Lecture 1

Review of Mathematics

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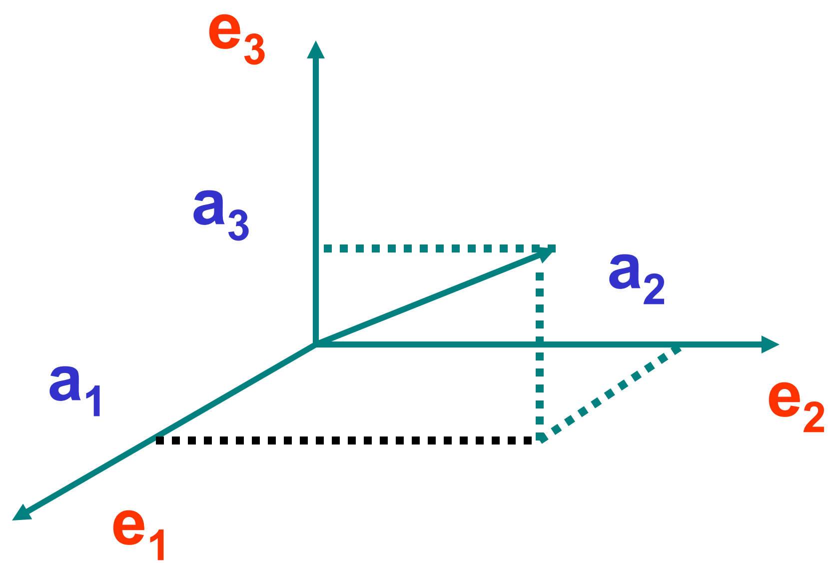
Vectors

Cartesian components of vectors

Let $\{e_1, e_2, e_3\}$ be three mutually perpendicular unit vectors which form a right handed triad. Then $\{e_1, e_2, e_3\}$ are said to form an *orthonormal basis*. The vectors satisfy:

$$|e_1| = |e_2| = |e_3| = 1$$

$$e_1 \times e_2 = e_3, e_1 \times e_3 = -e_2, e_2 \times e_3 = e_1$$



Vectors

We may express any vector a as a suitable combination of the unit vectors $\{e_{1,} e_{2\,,} e_3\}$. For example, we may write

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 = \sum_{i=1}^{3} a_i e_i$$

where $\{a_1, a_2, a_3\}$ are scalars, called the components of ***a*** in the basis $\{e_{1,} e_{2\,,} e_3\}$. The components of a have ***a*** simple physical interpretation. For example, if we calculate the dot product ***a.*** e_1, we find that

$$a \cdot e_1 = \left(a_1 e_1 + a_2 e_2 + a_3 e_3\right) \cdot e_1 = a_1$$

Recall that

$$a \cdot e_1 = |a||e_1| \cos\theta(a \cdot e_1)$$

$$a_1 = a \cdot e_1 = |a| \cos\theta(a \cdot e_1)$$

Vectors

Thus, a_1 represent the projected length of the vector ***a*** in the direction of e_1. This similarly applies to a_2, a_3 .

Change of basis

Let ***a*** be a vector and let $\{e_1, e_2, e_3\}$ be a Cartesian basis. Suppose that the components of ***a*** in the basis $\{e_1, e_2, e_3\}$ are known to be $\{a_1, a_2, a_3\}$

Now, suppose that we wish to compute the components of ***a*** in a second Cartesian basis, $\{r_1, r_2, r_3\}$. This means we wish to find components $\{\alpha_1, \alpha_2, \alpha_3\}$, such that $a = \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3$ to do so, note that

$$\alpha_1 = a \cdot r_1 = \alpha_1 e_1 \cdot r_1 + \alpha_2 e_2 \cdot r_1 + \alpha_3 e_3 \cdot r_1$$

$$\alpha_2 = a \cdot r_2 = \alpha_1 e_1 \cdot r_2 + \alpha_2 e_2 \cdot r_2 + \alpha_3 e_3 \cdot r_2$$

$$\alpha_3 = a \cdot r_3 = \alpha_1 e_1 \cdot r_3 + \alpha_2 e_2 \cdot r_3 + \alpha_3 e_3 \cdot r_3$$

Vectors

This transformation is conveniently written as a matrix operation

$$\alpha = [Q][a]$$

where $[\alpha]$ is a matrix consisting of the components of ***a*** in the basis $\{r_1, r_2, r_3\}$, $[a]$ is a matrix consisting of the components of ***a*** in the basis $\{a_1, a_2, a_3\}$, and $[Q]$ is a "rotation matrix" as follows

$$[\alpha] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad [a] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad [Q] = \begin{bmatrix} r_1 \cdot e_1 & r_1 \cdot e_2 & r_1 \cdot e_3 \\ r_2 \cdot e_1 & r_2 \cdot e_2 & r_2 \cdot e_3 \\ r_3 \cdot e_1 & r_3 \cdot e_2 & r_3 \cdot e_3 \end{bmatrix}$$

Using index notation $\alpha_i = Q_{ij} a_j, \; Q_{ij} = r_i \cdot e_j$

Gradient of a Vector Field

Let ***v*** be a vector field in three dimensional space. The gradient of ***v*** is a tensor field denoted by grad(***v***) or ∇***v*** , and is defined so that

$$(\nabla v)\cdot\alpha = \lim_{\varepsilon\to 0}\frac{v(r+\varepsilon\alpha)-v(r)}{\varepsilon}$$

for every position **r** in space and for every vector α**.**

Let $\{e_1, e_2, e_3\}$ **be a Cartesian basis with origin O in three dimensional space. Let** $r = x_1 e_1 + x_2 e_2 + x_3 e_3$ **denote the position vector of a point in space. The gradient of *v*** in this basis is given by

$$\nabla v = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

$$[\nabla v]_{ij} \equiv \frac{\partial v_i}{\partial x_j}$$

Divergence of a Vector Field

Let ***v*** be a vector field in three dimensional space. The divergent of ***v*** is a scalar field denoted by div(***v***) or ▽•***v*** , and is defined so that

$$div(v) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

Formally, it is defined as trace[grad(***v***)].

$$\nabla v = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

$$\nabla \cdot v = Tr(\nabla v) = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}$$

Curl of a Vector Field

Let ***v*** be a vector field in three dimensional space. The curl of ***v*** is a vector field denoted by **curl**(***v***) or $\nabla\times$***v*** , and it is best defined in terms of its components in a given basis.

$$r = x_1 e_1 + x_2 e_2 + x_3 e_3$$

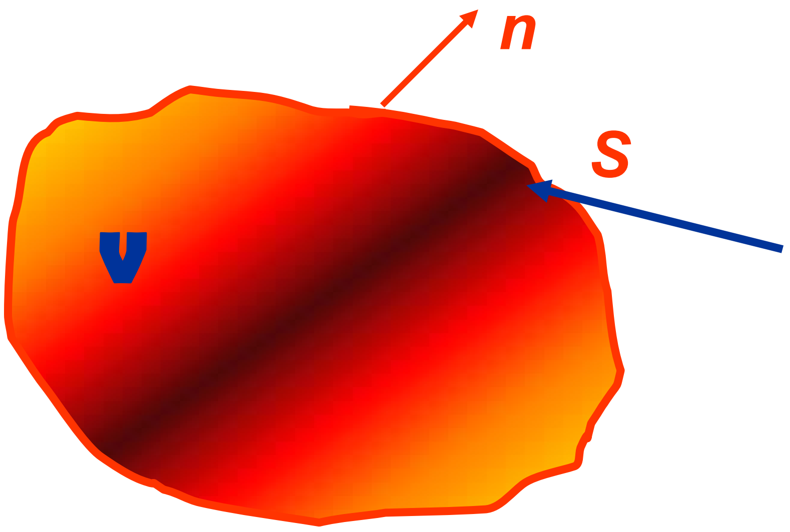
Express ***v*** as a function of the components of **r** ***v = v(x₁,x₂,x₃).*** The curl of ***v*** in this base is then given by

$$\nabla\times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)e_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)e_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)e_3$$

$$[\nabla v]i \equiv \varepsilon_{ijk}\frac{\partial v_j}{\partial x_k}$$

The Divergence Theorem

Let **V** be a closed region in three dimensional space, bounded by an oreintable surface **S.** Let n denote the unit vector normal to S, taken so that n points out of V. Let u be a vector field which is continuous and has continuous first partial derivatives in some domain containing T. Then



$$\int_V div(u)dV = \int_S u \cdot n dA$$

expressed in index notation:

$$\int_V \frac{\partial u_i}{\partial x_i} dV = \int_S u_i n_i dA$$

Integrals

Examples

$$\int_0^1 x^2 e^x dx$$

$$\begin{cases} u = x^2 \\ dv = e^x \end{cases}$$

After integration and differentiation, we get

$$\begin{cases} du = 2x dx \\ v = e^x \end{cases}$$

$$\int_0^1 x^2 e^x dx = x^2 e^x \Big|_0^1 - \int_0^1 2x e^x dx \quad \begin{cases} u = x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = e^x \end{cases}$$

$$\int_0^1 x e^x dx = x e^x \Big|_0^1 - e^x \Big|_0^1$$

$$\Rightarrow \int_0^1 x^2 e^x dx = x^2 e^x \Big|_0^1 - 2x e^x \Big|_0^1 + 2e^x \Big|_0^1$$

$$\int_0^1 x^2 e^x dx = e - 2$$

Integrals

Evaluate $$\int x\tan^{-1}(x)dx$$

$$\begin{cases} u = \tan^{-1}(x) \\ dv = xdx \end{cases} \Rightarrow \begin{cases} du = \dfrac{1}{1+x^2}dx \\ v = \dfrac{1}{2}x^2 \end{cases}$$

$$\int x\tan^{-1}(x)dx = \frac{1}{2}x^2\tan^{-1}(x) - \int \frac{1}{2}\frac{x^2}{1+x^2}dx$$

$$\int \frac{x^2}{1+x^2}dx = \int \frac{x^2+1-1}{1+x^2}dx = \int\left(1-\frac{1}{1+x^2}\right)dx = x - \tan^{-1}(x) + C$$

$$\int x\tan^{-1}(x)dx = \frac{1}{2}x^2\tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + C$$

Integrals – trig substitution

Evaluate $\int x^3 \sqrt{4-x^2}\,dx$

$$set\ x = 2\sin(t) \Rightarrow dx = 2\cos(t)dx$$

$$\int x^3 \sqrt{4-x^3}\,dx = \int 8\sin^3(t)\sqrt{4-4\sin^2(t)}\,2\cos(t)dt$$

$$\int x^3 \sqrt{4-x^3}\,dx = 32\int \sin^3(t)\cos^2(t)dt$$

$$\int \sin^3(t)\cos^2(t)dt = \int (1-\cos^2(t))\cos^2(t)\sin(t)dt$$

$$v = \cos(t) \Rightarrow dv = -\sin(t)dt$$

$$\int (1-\cos^2(t))\cos^2(t)\sin(t)dt = -\int (1-v^2)v^2 dv = -\frac{v^3}{3} + \frac{v^5}{5} + C$$

$$\int x^3 \sqrt{4-x^3}\,dx = -32\frac{v^3}{3} + 32\frac{v^5}{5} + C = -\frac{4(4-x^2)^{3/2}}{3} + \frac{(4-x^2)^{5/2}}{5} + C$$

Matrices

Consider $$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

J is 3x4 matrix composed of 3 rows and 4 columns.

