submarine communication



# Characteristic or Intrinsic Impedance $Z_m$

Define this via the material as before:

$$Z_m = \sqrt{\frac{\mu_0}{\varepsilon_c}} = \sqrt{\frac{\mu_0}{\varepsilon - j\frac{\sigma}{\omega}}}$$

But again, the conduction current predominates, which means the second term in the denominator is large. With this approximation we can arrive at:

$$\boxed{Z_m = (1+j)\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{1+j}{\sigma\delta}}$$

For copper at 10GHz $Z_m = 0.026(1+j)\ \Omega$

# Reflection from a metal surface

So a reflection coefficient at metal-air interface is

$$\rho = \frac{Z_m - Z_0}{Z_m + Z_0} \approx -1 \quad \text{since } Z_m << Z_0$$

We also note that as $\sigma \rightarrow \infty$, $Z_m \rightarrow 0$ and that $\rho = -1$ for the case of the perfect conductor. Thus the boundary condition for a PEC is satisfied in the limit.

The transmission coefficient into the metal is given by $\tau = 1+\rho$

# Conductors and dielectrics

Materials can behave as either a dielectric or a conductor depending on the frequency.

recall $\nabla \times H = \sigma E + j\omega\varepsilon E$

Displacement current density

Conduction current density

<u>3 choices</u>

$\omega\varepsilon >> \sigma$ displacement current >> conductor current $\Rightarrow$ dielectric

$\omega\varepsilon \approx \sigma$ displacement current $\approx$ conductor current $\Rightarrow$ quasi conductor

$\omega\varepsilon << \sigma$ displacement current << conductor current $\Rightarrow$ conductor

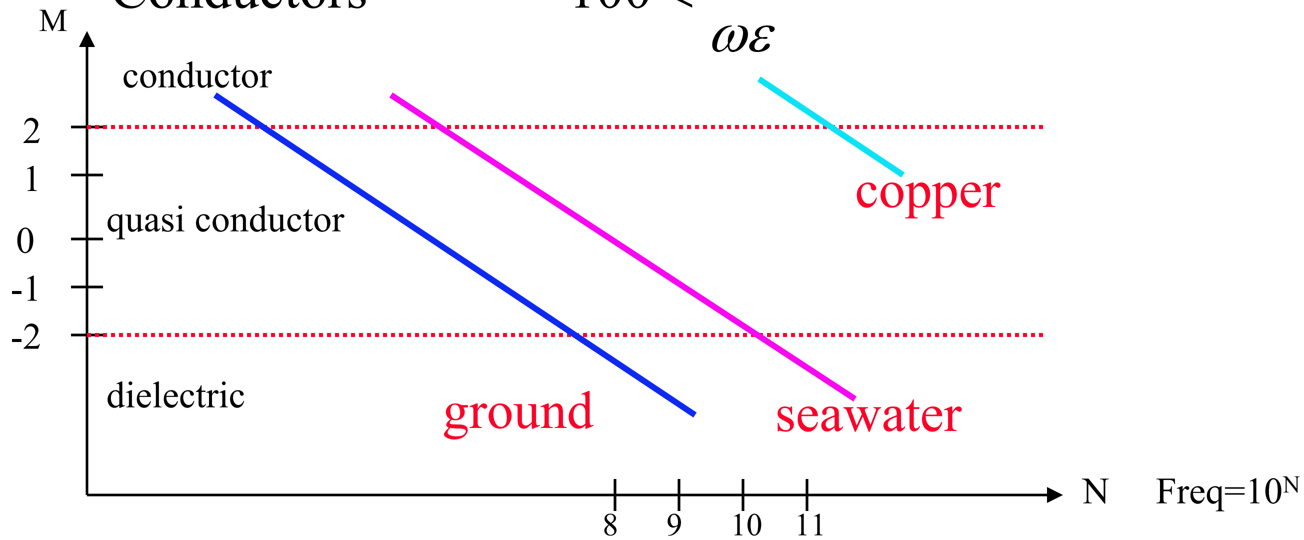
# A rule for determining whether dielectric or conductor

Dielectrics $$\frac{\sigma}{\omega\varepsilon} < \frac{1}{100}$$

Quasi Conductors $$\frac{1}{100} < \frac{\sigma}{\omega\varepsilon} < 100$$

Conductors $$100 < \frac{\sigma}{\omega\varepsilon}$$

$$\frac{\sigma}{\omega\varepsilon} = 10^M$$



# General case: (both conduction & displacement currents)

From 4.4, we now retain both terms:

$$\gamma^2 = j\omega\mu\sigma - \mu\varepsilon\omega^2 = -\omega^2\mu\varepsilon\left[1 + \frac{\sigma}{j\omega\varepsilon}\right]$$

If we now let $\gamma = \alpha + j\beta$, square it and equate real and imaginary parts and then solve simultaneously for $\alpha$ and $\beta$. We obtain:

$$\boxed{\begin{aligned} \alpha &= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]\right\}^{\frac{1}{2}} \quad \text{Np/m} \\ \beta &= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]\right\}^{\frac{1}{2}} \quad \text{rad/m} \end{aligned}}$$

