Microwave Physics and Techniques

Solutions to Homework Set#2 – June 18, 2003

Problem1.

Minimum power loss of 10dB in a layer of 1m thickness allows us to determine the attenuation constant α:

$$10\,log\,e^{-20\alpha} = -10 \Rightarrow \alpha = \frac{ln 10}{2}$$

The effective complex dielectric constant

$$\varepsilon_{eff} = \varepsilon_0\left(1 - \frac{x}{1 - jz}\right) = \varepsilon_0\left(1 - \frac{x(1 + jz)}{(1 + z^2)}\right) = \varepsilon_0 \frac{1 + z^2 - x - jxz}{1 + z^2}$$

$$= \underbrace{\varepsilon_0 \frac{1 + z^2 - x}{1 + z^2}}_{\varepsilon'_{eff}} - j\underbrace{\varepsilon_0 \frac{xz}{1 + z^2}}_{\varepsilon''_{eff}} \Rightarrow \varepsilon_{eff} = \varepsilon'_{eff} - j\varepsilon''_{eff}$$

Here $x = \frac{\omega_p^2}{\omega^2} = \frac{f_p^2}{f^2}, z = \frac{\nu}{\omega}$

We have

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{1/2} = \omega\sqrt{\frac{\mu\varepsilon'_{eff}}{2}}\left[\sqrt{1 + \left(\frac{\varepsilon''_{eff}}{\varepsilon'_{eff}}\right)^2} - 1\right]^{1/2}$$

$$= \frac{\omega}{c}\sqrt{\frac{1 + z^2 - x}{2(1 + z^2)}}\left[\sqrt{1 + \left(\frac{xz}{1 + z^2 - x}\right)^2} - 1\right]^{1/2}$$

Substituting numerical values of $\nu = 10^{11}\ s^{-1}$ and

$$f_p = 9\sqrt{N} = 9\sqrt{10^{19}\,m^{-3}} \approx 9\sqrt{10}10^9\,Hz,$$

We can solve for $f \Rightarrow f \approx 343 GHz$

Since r>>L, we can use far-zone field expressions. The maximum time-average radiated power density occurs at $\theta=90^0$ and is given by

$$|S_{av}|_{max} \approx \frac{|E_\theta|^2_{max}}{2\eta_{air}}$$

For far fields (Fraunhofer), we have

$$H_\phi \approx j\frac{Idl}{4\pi}\left(\frac{e^{-\beta r}}{r}\right)\beta \sin\theta$$

$$E_\theta \approx j\frac{Idl}{4\pi}\left(\frac{e^{-\beta r}}{r}\right)\eta\beta \sin\theta$$

$$E_z \approx 0$$

So $|E_\theta|_{max} = \frac{Idl\eta_{air}\beta}{4\pi r} = \frac{Idl\eta_{air}}{2\lambda r}$. Substituting I=10A and dl/λ=0.01, the maximum time-average radiated power density at r=100 m can be calculated as

$$|E_\theta|_{max} = \frac{(10)(0.01)(377)}{2(100)} = 0.1885V-m$$

$$\Rightarrow |S_{av}|_{max} \approx \frac{(0.1885)^2}{2(377)} \approx 47.1\mu W-m^{-2}$$

b) Repeating part (a) at r=1km, we find

$$|S_{av}|_{max} \approx 0.471\mu W-m^{-2}$$

b) Repeating part (a) at r=10km, we find

$$|S_{av}|_{max} \approx 4.71\mu W-m^{-2}$$

Using the power density the total power radiated by the antenna can be calculated as

$$P_{rad} = 4\pi r^2 |S_{av}| = 4\pi(10^8)\left(\frac{20\times10^{-12}}{10^{-4}}\right) = 80\pi\, W$$

Since the input power of the antenna is given to be 25W, the antenna gain is

$$G_{anten} = \frac{P_{rad}}{P_{in}} = \frac{80\pi\, W}{25W} \approx 10.5$$

Problem4.