When the numbers of rows and columns are equal, the matrix is called a square matrix. A square matrix of order n is an (nxn) matrix.

Matrices operation

Vector, $$p = \begin{bmatrix} a & b & c & d \end{bmatrix}$$ is a 1x4 row matrix.

Vector, $$q = \begin{bmatrix} k \\ l \\ m \\ n \end{bmatrix}$$ is a4x1 column matrix.

Matrices operation

1. Addition

$$\text{consider } P = \begin{bmatrix} \alpha & \beta \\ \mu & \gamma \end{bmatrix} \text{ and } Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$\text{then } T = P + Q \text{ is a } 2 \times 2 \text{ matrix with :}$$

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \text{with } t_{11} = \alpha + a,\ t_{12} = \beta + b$$

$$t_{21} = \mu + c,\ t_{22} = \gamma + d$$

$$T = \begin{bmatrix} \alpha + a & \beta + b \\ \mu + c & \gamma + d \end{bmatrix}$$

Matrices operation

If λ is a constant then,

$$\lambda\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

$$\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{is}\, 2\times 2\, identity\;\; matrix$$

Matrices operation

$$\underbrace{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}_{2\times 3} \underbrace{\begin{pmatrix} \alpha \\ \beta \\ \nu \end{pmatrix}}_{3\times 1} = \underbrace{\begin{pmatrix} a\alpha + b\beta + c\nu \\ d\alpha + e\beta + f\nu \end{pmatrix}}_{2\times 1}$$

$$\cancel{\underbrace{\begin{pmatrix} \alpha \\ \beta \\ \nu \end{pmatrix}}_{3\times 1} \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}_{2\times 3}}$$

Matrices operation

An n x n matrix A is called invertible iif there exists an n x n matrix B such that

$$\boxed{AB = BA = I_n}$$

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 3/2 \\ 1 & -1 \end{pmatrix}$$

$$AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$notation \quad AA^{-1} = A^{-1}A = I_n \quad (A \text{ is a } n \times n \text{ matrix})$$

$$\boxed{\left(\mathrm{A}^{-1}\right)^{-1} = A} \qquad \boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

Matrices operation

Let A be a n x m matrix defined by α_{ij}, then the transpose of A, denoted A^T is the m x n matrix defined by δ_{ij} where $\delta_{ij} = \alpha_{ji}$.

1. $(X+Y)^T = X^T + Y^T$
2. $(XY)^T = Y^T X^T$
3. $(X^T)^T = X$

Matrices operation

Consider a square matrix A and define the sequence of matrices

$$A_n = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ... + \frac{1}{n!}A^n$$

as $n \to \infty$,

$$e^A = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ... + \frac{1}{n!}A^n + ...$$

one can write this in series notation as

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!}A^n$$

Matrices operation

Determinants

Consider the matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A is invertible if and only if $ad-bc\neq 0$

This number is called the determinant of A.

Determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc.$

Properties:

$$\det A = \det A^T, \quad \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ b & d \end{vmatrix} = ad \;, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

$$\begin{vmatrix} \lambda a & \lambda b \\ c & d \end{vmatrix} = \lambda \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ \lambda c & \lambda d \end{vmatrix}, \quad \det(AB) = \det(A)\det(B)$$

Matrices operation

In general,

$$\det(A) = \sum_{j=1}^{j=n} a_{ij} A_{ij} \qquad \text{for any fixed } i$$

$$\det(A) = \sum_{i=1}^{i=n} a_{ij} A_{ij} \qquad \text{for any fixed } j$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Eigenvalues and Eigenvectors

Let A be a square matrix. A non-zero vector C is called an eigenvector of A iff $\exists$ a number (real or complex) $\lambda \ni AC = \lambda C$

If λ exists, it is called an eigenvalue of A.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

$$AC_1 = 0C_1,\ AC_2 = -4C_2, \text{ and } AC_3 = 3C_3$$

$$where \quad C_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, C_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, and \ C_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

Computing eigenvalues

$$AC = \lambda C$$

$$AI_n C = \lambda I_n C \Rightarrow AI_n C - \lambda I_n C = 0$$

$$\left(AI_n - \lambda I_n\right)C = 0 \Rightarrow \left(A - \lambda I_n\right)C = 0$$

This is a linear system for which the matrix coefficient is $A - \lambda I_n$.
This system has one solution if and only if the matrix coefficient is invertible,I.e. $\det\left(\mathrm{A} - \lambda \mathrm{I_n}\right) \neq 0$

Since the zero-vector is a solution and C is not the zero vector, we must have

$$\det\left(\mathrm{A} - \lambda \mathrm{I_n}\right) = 0$$

Computing eigenvalues

Consider matrix A:

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix}. \ \det\left(A - \lambda I_n\right) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -2 & 0-\lambda \end{vmatrix} = (1-\lambda)(0-\lambda) - 4 = 0$$

$$\textit{which is equivalent to the quadratic equation}$$

$$\lambda^2 - \lambda - 4 = 0$$

$$\textit{solutions}: \ \lambda = \frac{1+\sqrt{17}}{2}, \ \textit{and} \ \lambda = \frac{1-\sqrt{17}}{2}$$

Computing Eigenvalues

$$\det(A-\lambda I_n)=\det(A-\lambda I_n)^T=\det(A^T-\lambda I_n).$$

$$\textit{for any square matrix of order } 2,\ A=\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic polynomial is given by

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}=(a-\lambda)(d-\lambda)-bc=0$$

$$\Rightarrow \lambda^2-(a+b)\lambda+ad-bc=0.$$

The number $(a+b)$ is called the trace of A (denoted tr(A)),

and $(ad-bc)$ is the Determinant of A. $\lambda^2-tr(A)\lambda+\det(A)=0.$

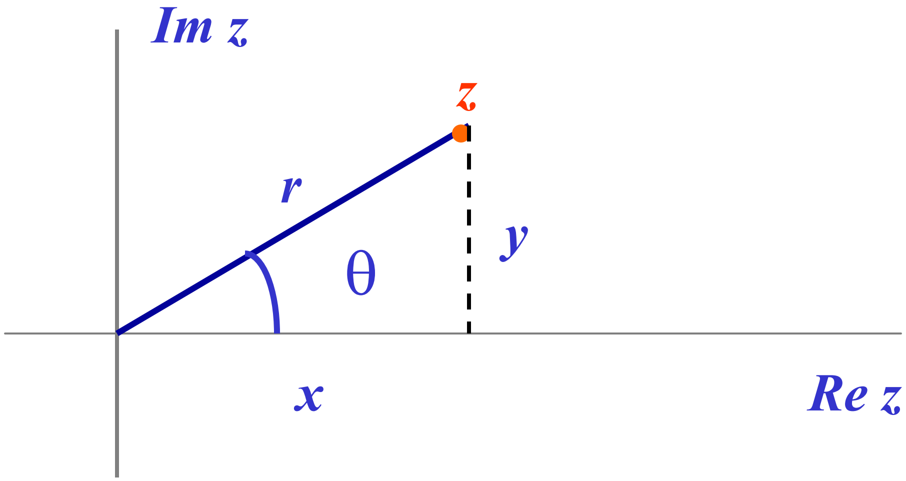
Complex Variables

Standard notation: $$z = x + iy = re^{i\theta}$$

where
$$x, y, r, \text{ and } \theta \text{ are real}, i^2 = -1$$
$$\text{and } e^{i\theta} = \cos\theta + i\sin\theta$$

x and *y* are the *real (Re z)* and *imaginary (Im z) part of z*, respectively.

$r = |z|$ is the *magnitude*, and $,\ \theta$ is the phase or *argument arg z*.



Complex Variables

The ***complex conjugate*** of *z* is denoted by z^*; $z^*=x-iy$.

A function *W(z)* of the complex variable z is itself a complex number whose real and imaginary parts *U* and *V* depend on the position of z in the xy-plane. $W(z) = U(x,y) + iV(x,y)$.

$$W(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$$U = x^2 - y^2 \qquad V = 2xy$$

$$or \quad W = z^2 = r^2 e^{2i\theta}$$

Complex Functions

1. Exponential

$$\exp(z) = e^{z} \quad \text{with} \quad z = x + iy$$

$$\exp(z) = e^{x}(\cos y + i\sin y)$$

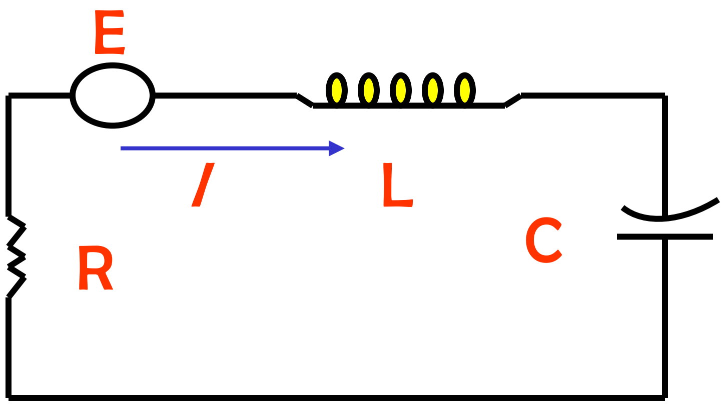
$$\frac{d}{dz}\exp(z) = \exp(z)$$

$$\text{if } z = x + iy \text{ and } w = u + iv, \text{ then}$$

$$\begin{aligned} \exp(z+w) &= e^{x+u}\left[\cos(y+v) + i\sin(y+y)\right] \\ &= e^{x}e^{u}\left[\cos y\cos v - \sin y\sin v + i(\sin y\cos v + \cos y\sin v)\right] \\ &= e^{x}e^{u}(\cos y + i\sin y)(\cos v + i\sin v) \\ &= \exp(z)\exp(w) \end{aligned}$$

Complex Functions

Circuit problem



$$V_R = RI$$

$$V_L = L\frac{di}{dt}$$

$$i_C = C\frac{dV}{dt}$$

$$V(t) = A\sin(\omega t + \phi) \Rightarrow V = \mathrm{Im}(Ae^{i\phi}e^{i\omega t}) = \mathrm{Im}(Be^{i\omega t})$$

$$I = \mathrm{Im}(Ce^{i\omega t})$$

$$\frac{d}{dt}Ae^{i\omega t} = i\omega Ae^{i\omega t}. \quad if \;\; I = be^{i\omega t},$$

$$\Rightarrow V = i\omega LI \;\; (for\ inductor)\ and\ i\omega VC = I, or\ V = \frac{I}{i\omega C} \;\; for\ a\ capacitor.$$

Complex Functions

Kirchoff's law:

$$i\omega LI + \frac{I}{i\omega C} + RI = ae^{i\omega t} \qquad \left(\mathrm{E} = \mathrm{a}\ \mathrm{e}^{i\omega t}\right)$$