# Approximations

By taking a binomial expansion of the term under the radical and simplifying, we can obtain:

| | Good dielectric | Good conductor |
|---|---|---|
| $\alpha$ | $\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$ | $\sqrt{\frac{\omega\mu\sigma}{2}}$ |
| $\beta$ | $\omega\sqrt{\mu\varepsilon}$ | $\sqrt{\frac{\omega\mu\sigma}{2}}$ |
| $Z_w$ | $\sqrt{\frac{\mu}{\varepsilon}}$ | $\sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$ |

## Example Problem 1:

An FM radio broadcats signal traveling in the y-dirrection in air has a magnetic field given by the phasor

$$H(y) = 2.92 \times 10^{-3} e^{-j0.68\pi y}(-\hat{x} + \hat{z}j)A - m^{-1}$$

(a) Determine the frequency (in MHZ) and wavelength (in m).

(b) Find the corresponding $E(y)$.

(a) we have

$$\beta = \omega\sqrt{\mu_o \varepsilon_o} = 0.68\pi \, rad - m^{-1}$$

from which

$$f = \frac{\omega}{2\pi} \approx 102 MHz$$

$$\nabla \times H = \hat{x}\frac{\partial H_z}{\partial y} - \hat{z}\frac{\partial H_x}{\partial y} = j\omega\varepsilon_o E$$

$$\Rightarrow E(y) \approx 1.1 e^{-j0.68\pi y}(-\hat{x}j - \hat{z})V - m^{-1}$$

# Example Problem 2:

A uniform plane wave of frequency 10 GHz propagates in a sufficiently large sample of gallium arsenide (GaAs, $\varepsilon_r \approx 12.9, \mu_r \approx 1$, $tan\delta_c \approx 5\text{x}10^{-5}$),which is a commonly substrate material for high-speed solid-state devices. Find (a) the attenuation constant $\alpha$ in np-m$^{-1}$,(b) phase velocity $\nu_p$ in m-s$^{-1}$,and (c) intrinsic impedance $\eta_c$ in $\Omega$.

Since $tan\delta_c = 5\times10^{-4} << 1$, we can use the approx for a good dielectric

(a) We have

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\omega\varepsilon tan\delta_c}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{2\pi\times10^{10}\times5\times10^{-4}}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$= \frac{2\pi\times10^{10}\times5\times10^{-4}\sqrt{\mu_r\varepsilon_r}\sqrt{\mu_0\varepsilon_0}}{2}$$

$$= \frac{2\pi\times10^{10}\times5\times10^{-4}}{2\times3\times10^{8}}\sqrt{12.9} \approx 0.188\, np - m^{-1}$$

## Example Problem 2:

(b) Since *phase* velocity $\nu_p = \dfrac{\omega}{\beta}$

where $\beta \approx \omega\sqrt{\mu\varepsilon}$, we have

$\nu_p \approx \dfrac{1}{\sqrt{\mu\varepsilon}} \approx \dfrac{3\times 10^8}{\sqrt{12.9}} \approx 8.35\times 10^7\, m-s^{-1}$.Note that the

phase velocity is ~ $3.59$ times slower that in the air.

(c) The intrinsic impedance $\eta_c \approx \sqrt{\dfrac{\mu}{\varepsilon}} \approx \dfrac{377}{\sqrt{12.9}} \approx 105\Omega$.

Note that the intrinsic impedance is ~ $3.59$ times smaller that that in air.

# Example Problem3:

A recent survey conducted in USA indicates that ~50% of the population is exposed to average power densities of approximately 0.005 $\mu$W-(cm)$^{-2}$due to VHF and UHF broadcast radiation. Find the corresponding amplitude of the electric and magnetic fields.

Consider the uniform plane wave propagating in a lossless medium :

$$E_x = E_0 \cos(\omega t - \beta z)$$

$$H_y = \frac{1}{\eta} E_0 \cos(\omega t - \beta z)$$

where $\beta = \omega\sqrt{\mu\varepsilon}$ and $\eta = \sqrt{\mu / \varepsilon}$. The Poynting vector for this wave is given by

$$\overline{\mathrm{P}} = \bar{E} \times \overline{H} = \hat{z} E_0 \left(\frac{E_0}{\eta}\right) \cos^2(\omega t - \beta z) = \hat{z} \frac{E_0{}^2}{2\eta}\left[1 + \cos 2(\omega t - \beta z)\right]$$

Example Problem3:

$$S_{av} = \frac{1}{T_p}\int_0^{T_p} \overline{\mathrm{P}}(z,t)dt = \frac{1}{T_p}\int_0^{T_p} \hat{z}\frac{E_0{}^2}{2\eta}\left[1+cos2(\omega t-\beta z)\right]dt$$

$$\Rightarrow \boxed{S_{av} = \hat{z}\frac{E_0{}^2}{2\eta}}$$

$$S_{av} = \hat{z}\frac{E_0{}^2}{2\eta} = 0.005\mu W-(cm)^{-2}$$

so $E_0 \approx \sqrt{2\times 377\times 5\times 10^{-9}/10^{-4}} \approx 194mV-m^{-1}$

$$H_0 = \frac{E_0}{\eta} = \frac{194mV-m^{-1}}{377\Omega} = 515\mu A-m^{-1}$$