The total time-average power radiated by the Hertzian dipole antenna between the $\pm 45^0$ equatorial plane can be calculated as

$$\begin{aligned}
[P_{rad}]_{45^0 \le \theta < 135^0} &= \frac{(Idl)^2}{32\pi^2}\eta\beta^2(2\pi)\int_{\pi/4}^{3\pi/4} sin^3\theta d\theta \\
&= \frac{(Idl)^2}{32\pi^2}\eta\beta^2(2\pi)\left[-\frac{cos\,\theta}{3}\left(2+sin^2\theta\right)\right]_{\pi/4}^{3\pi/4} \\
&= \frac{(Idl)^2}{32\pi^2}\eta\beta^2(2\pi)\left[\frac{1}{3\sqrt{2}}(2.5)+\frac{1}{3\sqrt{2}}(2.5)\right] \\
&= \frac{5(Idl)^2}{48\pi\sqrt{2}}\eta\beta^2
\end{aligned}$$

Use:

$$\int sin^3\theta d\theta = \int sin^2\theta\, sin\,\theta d\theta = \int\left(cos^2\theta - 1\right)d(cos\,\theta)$$

$$= \frac{cos^3\theta}{3} - cos\,\theta = \frac{cos\,\theta}{3}\left(cos^2\theta - 3\right) = \frac{cos\,\theta}{3}\left(1 - sin^2\theta - 3\right)$$

$$= -\frac{cos\,\theta}{3}\left(2 + sin^2\theta\right)$$

The fraction of the total power radiated between $45^\circ \le \theta < 135^\circ$ can be found as

$$\frac{[P_{rad}]_{45^0 \le \theta < 135^0}}{P_{rad}} = \frac{5}{4\sqrt{2}} \approx 0.884 \;\; or \;\; 88.4\%$$

The penetration depth at 2.45 GHz with $\varepsilon_r' = 52.4 \; and \; tan\,\delta_c = 0.33(\mu_r = 1)$ can be calculated as

$$d = \frac{1}{\alpha} = \frac{\sqrt{2}}{\omega\sqrt{\varepsilon_r'}\sqrt{\mu_0\varepsilon_0}\left[\sqrt{1+tan^2\,\delta_c} - 1\right]^{1/2}}$$

$$= \frac{\sqrt{2}\left(3\times 10^8\right)}{2\pi\left(2.45\times 10^9\right)\sqrt{52.4}\left[\sqrt{1+\left(0.33^2\right)} - 1\right]^{1/2}} \approx 0.0165m = 1.65cm.$$

Therefore assuming that the hamburger is uniformly cooked if its thickness is about 1 to 1.5 times its penetration depth, the maximum thickness of the hamburger to be heated uniformly at 3.45 GHz is 1.5d ≈ 2.48 cm.

b) Following a similar approach as in part a) the penetration depth in the hamburger slice at 915 MHz with $\varepsilon_r' = 54.4 \; and \; tan\,\delta_c = 0.41(\mu_r = 1)$ is

$$d = \frac{\sqrt{2}\left(3\times 10^8\right)}{2\pi\left(915\times 10^6\right)\sqrt{54.4}\left[\sqrt{1+\left(0.41^2\right)} - 1\right]^{1/2}} \approx 0.0351m = 3.51cm.$$

Therefore the maximum thickness of the hamburger to be heated uniformly at 915 MHz is 1.5d ≈ 5.26 cm. As mentioned, microwave ovens operating at 915 MHz allow the user to cook with larger dimensions uniformly compared to microwave ovens operating at 2.45 GHz.

Problem6.

Note that since for normal incidence, 1+Γ=T, the reflection and transmission coefficients must, Γ=-0.5 and T=0.5. For a lossless nonmagnetic case, the reflection and transmission coefficients can be simplified as

$$\Gamma = \frac{\sqrt{\varepsilon_{1r}} - \sqrt{\varepsilon_{2r}}}{\sqrt{\varepsilon_{1r}} + \sqrt{\varepsilon_{2r}}} = -0.5, \; T = \frac{2\sqrt{\varepsilon_{1r}}}{\sqrt{\varepsilon_{1r}} + \sqrt{\varepsilon_{2r}}} = 0.5$$

Dividing these two with one another gives:

$$0.5 - 0.5\frac{\sqrt{\varepsilon_{2r}}}{\sqrt{\varepsilon_{1r}}} = -1 \Rightarrow \frac{\varepsilon_{2r}}{\varepsilon_{1r}} = 9$$

Problem7.

$$\bar{J} = \sigma\bar{E}, \nabla\cdot\varepsilon\bar{E} = \rho = \nabla\cdot\left(\frac{\varepsilon}{\sigma}\right)\bar{J}$$

$$\Rightarrow \nabla\cdot\varepsilon\bar{E} = \nabla\cdot\left(\frac{\varepsilon}{\sigma}\right)\bar{J} = \left(\frac{\varepsilon}{\sigma}\right)\nabla\cdot\bar{J} = -\left(\frac{\varepsilon}{\sigma}\right)\frac{\partial\rho}{\partial t}$$

$$\Rightarrow \rho = \rho_0 e^{-\alpha t/\varepsilon}.$$

For copper $\tau = \varepsilon/\sigma = 1.52\times10^{-19}$. For sea water $\tau = \varepsilon/\sigma = 1.77\times10^{-10}$

$f = 1/\tau = 6.58\times10^{18}\ sec^{-1}$ for copper and $5.59\times10^{9}\ sec^{-1}$ for sea water.