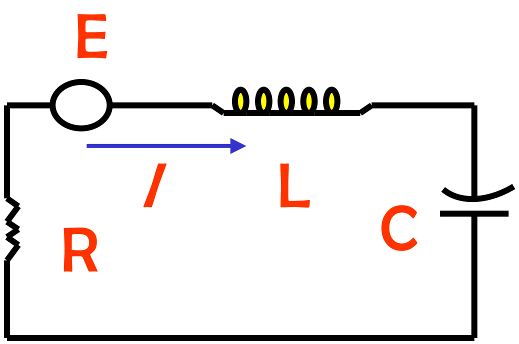
$$i\omega Lb + \frac{b}{i\omega C} + Rb = a$$

$$\Rightarrow b = \frac{a}{R + i\left(\omega L - \frac{1}{\omega C}\right)}$$

$$b = \frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i\phi}, \qquad \tan\phi = \frac{\omega L - \frac{1}{\omega C}}{R}$$

$$I = \mathrm{Im}(be^{i\omega t}) = \mathrm{Im}\left(\frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{i(\omega t + \phi)}\right)$$

$$= \frac{a}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t + \phi)$$



Differential Equations

1st order DE has the following form:

$$\frac{dy}{dx} + P(x)y = q(x)$$

The general solution is given by

$$y = \frac{\int u(x)q(x) + C}{u(x)},$$

U(x) is called the integrating factor.

$$u(x) = \exp\left(\int p(x)dx\right)$$

Differential Equations

Example 1

Find the particular solution of $y' + \tan(x)y = \cos^2(x), \;\; y(0) = 2.$

- step 1: identify *p(x)* and *q(x).*

$$p(x) = \tan(x) \quad and \; q(x) = \cos^2(x)$$

- step 2: Evaluate the integrating factor

$$u(x) = e^{\int \tan(x)\,dx} = e^{-\ln(\cos(x))} = e^{\ln(\sec(x))} = \sec(x)$$

- We have

$$\int \sec(x)\cos^2(x)dx = \int \cos(x)dx = \sin(x)$$

$$y = \frac{\sin(x) + C}{\sec(x)} = (\sin(x) + C)\cos(x), \;\; y(0) = C = 2$$

$$y = (\sin(x) + 2)\cos(x)$$

Example 2

Find solution to

$$\cos^2(t)\sin(t)y' = -\cos^3(t)y + 1, \quad y(\pi/4) = 0.$$

Rewrite the equation:

$$y' = -\frac{\cos^3(t)}{\cos^2(t)\sin(t)}y + \frac{1}{\cos^2(t)\sin(t)} = -\frac{\cos(t)}{\sin(t)}y + \frac{1}{\cos^2(t)\sin(t)}$$

$$\rightarrow y' + \frac{\cos(t)}{\sin(t)}y = \frac{1}{\cos^2(t)\sin(t)}$$

Hence the integration factor is given by

$$\boxed{u(t) = e^{-\int \frac{\cos(t)}{\sin(t)}dt} = e^{\ln|\sin(t)|} = \sin(t)}$$

Differential Equations

Example 2

The general solution can be obtained as

$$y = \frac{\int \sin(t) \frac{1}{\cos^2(t)\sin(t)} dt + C}{\sin(t)}$$

Since we have

$$\int \sin(t) \frac{1}{\cos^2(t)\sin(t)} dt = \int \frac{1}{\cos^2(t)} dt = \tan(t),$$

Differential Equations

Example 2

We get

$$y = \frac{\tan(t) + C}{\sin(t)} = \frac{1}{\cos(t)} + \frac{C}{\sin(t)} = \sec(t) + C\csc(t)$$

The initial condition $y(\frac{\pi}{4}) = 0$ implies

$$\sqrt{2} + C\sqrt{2} = 0, \Rightarrow C = -1$$

$$\boxed{y(t) = \sec(t) - \csc(t)}$$

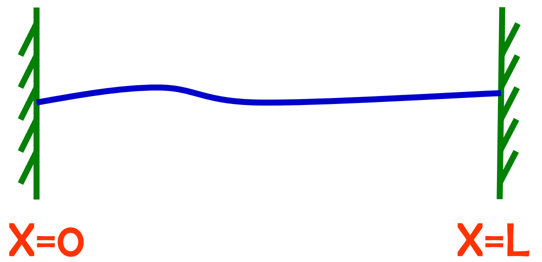
Separation of Variables-PDE

This method can be applied to partial differential equations, especially with constant coefficients in the equation. Consider one-dim wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(x,t) \textit{ is the displacement (deflection) of the stretched string.}$$

$$u(0,t)=0 \quad u(L,t)=0 \quad \forall t \qquad \textbf{(BC's)}$$

$$u(x,0)=f(x) \text{ and } \frac{\partial u}{\partial t}(\mathrm{x},0)=\mathrm{g(x)} \qquad \textbf{(IC's)}$$



Basic idea:

1. Apply the method of separation to obtain two *ordinary* DE's
2. Determine the solutions that satisfy the bc's.
3. Use Fourier series to superimpose the solutions to get final solution that satisfies both the wave equation and the initial conditions.

Separation of Variables-PDE

We seek a solution of the form

$$u(x,t) = \mathrm{X}(x)\mathrm{T}(t)$$

Differentiating, we get

$$\frac{\partial u}{\partial t} = \mathrm{X}(x)\dot{\mathrm{T}}(t) \qquad \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial^2 u}{\partial t^2} = \mathrm{X}(x)\ddot{\mathrm{T}}(t)$$

and

$$\frac{\partial u}{\partial x} = \mathrm{X}'(x)\mathrm{T}(t) \qquad \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = \mathrm{X}''(x)\mathrm{T}(t)$$

Thus the wave equation becomes

$$\mathrm{X}''(x)\mathrm{T}(t) = \frac{1}{c^2}\mathrm{X}(x)\ddot{\mathrm{T}}(t),$$

dividing by the product $\mathrm{X}(x)\mathrm{T}(t)$

$$\frac{\mathrm{X}''}{\mathrm{X}} = \frac{\ddot{\mathrm{T}}}{c^2\mathrm{T}}$$

Separation of Variables-PDE

$$\frac{X''}{X} = \frac{\ddot{T}}{c^2 T} = \textit{constant} = c$$

$$X'' = cX$$

$$\ddot{T} = cT$$

We allow the constant to take any value and then show that only certain values are allowed to satisfy the boundary conditions. We consider the three possible cases for *c*, namely $c = p^2$ positive, $c = 0$, and $c = -p^2$. These give us three distinct types of solution that are restricted by the initial and boundary conditions.

With $c = 0$

$$X'' = 0 \qquad \Rightarrow X(x) = Ax + B$$

$$\ddot{T} = 0 \qquad \Rightarrow T(t) = Dt + E$$

Separation of Variables-PDE

$c = p^2$

$$X'' - p^2 X = 0$$

$$\ddot{T} - c^2 p^2 T = 0$$

$$X(x) = e^{\lambda x}, \qquad \Rightarrow X''(x) = \lambda^2 e^{\lambda x} = \lambda^2 X(x)$$

$$\lambda^2 X - p^2 X = 0, \qquad \Rightarrow \lambda^2 = p^2 \qquad \Rightarrow \lambda = \pm p$$

Solution:

$$X(x) = Ae^{px} + Be^{-px}$$

BC's in x⇒A=0, B=0. Trivial solution

Separation of Variables-PDE

$$c = -p^2$$

$$X'' + p^2 X = 0$$

$$\ddot{T} + c^2 p^2 T = 0$$

$$X(x) = e^{\lambda x}$$

where $\lambda^2 = -p^2 \quad \Rightarrow \lambda = \pm ip$

Thus the solution is $\quad X(x) = A\cos px + B\sin px$

BC at $x = 0 \Rightarrow A = 0,$ *at* $x = L \;\; X(L) = B\sin pL$

if $B = 0$*, we have the trivial solution. Non-trivial solution* $\Rightarrow \sin pL = 0$

$\Rightarrow pL = n\pi,\; n$ *is an integer.*

Separation of Variables-PDE

Similarly;

$$\mathrm{T}(t) = D\cos pct + E\sin pct$$

$p = n\pi/L$. *Thus, a solution for* $u(x,t)$ *is*

$$u(x,t) = A\sin\frac{n\pi}{L}x\left(D\cos\frac{n\pi c}{L}t + E\sin\frac{n\pi c}{L}t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi}{L}x\left(D_n\cos\frac{n\pi c}{L}t + E_n\sin\frac{n\pi c}{L}t\right)$$

We can set $A=1$ without any loss of generality.

Separation of Variables-PDE

Applying IC's. Setting t=0.

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{L}x$$

since $\sin(0) = 0$ *and* $\cos(0) = 1,$

$$f(x) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{L}x$$

To determine the constants, D_n, we multiply both sides of the equation by $\sin\frac{m\pi}{L}x$ and integrate from *x=0 to x=L.*

$$\int_0^L f(x)\sin\frac{m\pi}{L}x\,dx = \int_0^L\left(\sum_{n=1}^{\infty} D_n \sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\right)dx.$$

$$\int_0^L f(x)\sin\frac{m\pi}{L}x\,dx = \sum_{n=1}^{\infty}\left(\int_0^L D_n \sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\right)dx$$

Separation of Variables-PDE

Using orthogonality condition:

$$\int_0^L f(x)\sin\frac{m\pi}{L}x\,dx = D_m\frac{L}{2}.$$

Replacing m by n:

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx.$$

the other IC requires the time derivative of $u(x,t)$.

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{n\pi c}{L}\sin\frac{m\pi}{L}x\left(E_n\cos\frac{n\pi c}{L}t - D_n\sin\frac{n\pi c}{L}t\right).$$

at $t = 0$,

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}E_n\sin\frac{n\pi}{L}x.$$

Separation of Variables-PDE

using IC,

$$g(x)=\sum_{n=1}^{\infty}\frac{n\pi c}{L}E_n\sin\frac{n\pi}{L}x.$$

Repeat the same procedure

$$\int_0^L g(x)\sin\frac{m\pi}{L}x\,dx=\frac{m\pi c}{L}E_m\frac{L}{2}.$$

$$\Rightarrow E_n=\frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi}{L}x\left(D_n\cos\frac{n\pi c}{L}t+E_n\sin\frac{n\pi c}{L}t\right).$$

Fourier series

$$A_0 + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx)).$$

A Fourier polynomial is an expression of the form

$$F_n(x) = a_0 + (a_1 \cos(x) + b_1 \sin(x)) + \ldots + (a_n \cos(nx) + b_n \sin(nx))$$

Which may be written as

$$F_n = a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)).$$

The constants $a_0, a_i \; and \; b_i, i = 1, \ldots, n,$ are called the coefficients of $F_n(x)$.

Fourier series

The Fourier polynomials are 2 π-periodic functions.

$$F_n = a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)).$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x)dx,$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F_n(x)\cos(kx)dx, 1 \le k \le n$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F_n(x)\sin(kx)dx, 1 \le k \le n$$

Fourier series

Example

Find the Fourier series of the function $f(x) = x, \ -\pi \le x \le \pi.$

Since $f(x)$ is odd, then $a_n = 0, for\ n \ge 0.$ For any $n \ge 1,$ we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)dx = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n = -\frac{2}{n}\cos(n\pi) = \frac{2}{n}(-1)^{n+1}.$$

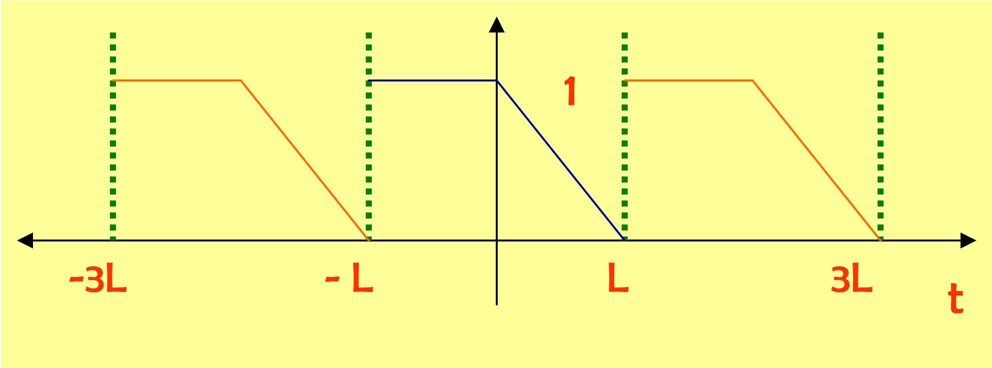
$$Hence \quad f(x) \sim 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3}\cdots\right).$$

Fourier series

Example

Find the Fourier series of the function with period 2L defined by

$$f(t)=\begin{cases} 1 & -L<t<0 \\ 1-\dfrac{t}{L} & 0<t<L \end{cases}$$



$$T=2L,\ \omega=\frac{2\pi}{T}=\frac{\pi}{L}$$

Fourier series given by

$$f(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega t)+bn\sin(n\omega t)$$

Fourier series

Coefficients found by evaluating

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt\,,\ b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt$$

Calculating

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{1}{L}\left\{\int_{-L}^{0}\cos\left(\frac{n\pi t}{L}\right)dt + \int_{0}^{L}\left(1-\frac{t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)dt\right\}$$

$$= \frac{1}{L}\left\{\left[\frac{L}{n\pi}\sin\left(\frac{n\pi t}{L}\right)\right]_{-L}^{0} + \left[\left(1-\frac{t}{L}\right)\frac{L}{n\pi}\sin\left(\frac{n\pi t}{L}\right)\right]_{0}^{L} + \int_{0}^{L}\frac{1}{L}\frac{L}{n\pi}\sin\left(\frac{n\pi t}{L}\right)dt\right\}$$

Fourier series

$$a_n = \frac{1}{n\pi L}\int_0^L \sin\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{1}{n\pi L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi t}{L}\right)\right]_0^L$$

$$= \frac{1-\cos(n\pi)}{n^2\pi^2} = \frac{1-(-1)^n}{n^2\pi^2}$$

$$\therefore\ a_n = 0\ \ if\ \ n\ is\ even,\ \ a_n = \frac{2}{n^2\pi^2}\, if\ \ n\ is\ odd,$$

$$\Rightarrow a_{2m} = 0,\ a_{2m+1} = \frac{2}{(2m+1)^2\pi^2}$$

Fourier series

Calculate a_0

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$= \frac{1}{L}\left\{\int_{-L}^{0} 1dt + \int_{0}^{L} 1 - \frac{t}{L}dt\right\}$$

$$= \frac{1}{L}\left\{\left[t\right]_{-L}^{0} + \left[t - \frac{t^2}{2L}\right]_{0}^{L}\right\}$$

$$= \frac{1}{L}\left\{L + L - \frac{L^2}{2L}\right\} = \frac{3}{2}$$

$$\therefore a_0 = 3/2$$

Fourier series

Calculating b_n

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{1}{L}\left\{\int_{-L}^{0}\sin\left(\frac{n\pi t}{L}\right)dt + \int_{0}^{L}\left(1-\frac{t}{L}\right)\sin\left(\frac{n\pi t}{L}\right)dt\right\}$$

$$= \frac{1}{L}\left\{\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi t}{L}\right)\right]_{-L}^{0} - \left[\left(1-\frac{t}{L}\right)\frac{L}{n\pi}\cos\left(\frac{n\pi t}{L}\right)\right]_{0}^{L} - \int_{0}^{L}\frac{1}{L}\frac{L}{n\pi}\cos\left(\frac{n\pi t}{L}\right)dt\right\}$$

$$= \frac{\cos(n\pi)-1}{n\pi} + \frac{1}{n\pi} - \frac{1}{n\pi L}\int_{0}^{L}\cos\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{(-1)^n}{n\pi} - \frac{1}{n\pi L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi t}{L}\right)\right]_{0}^{L} = \frac{(-1)^n}{n\pi} \qquad \therefore b_n = \frac{(-1)^n}{n\pi}$$

Fourier series

We now know that

$$a_{2n} = 0, \qquad a_{2n+1} = \frac{2}{(2n+1)^2 \pi^2} \qquad n = 1,2,3,\ldots$$

$$a_0 = \frac{3}{2}$$

$$b_n = \frac{(-1)^n}{n\pi} \qquad n = 1,2,3,\ldots$$

$$f(t) \sim \frac{3}{4} + \sum_{n=1}^{\infty} \frac{2}{(2n+1)^2 \pi^2} \cos\left(\frac{(2n+1)\pi t}{L}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin\left(\frac{n\pi t}{L}\right)$$

Fourier transform

The continuous time Fourier transform of $x(t)$ is defined as

$$\chi(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt,$$

and the inverse transform is defined as

$$x(t) = \int_{-\infty}^{\infty} \chi(f) e^{i2\pi f t} df$$

A common notation is to define the Fourier transform in terms of $i\omega$ as

$$X(i\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt,$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(i\omega) e^{i\omega t} d\omega$$

Fourier transform properties symmetry

$$\chi(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$$

$$\chi(f) = \int_{-\infty}^{\infty} (x_e(t) + x_o(t))(\cos(2\pi ft) - i\sin(2\pi ft))dt.$$

The odd components of the integrand contribute zero to the integral. Hence

$$\chi(f) = \int_{-\infty}^{\infty} x_e(t)(\cos(2\pi ft) + i\int_{-\infty}^{\infty} - x_o(t)\sin(2\pi ft)dt,$$

$$\chi(f) = \chi_r(f) + i\chi_i(f),$$

where

$$\chi_r(f) = \int_{-\infty}^{\infty} x_e(t)\cos(2\pi ft)dt,$$

$$\chi_i(f) = -\int_{-\infty}^{\infty} x_o(t)\sin(2\pi ft)dt.$$

Odd and Even Functions

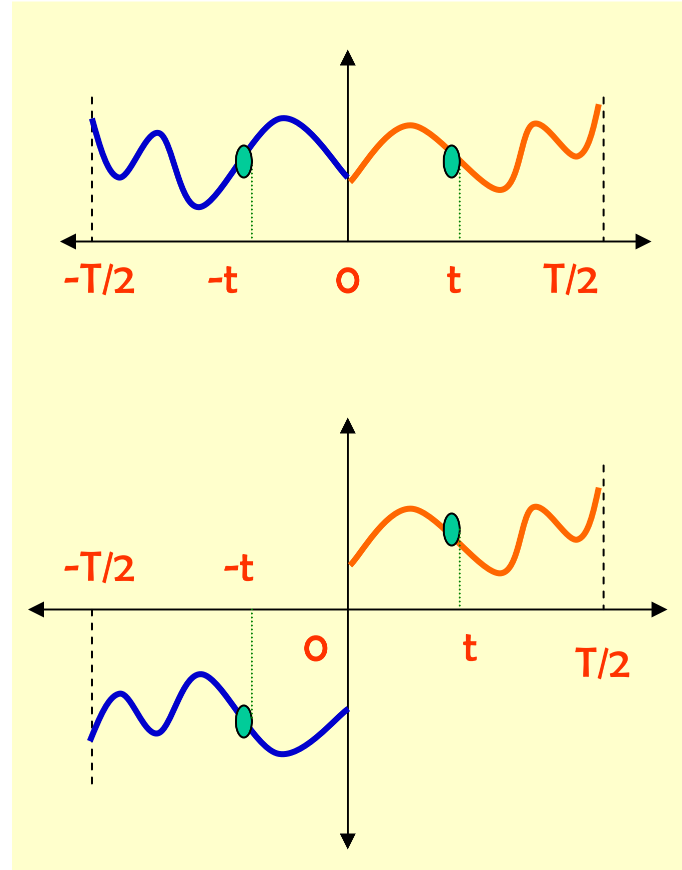
Even	Odd
$f(-t) = f(t)$	$f(-t) = -f(-t)$
Symmetric	Anti-symmetric
Cosines	Sines
Transform is real* imaginary	Transform is

*for real-valued signals

- Important property of even and odd functions for any L,

$$\int_{-L}^{L} f(t)dt = 2\int_{0}^{L} f(t)dt \qquad \textit{If f is even}$$

$$\int_{-L}^{L} f(t)dt = 0 \qquad \textit{If f is odd}$$



Complex form of the Fourier series

Recall that

$$e^{-i\theta} = \cos\theta + i\sin\theta \xrightarrow{\text{Complex conjugate}} e^{i\theta} = \cos\theta - i\sin\theta$$

This gives

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad \text{and} \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Hence

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right) \quad \text{and} \quad \sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$$

Complex form of the Fourier series

Now consider the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{inx} + e^{-inx}\right) + \frac{b_n}{2i}\left(e^{inx} - e^{-inx}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1}{2}\left(a_n - ib_n\right)e^{inx} + \frac{1}{2}\left(a_n + ib_n\right)e^{-inx}$$

$$\Rightarrow f(x) = c_0 + \sum_{n=1}^{\infty} c_n\left(e^{inx} + k_n e^{-inx}\right)$$

$$c_n = \frac{1}{2}\left(a_n - ib_n\right), \quad k_n = c_n^*$$

Complex form of the Fourier series

Remembering that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Hence

$$c_n = \frac{1}{2}\left\{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx - \frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx\right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx$$

Similarly

$$k_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\, dx$$

Complex form of the Fourier series

and note that $k_n = c_{-n}$. Then we have

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} c_{-n} e^{-inx} = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=-1}^{-\infty} c_n e^{inx}$$

Finally noting that $e^{i(0)x} = 1$ we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{2} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

This the ***complex form*** of the Fourier series for $f(x)$. c_n are the complex Fourier coefficients.

Complex form of the Fourier series

Example: Find the general solution to

$$y'' + \omega^2 y = r(t)$$

where

$$r(t) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} sin(2n-1)t$$

We have

$$y'' + \omega^2 y = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} sin(2n-1)t$$

Consider the equation

$$y_n'' + \omega^2 y_n = \frac{1}{n^2} sin\, nt \qquad (n = 1,3,5,\ldots)$$

We find the general solution to this equation.