Problem8.

$\nabla\times\bar{H} = j\omega\varepsilon\bar{E} + \sigma\bar{E}.$ Displacement current can be neglected when

$\omega\varepsilon << \sigma$ or $f << \sigma/2\pi\varepsilon, \Rightarrow$ i.e. $f << 1/2\pi\tau.$

Problem9. (Colin 2.15)

Consider equation 2.92 (Collin, P.50)

$$sin\,\theta_1 = \eta\, sin\,\theta_3$$

Here in place of this equation we have $sin\,\theta_1 = \eta\, sin\,\theta_3$ where now $\eta = \sqrt{\frac{\mu}{\mu_0}}.$

Similarly in place of (eq. 2.93):

$$(E_1 + E_2)cos\,\theta_1 = E_3\sqrt{1 - sin^2\theta} = E_3\frac{\sqrt{\varepsilon_r - sin^2\theta_1}}{\eta}$$

We get $(E_1 + E_2)cos\,\theta_1 = E_3\left[\left(\mu_r - sin^2\theta_1\right)/\mu_r\right]^{1/2}$ and for (2.94)

$E_1 - E_2 = E_3\mu_r$ where again $\mu_r = \mu/\mu_0$.

Thus we get $(1+\Gamma)cos\,\theta_1 = T\left[\left(\mu_r - sin^2\theta_1\right)/\mu_r\right]^{1/2}$

$$1 - \Gamma = T\mu_r^{-1/2}$$

$$\Gamma = \frac{\left(\mu_r - sin^2\theta_1\right)^{1/2} - cos\,\theta_1}{\left(\mu_r - sin^2\theta_1\right)^{1/2} + cos\,\theta_1} \quad and \quad T = \frac{2\mu_r^{1/2}cos\,\theta_1}{\left(\mu_r - sin^2\theta_1\right)^{1/2} + cos\,\theta_1}$$

A Brewster angle that makes Γ=0 does not exist.

Problem10: (Collin 2.18)

We have

$$\begin{cases} E_1 e^{-jk_0 z} + E_2 e^{jk_0 z} = E_x \\ Y_0 E_1 e^{-jk_0 z} - Y_0 E_2 e^{jk_0 z} = H_y \end{cases} z \le -d$$

$$\begin{cases} E_3 e^{-jk_1 z} + E_4 e^{jk_1 z} = E_x \\ Y_1 E_3 e^{-jk_1 z} - Y_1 E_4 e^{jk_1 z} = H_y \end{cases} -d \le z \le -0$$

$$\begin{cases} E_5 e^{-jk_2 z} = E_x \\ Y_2 E_5 e^{-jk_2 z} = H_y \end{cases} z \ge 0$$

B.C's give:

$$E_1 e^{jk_0 d} + E_2 e^{-jk_0 d} = E_3 e^{jk_1 d} + E_4 e^{-jk_1 d}$$

$$E_1 e^{jk_0 d} - E_2 e^{-jk_0 d} = Y_1 Z_0 \left(E_3 e^{jk_1 d} - E_4 e^{-jk_1 d} \right)$$

$$E_3 + E_4 = E_5$$

$$E_3 - E_4 = Y_2 Z_1 E_5$$

For $k_1 d = \pi/2 = 2\pi d/\lambda_1 = 2\pi d\,(\varepsilon_{1r})^{1/2}/\lambda_0$

$$E_1 - E_2 = E_3 - E_4 = Y_1 Z_1 E_5$$

$$E_1 + E_2 = Y_1 Z_0 (E_3 + E_4) = Y_1 Z_0 E_5$$

Subtracting

$$\Rightarrow 2E_2 = (Y_1 Z_0 - Y_2 Z_1) E_5$$

$$\Rightarrow E_2 = 0 \;\; if \;\; Y_1 Z_0 - Y_2 Z_1 = 0 \Rightarrow Z_1 = (Z_0 Z_2)^{1/2} \Rightarrow \varepsilon_1 = (\varepsilon_0 \varepsilon_2)^{1/2}$$