Complex form of the Fourier series

The general solution of the homogeneous form is

$$y_h = A\cos\omega t + B\sin\omega t$$

For a particular solution try $\quad y_n = A_n\cos nt + B_n\sin nt$

Differentiating and substituting gives

$$\left(-n^2+\omega^2\right)A_n\cos nt + \left(-n^2+\omega^2\right)B_n\sin nt = \frac{1}{n^2}\sin nt$$

(assuming $\omega \neq n$ for n odd) we have

$$A_n = 0, \quad B_n = \frac{1}{n^2\left(\omega^2 - n^2\right)}$$

Thus the particular solution is

$$y_n = \frac{1}{n^2\left(\omega^2 - n^2\right)}\sin nt$$

Complex form of the Fourier series

Since $$y'' + \omega^2 y = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} sin(2n-1)t$$ is linear, the general

solution is a superposition

$$y_1 + y_3 + y_5 + \ldots. + y_h$$

Therefore

$$y = A\cos\omega t + B\sin\omega t + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2\left[\omega^2 - (2n-1)^2\right]} sin(2n-1)t.$$

Convolution Theorem

Let *F*, *G*, *H* denote the Fourier Transforms of signals *f*, *g*, and *h* respectively.

$g = f*h$	$g = fh$
implies	implies
$G = FH$	$G = F*H$

Convolution in one domain is multiplication in the other and vice versa.

$$\Im(f(t) * g(t)) = \Im(f(t))\Im(g(t))$$
$$\Im(f(t)g(t) = \Im(f(t)) * \Im(g(t))$$

Convolution

$$\Im(f(t) * g(t)) = \Im\left(\int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau\right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau \, e^{-i2\pi\omega t} dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-\tau) g(\tau) e^{-i2\pi\omega t} d\tau dt$$

$$\Im(f(t) * g(t)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-\tau) g(\tau) e^{-i2\pi\omega t} d\tau dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) g(\tau) e^{-i2\pi\omega (u+\tau)} d\tau du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-i2\pi\omega u} g(\tau) e^{-i2\pi\omega\tau} d\tau du$$

$$= \int_{-\infty}^{\infty} f(u) e^{-i2\pi\omega u} du \int_{-\infty}^{\infty} g(\tau) e^{-i2\pi\omega\tau} d\tau$$

$$\Im(f(t) * g(t)) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\omega t} dt \int_{-\infty}^{\infty} g(t) e^{-i2\pi\omega t} dt$$

$$= \Im(f(t)) \Im(g(t))$$

Convolution

$$\Im(f(t) * g(t)) = \Im(f(t))\,\Im(g(t))$$

$$\Im(f(t)g(t)) = \Im(f(t)) * \Im(g(t))$$

Green's functions

Consider a general linear operator L

If on the closed interval $a \le x \le b$ we have a two point boundary problem for a general linear differential equation of the form:

$$Ly = f(x),$$

Where the highest derivative in L is order n and with general homogeneous boundary conditions at x=a and x=b on linear combinations of y and n-1 of its derivatives:

$$A\left(y(a), y'(a), \ldots, y^{(n-1)}(a)\right)^T + B\left(y(b), y'(b), \ldots, y^{(n-1)}(b)\right)^T = 0$$

Where A and B are $n \times n$ constant coefficient matrices, then knowing L, A and B, we can form a solution of the form:

$$y(x) = \int_a^b f(s) g(x,s)\, ds$$

Green's functions

This is desirable as

• once $g(x, s)$ is known, the solution is defined for all f including

- forms of *f* for which no simple explicit integrals can be written
- piecewise continuous forms of *f*

• numerical solution of the quadrature problem is more robust than direct numerical solution of the original differential equation

•The solution will automatically satisfy all boundary conditions

•The solution is useful in experiments in which the system dynamics are well characterized (e.g. mass spring damper).

Green's functions

We take g(x, s) to be the Green's function for the linear differential operator L if it satisfies the following conditions:

1. L $g(x, s)=\delta(x-s)$
2. $g(x, s)$ satisfies all boundary conditions given on x
3. $g(x, s)$ is a solution of L g=0 on $a \leq x < s$ and $s < x \leq b$
4. $g(x, s), g'(x, s), \ldots, g^{(n-2)}(x, s)$ are continuous for $[a, b]$
5. $g^{(n-1)}(x, s)$ is continuous for $[a, b]$ except at $x=s$ where it has a jump of $\dfrac{-1}{P_n(s)}$

Green's functions

Consider:

$$L = P_2(x)\frac{d^2}{dx^2} + P_1(x)\frac{d}{dx} + P_o(x)$$

Then we have

$$P_2(x)\frac{d^2 g}{dx^2} + P_1(x)\frac{dg}{dx} + P_o(x)g = \delta(x-s)$$

$$\frac{d^2 g}{dx^2} + \frac{P_1(x)}{P_2(x)}\frac{dg}{dx} + \frac{P_o(x)}{P_2(x)}g = \frac{\delta(x-s)}{P_2(x)}$$

Now we integrate both sides with respect to *x* in a small neighborhood enveloping *x*=*s.*

$$\int_{s-\varepsilon}^{s+\varepsilon}\frac{d^2 g}{dx^2}dx + \int_{s-\varepsilon}^{s+\varepsilon}\frac{P_1(x)}{P_2(x)}\frac{dg}{dx}dx + \int_{s-\varepsilon}^{s+\varepsilon}\frac{P_o(x)}{P_2(x)}g\,dx = \int_{s-\varepsilon}^{s+\varepsilon}\frac{\delta(x-s)}{P_2(x)}dx$$

Green's functions

$$\int_{s-\varepsilon}^{s+\varepsilon} \frac{d^2 g}{dx^2} dx + \frac{P_1(x)}{P_2(x)} \int_{s-\varepsilon}^{s+\varepsilon} \frac{dg}{dx} dx + \frac{P_o(x)}{P_2(x)} \int_{s-\varepsilon}^{s+\varepsilon} g \, dx = \frac{1}{P_2(x)} \int_{s-\varepsilon}^{s+\varepsilon} \delta(x-s) dx$$

Integrating

$$\left.\frac{dg}{dx}\right|_{s+\varepsilon} - \left.\frac{dg}{dx}\right|_{s-\varepsilon} + \frac{P_1(s)}{P_2(s)} \left(g|_{s+\varepsilon} - g|_{s-\varepsilon} \right) + \frac{P_o(s)}{P_2(s)} \int_{s-\varepsilon}^{s+\varepsilon} g \, dx = \frac{1}{P_2(s)} H(x-s)\Big|_{s-\varepsilon}^{s+\varepsilon}$$

Since *g* is continuous, this reduces to

$$\left.\frac{dg}{dx}\right|_{s+\varepsilon} - \left.\frac{dg}{dx}\right|_{s-\varepsilon} = \frac{1}{P_2(s)}$$

This is consistent with the final point that the second highest derivative of *g* suffers a jump at *x*=*s.*

Green's functions

Next, we show that applying this definition of *g(x, s)* to our desired result lets us recover the original differential equation, rendering *g(x, s)* to be appropriately defined. This can be easily shown by direct substitution:

$$y(x) = \int_a^b f(s) g(x,s)\, ds$$

$$L_y = L \int_a^b f(s) g(x,s)\, ds$$

L behaves as $\dfrac{\partial^n}{\partial x^n}$, via Leibritz's Rule:

$$= \int_a^b f(s) L g(x,s)\, ds$$

$$= \int_a^b f(s) \delta(x-s)\, ds$$

$$= f(x)$$

This analysis can be extended in a straightforward manner to more arbitrary systems with inhomogeneous boundary conditions using matrix methods

Green's functions

Example: Find the Green's function and the corresponding solution integral of the differential equation

$$\frac{d^2 y}{dx^2} = f(x)$$

subject to boundary conditions

$$y(0) = 0, \qquad y(1) = 0$$

Verify the solution integral if *f (x)=6x*

Here $$L = \frac{d^2}{dx^2}$$

1) Break the problem up into two domains: a) x<s, b) x>s, 2) Solve Lg=0 in both domains, four constraints arise, 3) Use boundary conditions for two constants, 4) Use conditions at x-s: continuity of g and a jump of *dg/dx* , for the other two constants.

Green's functions

Example:

a) x<a

$$\frac{d^2 g}{dx^2} = 0$$

$$\frac{dg}{dx} = C_1$$

$$g = C_1 x + C_2$$

$$g(0) = 0 = C_1(0) + C_2$$

$$C_2 = 0$$

$$g(x,s) = C_1 x, \quad x < s$$

Green's functions

Example:

b) x> s

$$\frac{d^2 g}{dx^2} = 0$$

$$\frac{dg}{dx} = C_3$$

$$g = C_3 x + C_4$$

$$g(1) = 0 = C_3(1) + C_4$$

$$C_4 = -C_3$$

$$g(x,s) = C_3(x-1), \quad x > s$$

Continuity of g(x, s) when x=s:

$$C_1 s = C_3(s-1)$$

$$C_1 = C_3 \frac{s-1}{s}$$

$$g(x,s) = C_3 \frac{s-1}{s} x, \quad x < s$$

$$g(x,s) = C_3(x-1), \quad x > s$$

Green's functions

Example:

b) x> s

Jumping in *dg/dx* at x=s (note $P_2(x)=1$):

$$\left.\frac{dg}{dx}\right|_{s+\varepsilon} - \left.\frac{dg}{dx}\right|_{s-\varepsilon} = 1$$

$$C_3 - C_3 \frac{s-1}{s} = 1$$

$$C_3 = s$$

$$g(x,s) = x(s-1), \quad x < s$$

$$g(x,s) = s(x-1), \quad x > s$$

Green's functions

Note some properties of g(x, s) which are common in such problems:

- It is broken into two domains
- It is continuous in and through both domains
- Its n-1 (here n=2), so first) derivative is discontinuous at x=s
- It is symmetric in s and x across the two domains
- It is seen by inspection to satisfy both boundary conditions

The general solution in integral form can be written by breaking the integral into two pieces as

$$y(x) = \int_0^x f(s)s(x-1)ds + \int_x^1 f(s)x(s-1)ds$$

$$y(x) = (x-1)\int_0^x f(s)sds + x\int_x^1 f(s)(s-1)ds$$

Green's functions

Now evaluate the integral if f(x)=6x (thus f (s)=6s).

$$y(x) = (x-1)\int_0^x (6s)s\,ds + x\int_x^1 (6s)(s-1)\,ds$$

$$= (x-1)\int_0^x \left(6s^2\right)ds + x\int_x^1 \left(6s^2 - 6s\right)ds$$

$$= (x-1)\left[2s^3\right]_0^x + x\left[2s^3 - 3s^2\right]_x^1$$

$$= (x-1)\left(2x^3 - 0\right) = x\left[(2-3) - \left(2x^3 - 3x^2\right)\right]$$

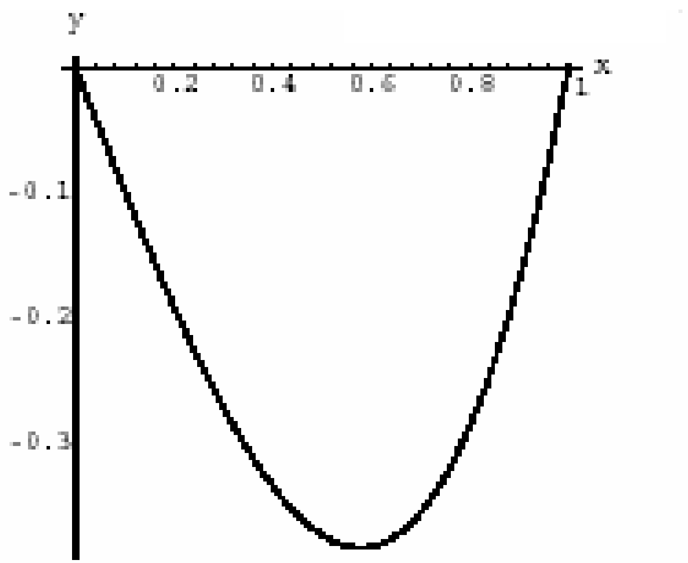
$$= 2x^4 - 2x^3 - x - 2x^4 + 3x^3$$

$$y(x) = x^3 - x$$

Green's functions

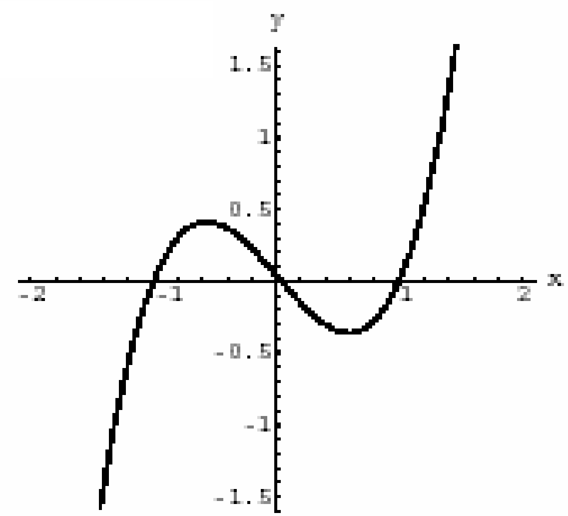
Note the original differential equation and both boundary conditions are automatically satisfied by the solution.

$$y'' = 6x, \quad y(0) = 0, y(1) = 0$$



$y(x) = x^3 - x$

in domain of interest $0 < x < 1$



$y(x) = x^3 - x$

in expanded domain, $-2 < x < 2$

Bessel equation

Bessel's differential equation is as follows, with it being convenient to define $\lambda = -\nu^2$.

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(\mu^2 x^2 - \nu^2\right) y = 0$$

We find that

$$p(x) = x$$

$$r(x) = \frac{1}{x}$$

$$q(x) = \mu^2 x$$

$$a(x)\frac{d^2 y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y + \lambda y = 0$$

$$\alpha_1 y(a) + \alpha_2 y(a) = 0$$

$$\beta_1 y(b) + \beta_2 y(b) = 0$$

Linear homogeneous second order D.E with general homogeneous b.c.

Sturm-Liouville

Define the following functions:

$$p(x) = exp\left(\int \frac{b(s)}{a(s)} ds \right)$$

$$r(x) = \frac{1}{a(x)} exp\left(\int \frac{b(s)}{a(s)} ds \right)$$

$$q(x) = \frac{c(x)}{a(x)} exp\left(\int \frac{b(s)}{a(s)} ds \right)$$

With these definitions, the original equations are transformed to the type known as a Sturm-Liouville equation:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx} \right] + \left[q(x) + \lambda r(x) \right] y(x) = 0$$

$$\left[\frac{1}{r(x)} \left(\frac{d}{dx}\left(p(x)\frac{d}{dx} \right) + q(x) \right) \right] y(x) = -\lambda y(x)$$

Sturm-Liouville

Here the Sturm-Liouville linear operator $\boldsymbol{L_s}$ is

$$L_s = \frac{1}{r(x)}\left(\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right)$$

So we have $$L_s y(x) = -\lambda y$$

We thus require 0<x<∞, though in practice, it is more common to employ a finite domain such as 0<x<*l*. In the Sturm-Liouville form, we have

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\left[\mu^2 x-\frac{\nu^2}{x}\right]y(x)=0$$

$$\left[x\left(\frac{d}{dx}\left(x\frac{d}{dx}\right)+\mu^2 x\right)\right]y(x)=\nu^2 y(x)$$

Sturm-Liouville

The Sturm-Liouville linear operator is

$$L_s = x\left(\frac{d}{dx}\left(x\frac{d}{dx}\right) + \mu^2 x\right)$$

In some other cases it is more convenient to take $\lambda = \mu^2$ in which case we get

$$p(x) = x$$

$$r(x) = x$$

$$q(x) = -\frac{\nu^2}{x}$$

and the Sturm-Liouville form and operator are:

$$\left[\frac{1}{x}\left(\frac{d}{dx}\left(x\frac{d}{dx}\right) - \frac{\nu^2}{x}\right)\right] y(x) = -\mu^2 y(x)$$

$$L_s = \frac{1}{x}\left(\frac{d}{dx}\left(x\frac{d}{dx}\right) - \frac{\nu^2}{x}\right)$$

Bessel equation

The general solution is

$$y(x) = C_1 J_\nu(\mu x) + C_2 Y_\nu(\mu x) \qquad \text{if } \nu \text{ is an integer}$$

$$y(x) = C_1 J_\nu(\mu x) + C_2 J_{-\nu}(\mu x) \qquad \text{if } \nu \text{ is not an integer}$$

Where $J_\nu(\mu x)$ and $Y_\nu(\mu x)$ are called the Bessel and Neumann functions of order ν. Often *$J_\nu(\mu x)$ is known as a Bessel function of the first kind* and *$Y_\nu(\mu x)$ is known as a Bessel function of the second kind.* Both J_ν and Y_ν are represented by infinite *series* rather than finite series such as the series for Legendre polynomials.

The Bessel function of the first kind of order ν, $J_\nu(\mu x)$, is represented by

$$J_\nu(\mu x) = \left(\frac{1}{2}\mu x\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}\mu^2 x^2\right)^k}{k!\,\Gamma(\nu + k + 1)}$$

Bessel equation

The Neumann function $Y_\nu(\mu x)$ has a complicated representation. The representations for $J_0(\mu x)$ and $Y_0(\mu x)$ are

$$J_o(\mu x) = 1 - \frac{\left(\frac{1}{4}\mu^2 x^2\right)^1}{(1!)^2} + \frac{\left(\frac{1}{4}\mu^2 x^2\right)^2}{(2!)^2} + \ldots + \frac{\left(-\frac{1}{4}\mu^2 x^2\right)^n}{(n!)^2}$$

$$Y_o(\mu x) = \frac{2}{\pi}\left(ln\left(\frac{1}{2}\mu x\right) + \gamma\right) J_0(\mu x)$$

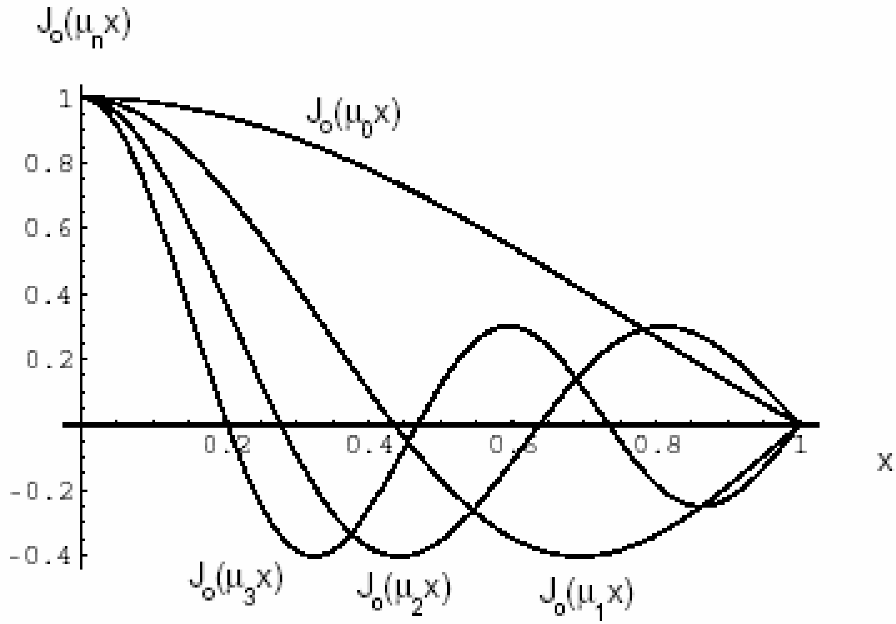
$$+ \frac{2}{\pi}\left(\frac{\left(\frac{1}{4}\mu^2 x^2\right)^1}{(1!)^2} - \left(1 + \frac{1}{2}\right)\frac{\left(\frac{1}{4}\mu^2 x^2\right)^2}{(2!)^2} \ldots\right)$$

It can be shown using term by term differentiation that

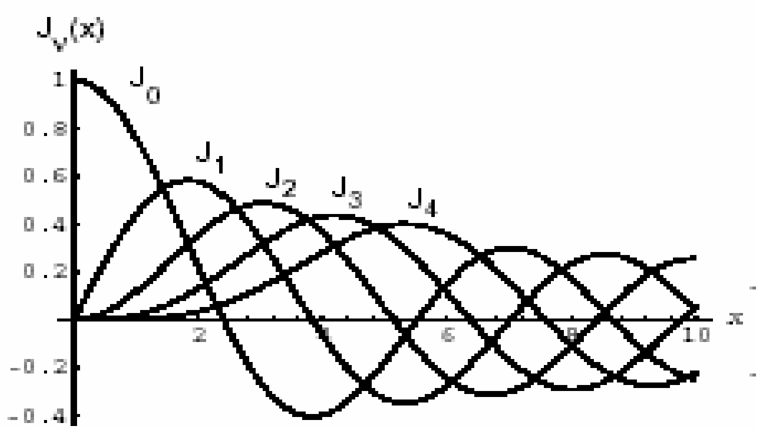
$$\frac{dJ_\nu(\mu x)}{dx} = \mu\frac{J_{\nu+1}(\mu x) - J_{\nu-1}(\mu x)}{2} \qquad \frac{dY_\nu(\mu x)}{dx} = \mu\frac{Y_{\nu+1}(\mu x) - Y_{\nu-1}(\mu x)}{2}$$

$$\frac{d}{dx}\left[x^\mu J_\nu(\mu x)\right] = \mu x^\nu J_{\nu-1}(x) \qquad \frac{d}{dx}\left[x^\mu Y_\nu(\mu x)\right] = \mu x^\nu Y_{\nu-1}(x)$$

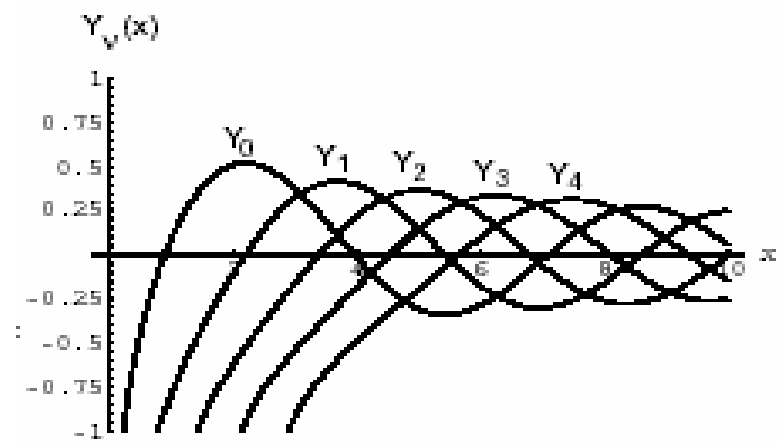
Bessel equation



Bessel functions $J_0(\mu_0 x)$, $J_0(\mu_1 x)$, $J_0(\mu_2 x)$, $J_0(\mu_3 x)$



Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, $J_3(x)$, $J_4(x)$



Neumann functions $Y_0(x)$, $Y_1(x)$, $Y_2(x)$, $Y_3(x)$, $Y_4(x)$

Bessel equation

The orthogonality condition for a domain $x \in [0,1]$, taken here for the case in which the eigenvalue is μ_l can be shown to be

$$\int_0^1 x\, J_\nu(\mu_i x) J_\nu(\mu_j x) dx = 0 \quad i \neq j$$

$$\int_0^1 x\, J_\nu(\mu_n x) J_\nu(\mu_n x) dx = \frac{1}{2}\left(J_{\nu+1}(\mu_n)\right)^2 \quad i = j$$

Here we must choose μ_i such that $J_\nu(\mu_i)=0$, which corresponds to a vanishing of the function at the outer limit $x = 1$. So the orthogonal Bessel function is

$$\varphi_n(x) = \frac{\sqrt{2x}\, J_\nu(\mu_n x)}{\left|J_{\nu+1}(\mu_n)\right|}$$

Bessel equation

Hankel functions, also known as *Bessel functions of the third kind* are defined

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$$

The modified Bessel equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - \left(x^2 + \nu^2\right) y = 0$$

The solution of which are the modified Bessel functions. It is satisfied by the modified Bessel functions. *The modified Bessel functions of the first kind* of order ν is

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

The modified Bessel functions of the second kind of order ν is

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_n^{(1)}(ix)$$

Vectors and Tensors

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3 = \sum_{i=1}^{3} u_i e_i = u_i e_i = u_i$$

using Einstein notation

Here u_1, u_2, u_3 are three Cartesian components of u.

Two additional symbols:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Kronecker delta

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if indices are in cyclical order } 1,2,3,1,2,\ldots \\ -1 & \text{if indices are not in cyclical order} \\ 0 & \text{if two or more indices are the same} \end{cases}$$

Levi-Civita density

Vectors and Tensors

The identity

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl}$$

relates the two.

We also have the following identities:

$$\delta_{ii} = 3$$

$$\delta_{ij} = \delta_{ji}$$

$$\delta_{ij}\delta_{ik} = \delta_{jk}$$

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

$$\varepsilon_{ijk}\varepsilon_{ljk} = 2\delta_{il}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$

$$\varepsilon_{ijk} = -\varepsilon_{jik}$$

$$\varepsilon_{ijk} = -\varepsilon_{kji}$$

$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$$

Vectors and Tensors

Regarding index notation:

- repeated index indicates summation on that index
- non-repeated index is known as free index
- number of free indices give the order of the tensor
  - u, uv, $u_i v_i w$, u_{ii}, $u_{ij} v_{ij}$ zeroth order tensor-scalar
  - u_i, $u_i v_{ij}$ second order tensor
  - u_{ijk}, $u_i v_j w_k$, $u_{ij} v_{km} w_m$ third order tensor
  - u_{ijkl}, $u_{ij} v_{kl}$ fourth order tensor
- indices cannot be repeated more than once
  - u_{iik}, u_{ij}, u_{iijj}, $v_i u_{jk}$ are proper
  - $u_i v_i w_i$, u_{iiij}, $u_{ij} v_{ii}$ are improper
- Cartesian components commute: $u_{ij} v_i w_{klm} = v_i w_{klm} u_{ij}$
- Cartesian indices do not commute: $u_{ijkl} \neq u_{jlik}$

Vectors and Tensors

Matrix representation: Tensors can be represented as matrices (but all matrices are not tensors!):

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

A simple way to choose a vector q_j associated with a plane of arbitrary orientation is to form the inner product of the tensor T_{ij} and the unit normal associated with the plane n_i:

$$q_j = n_i T_{ij} \qquad q = n \cdot T$$

Here n_i has components which are the direction cosines of the chosen direction.

Example: $n_i = (0,1,0)$

$$n \cdot T = \begin{pmatrix} 0, & 1, & 0 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} T_{21}, & T_{22}, & T_{23} \end{pmatrix}$$

$$\begin{aligned} n_i T_{ij} &= n_1 T_{1j} + n_2 T_{2j} + n_3 T_{3j} \\ &= (0) T_{1j} + (1) T_{2j} + (0) T_{3j} \\ &= (T_{21}, T_{22}, T_{23}) \end{aligned}$$

Vectors and Tensors

The *transpose* T_{ij}^{T} of T_{ij} is found by trading elements across the diagonal

$$T_{ij}^{T} = T_{ji}$$

$$T_{ij}^{T} = \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}$$

A tensor is *symmetric* if it is equal to its transpose, i.e.

$$T_{ij} = T_{ji}$$

A tensor is *anti-symmetric* if it is equal to the additive inverse of its transpose, i.e.

$$T_{ij} = -T_{ji}$$

The inner product of a *symmetric tensor* S_{ij} and *anti-symmetric tensor* A_{ij} can be shown to 0:

$$S_{ij} A_{ij} = 0$$

Vectors and Tensors

Example: Show this for a two-dimensional space. Take a general symmetric tensor to be

$$S_{ij} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Taking a general anti-symmetric tensor to be

$$A_{ij} = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$$

So

$$\begin{aligned} S_{ij} A_{ij} &= S_{11} A_{11} + S_{12} A_{12} + S_{21} A_{21} + S_{22} A_{22} \\ &= a(0) + bd - bd + c(0) \\ &= 0 \end{aligned}$$

Vectors and Tensors

An arbitrary tensor can be represented as the sum of a symmetric and anti-symmetric tensor:

$$T_{ij} = \frac{1}{2}T_{ij} + \frac{1}{2}T_{ij} + \frac{1}{2}T_{ji} - \frac{1}{2}T_{ji}$$

$$= \frac{1}{2}\left(T_{ij} + T_{ji}\right) + \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

so with

$$T_{(ij)} \equiv \frac{1}{2}\left(T_{ij} + T_{ji}\right)$$

$$T_{[ij]} \equiv \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

$$T_{ij} = \underbrace{T_{(ij)}}_{symmetric} + \underbrace{T_{[ij]}}_{anti\text{-}symmetric}$$

Vectors and Tensors

Example: Decompose the tensor given below into a combination of orthogonal basis vector and dual vector.

$$T_{ij} = \begin{pmatrix} 1 & 1 & -2 \\ 3 & 2 & -3 \\ -4 & 1 & 1 \end{pmatrix}$$

$$T_{(ij)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) = \begin{pmatrix} 1 & 1 & -3 \\ 2 & 2 & -1 \\ -3 & -1 & 1 \end{pmatrix} \quad \textbf{and} \quad T_{[ij]} = \frac{1}{2}\left(T_{ij} - T_{ji}\right) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}$$

First we get the dual vector:

$$d_i = \varepsilon_{ijk} T_{[jk]}$$

$$d_1 = \varepsilon_{1jk} T_{[jk]} = \varepsilon_{123} T_{[23]} + \varepsilon_{132} T_{[32]} = (1)(-2) + (-1)(2) = -4$$

$$d_2 = \varepsilon_{2jk} T_{[jk]} = \varepsilon_{213} T_{[13]} + \varepsilon_{231} T_{[31]} = (-1)(1) + (1)(-1) = -2$$

$$d_3 = \varepsilon_{3jk} T_{[jk]} = \varepsilon_{312} T_{[12]} + \varepsilon_{321} T_{[21]} = (1)(-1) + (-1)(1) = -2$$

Vectors and Tensors

We now find the eigenvalues and eigenvectors for the symmetric part.

$$\begin{vmatrix} 1-\lambda & 2 & -3 \\ 2 & 2-\lambda & -1 \\ -3 & -1 & 1-\lambda \end{vmatrix} = 0$$

We get the characteristic polynomial,

$$\lambda^3 - 4\lambda^2 - 9\lambda + 9 = 0$$

The eigenvalue and the associated normalized eigenvector for each root is

$$\lambda^{(1)} = 5.36488 \qquad n_i^{(1)} = \left(-0.630537 \quad -0.540358 \quad 0.557168\right)^T$$

$$\lambda^{(2)} = -2.14644 \qquad n_i^{(2)} = \left(-0.740094 \quad 0.202303 \quad -0.641353\right)^T$$

$$\lambda^{(3)} = 0.781562 \qquad n_i^{(3)} = \left(-0.233844 \quad 0.816754 \quad 0.527476\right)^T$$

It is easy to verify that each eigenvector is orthogonal. When the coordinates are transformed to be aligned with the principal axes, the magnitude of the vector with each face is the eigenvalue; this vector points in the same direction of the unit normal associated with the face.

Green's theorem

Let $\mathbf{u}=u_x\mathbf{i}+u_y\mathbf{j}$ be a vector field, ***C*** a closed curve, and ***D*** the region enclosed by ***C***, all in the x-y plane. Then

$$\oint_C u\cdot dr = \iint_D \left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)dxdy$$

Example: Show that the Green's theorem is valid if $\mathbf{u}=y\mathbf{i}+2xy\mathbf{j}$, and C consists of the straight line (0,0) to (1,0) to (1,1) to (0,0)

$$\oint_C u\cdot dr = \oint_{C1} u\cdot dr + \oint_{C2} u\cdot dr + \oint_{C3} u\cdot dr$$

Where C_1, C_2 , C_3, are the straight lines (0,0) to (1,0), (1,0) to (1,1), and (1,1) to (0,0), respectively.

$$C_1: \quad y=0, \quad dy=0, \quad x:[0,1], \quad u=0$$

$$C_2: \quad x=1, \quad dx=0, \quad y:[0,1], \quad u=yi+2yj$$

$$C_3: \quad x=y, \quad dx=dy, \quad x:[1,0], y:[1,0] \quad u=xi+2x^2j$$

Green's theorem

Thus

$$\oint_C u\cdot dr = \int_0^1 (0i+0j)\cdot(dx\, i) + \int_0^1 (yi+2yj)\cdot(dy\, j) + \int_1^0 \left(xi+2x^2\, j\right)\cdot(dx\, i+dx\, j)$$

$$= \int_0^1 2y\, dy + \int_1^0 \left(x+2x^2\right)dx$$

$$= \left[y^2\right]_0^1 + \left[\frac{1}{2}x^2+\frac{2}{3}x^3\right]_1^0 = 1-\frac{1}{2}-\frac{2}{3} = -\frac{1}{6}$$

On the other hand

$$\iint_D \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)dx\, dy = \int_0^1 \int_0^x (2y-1)dy\, dx = \int_0^1 \left(x^2 - x\right)dx$$

$$= -\frac{1}{6}$$

Gauss's theorem

Let ***S*** be a closed surface, and ***V*** the region enclosed within it, then

$$\int_S u \cdot n \, dS = \int_V \nabla \cdot u \, dV \qquad \int_S u_i n_i \, dS = \int_V \frac{\partial u_i}{\partial x_i} dV$$

Where ***dV*** an element of the volume and ***dS*** an element of the surface, and n (or n_i) an outward unit normal to it. It extends to tensors of arbitrary order:

$$\int_S T_{ijk\ldots} n_i dS = \int_V \frac{\partial T_{ijk\ldots}}{\partial x_i} dV$$

Gauss's theorem can be thought of as an extension of the familiar one-dimensional scalar result:

$$\phi(b) - \phi(a) = \int_a^b \frac{d\phi}{dx} dx$$

Here the end points play the role of the surface integral and the integral on x plays the role of the volume integral.

Gauss's theorem

Example: show that Gauss's theorem is valid if **u** = *x***i** + *y***j** and **S** is the closed surface which consists of a circular base and the hemisphere on unit radius with center at the origin and z≥0, that is $x^2 + y^2 + z^2 = 1$

In spherical coordinates, defined by

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$x = r \cos\theta$$

The hemispherical surface is described by ***r* = *1*.**

We split the surface integral into two parts

$$\int_S u \cdot n \, dS = \int_B u \cdot n \, dS + \int_H u \cdot n \, dS$$

Where ***B*** is the base and ***H*** is the curved surface of the hemisphere.

Gauss's theorem

$$\int_B u \cdot n \, dS$$ **= 0** since n=-k and ***u .n=0*** on ***B***. On ***H*** the unit normal is

$$n = \sin\theta\cos\phi i + \sin\theta\sin\phi j + (\;\;)k$$

$$u \cdot n = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi = \sin^2\theta$$

$$\int_H u \cdot n \, dS = \int_0^{2\pi}\int_0^{\pi/2} \sin^2\theta(\sin\theta \, d\theta \, d\phi) = 2\pi\int_0^{\pi/2}\left(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta\right)d\theta$$

$$= 2\pi\left(\frac{3}{4} - \frac{1}{12}\right) = \frac{4}{3}\pi$$

On the other hand if we use Gauss's theorem we find that

$$\nabla \cdot u = 2$$

$$\int_V \nabla \cdot u \, dV = \frac{4}{3}\pi$$

Since the volume of the hemisphere is $\frac{2}{3}\pi$

Green's identities

Applying Gauss's theorem to vector $u = \phi \nabla \Psi$, we get

$$\int_S \phi \nabla \Psi \cdot n \, dS = \int_V \nabla \cdot \left(\phi \nabla \Psi \right) dV$$

$$\int_S \phi \frac{\partial \Psi}{\partial x_i} n_i dS = \int_V \frac{\partial}{\partial x_i} \left(\phi \frac{\partial \Psi}{\partial x_i} \right) dV$$

From this we get Green's first identity

$$\int_S \phi \nabla \Psi \cdot n \, dS = \int_V \left(\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \right) dV$$

$$\int_S \phi \frac{\partial \Psi}{\partial x_i} n_i dS = \int_V \left(\phi \frac{\partial^2 \Psi}{\partial x_i \partial x_i} + \frac{\partial \phi}{\partial x_i} \frac{\partial \Psi}{\partial x_i} \right) dV$$

Green's identities

Interchanging ϕ and Ψ and subtracting, we get Green's second identity.

$$\int_S (\phi\nabla\Psi - \Psi\nabla\phi)\cdot n\,dS = \int_V \left(\phi\nabla^2\Psi - \Psi\nabla^2\phi\right)dV$$

$$\int_S \left(\phi\frac{\partial\Psi}{\partial x_i} - \Psi\frac{\partial\phi}{\partial x_i}\right)n_i dS = \int_V \left(\phi\frac{\partial^2\Psi}{\partial x_i\partial x_i} - \frac{\partial^2\phi}{\partial x_i\partial x_i}\right)dV$$

Stokes' theorem

Let ***S*** be an open surface, and the curve C its boundary. Then

$$\int_S (\nabla\times u)\cdot n\,dS = \oint_C u\cdot dr$$

$$\int_S \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}dS = \oint_C u_i\cdot dr_i$$

Eigenvalues and eigenvectors

If T is a linear operator, its eigenvalue problem consists of a nontrivial solution of the equation

$$Te = \lambda e$$

where e is called an eigenvector and λ is an eigenvalue.

1. **The eigenvalues of an operator and its adjoint are complex conjugates of each other.**
2. **The eigenvalues of a self-adjoint operator are real.**
3. **The eigenvectors of a self-adjoint operator corresponding to distinct eigenvalues are orthogonal.**
4. **The eigenvectors of any self-adjoint operator on vectors of a finite-dimensional vector space constitute a basis for the space.**

Eigenvalues and eigenvectors

Example: For $x \in \Re^2, A: \Re^2 \to \Re^2,$ find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$Ax = \lambda x$$

$$(A - \lambda I)x = 0 \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If we write

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then

$$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvalues and eigenvectors

By Cramer's rule we could write

$$x_1 = \frac{\det\begin{pmatrix} 0 & 1 \\ 0 & 2-\lambda \end{pmatrix}}{\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}} = \frac{0}{\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}}$$

$$x_2 = \frac{\det\begin{pmatrix} 2-\lambda & 0 \\ 1 & 0 \end{pmatrix}}{\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}} = \frac{0}{\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}}$$

An obvious, but uninteresting solution is the trivial solution $x_1=0$, $x_2=0$. Nontrivial solutions of x_1 and x_2 can only be obtained only if

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

Eigenvalues and eigenvectors

Which gives the characteristic equation

$$(2-\lambda)^2 - 1 = 0$$

solutions are $\lambda_1=1$ and $\lambda_2=3$.

For $\lambda=1$

$$\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\longrightarrow \quad x_1 + x_2 = 0$$

If we choose $x_1=1$ and $x_2 = -1$, the eigenvector corresponding to $\lambda=1$ is

$$e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda=3$, the equations are

$$\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\longrightarrow \quad -x_1 + x_2 = 0$$

$$e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Laplace Transform

Let $f(t)$ be a given function defined for all t ≥ 0. If the integral

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

exists, it is called the Laplace transform of $f(t)$. We denote it by $\mathcal{L}(f)$

$$\mathcal{L}(f) = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

The original function $f(t)$ is called the inverse transform of F (s) ;we denote it by

$\mathcal{L}^{-1}(F)$, so $f(t)=\mathcal{L}^{-1}(F)$.

Laplace Transform

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Some useful transforms:

$f(t)$	$\mathcal{L}(f)$	$f(t)$	$\mathcal{L}(f)$
1	$\frac{1}{s}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
t^2	$\frac{2!}{s^3}$	$\cosh \alpha t$	$\frac{s}{s^2-\alpha^2}$
$t^n (n=1,2,\ldots)$	$\frac{n!}{s^{n+1}}$	$\sinh \alpha t$	$\frac{\alpha}{s^2-\alpha^2}$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$		

Transform of derivatives

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\begin{aligned}\mathcal{L}(f'') &= s\mathcal{L}(f') - f'(0)\\ &= s\left[s\mathcal{L}(f) - f(0)\right] - f'(0)\\ &= s^2\mathcal{L}(f) - sf(0) - f'(0)\end{aligned}$$

In general:

$$\mathcal{L}\left(f^{(n)}\right) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

Transform of an integral of a function

If $f(t)$ is piecewise continuous

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}\mathcal{L}(f(t))$$

Hence if we write $\mathcal{L}(f(t)) = F(s)$ then

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{F(s)}{s}$$

Taking the inverse Laplace transform gives

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)d\tau.$$

Shifting theorems and the step function

If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}\left(e^{\alpha t} f(t)\right) = F(s-\alpha)$

Taking the inverse transform

$$\mathcal{L}^{-1}(F(s-\alpha)) = e^{\alpha t} f(t)$$

Example: Find $\mathcal{L}\left(e^{\alpha t} \cos \omega t\right)$

We know

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}$$

Using the above

$$\mathcal{L}\left(e^{\alpha t} \cos \omega t\right) = \frac{s-\alpha}{(s-\alpha)^2 + \omega^2}$$

Shifting on the t-axis

If $f(t)$ has the transform $\mathbf{F(s)}$ and $a>0$ then the function

$$\tilde{f}(t)=\begin{cases}0 & \text{if } \quad t<a \\ f(t-a) & \text{if } \quad t>a\end{cases}$$

has the transform

$$e^{-as}F(s)$$

Thus if we know **F(s)** is the transform of *f (t)* then we get the transform by multiplying **F(s)** by e^{-as}.

Laplace transform

Example: Using Laplace transform solve

$$y'' + y = 2\cos t, \qquad y(0) = 2, \quad y'(0) = 0$$

Taking Laplace transform of the differential equation. Define $Y(s) = \mathcal{L}(y)$.

$$\left[s^2 \mathcal{L}(y) - sy(0) - y'(0)\right] + \mathcal{L}(y) = \mathcal{L}(2\cos t)$$

$$\Rightarrow (s^2+1)\mathcal{L}(y) - 2s = \frac{2s}{s^2+1} \Rightarrow \mathcal{L}(y) = \frac{2s}{s^2+1} + \frac{2s}{(s^2+1)^2}$$

We have a complex and repeated complex factor.

$$\mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) = 2\cos t$$

$$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = t\sin t$$

$$\Rightarrow \quad y(t) = 2\cos t + t\sin t$$

Laplace transform

Example: Solve

$$y_1'' = y_1 + 3y_2 \qquad y_1(0) = 2, \; y_1'(0) = 3$$

$$y_2'' = 4y_1 - 4e^t \qquad y_2(0) = 1, \; y_2'(0) = 2$$

Define $F = \mathcal{L}(y_1)$, $G = \mathcal{L}(y_2)$ and take the LAplace transform of both equations

$$\mathcal{L}(y_1'') = \mathcal{L}(y_1) + 3\mathcal{L}(y_2)$$

$$\Rightarrow s^2\mathcal{L}(y_1) - sy_1(0) - y_1'(0) = \mathcal{L}(y_1) + 3\mathcal{L}(y_2)$$

$$\Rightarrow (s^2 - 1)F - 3G = 2s + 3$$

$$\mathcal{L}(y_2'') = 4\mathcal{L}(y_1) - 4\mathcal{L}(e^t)$$

$$\Rightarrow s^2\mathcal{L}(y_2) - sy_2(0) - y_2'(0) = 4\mathcal{L}(y_1) - 3\mathcal{L}(e^t)$$

$$\Rightarrow s^2 G - 4F = s + 2 - \frac{4}{s-1}$$

Laplace transform

Where we have used $\mathcal{L}\left(e^{t}\right)=\dfrac{1}{s-1}$. Solving for F and G

$$F=\frac{1}{s-2}+\frac{1}{s-1},\quad G=\frac{1}{s-2}$$

Then

$$y_1=\mathcal{L}^{-1}\left(F\right)=e^{2t}+e^{t},\ \ y_2=\mathcal{L}^{-1}\left(G\right)=e^{2t}.$$