Microwave Physics and Techniques

Solutions to Homework Set#1 – June 17, 2003

Problem1.

$$x(t) = 2b\cos t$$
$$y(t) = b\sin t$$

since

$$\cos^2 t + \sin^2 t = 1,$$ We can write

$$\frac{x^2}{(2b)^2} + \frac{y^2}{b^2} = 1$$

$$\text{or} \quad F(x,y) = x^2 + 4y^2 = 4b^2 = const$$

thus

$$\frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y} = 0 \Rightarrow 2x\dot{x} + 8y\dot{y} = 0$$

with

$$p(x,y) = 2x\sin y \quad \text{and} \quad q(x,y) = x^2\cos y + \sin y$$

We get

$$\frac{\partial p}{\partial y} = 2x\cos y = \frac{\partial q}{\partial x}$$

$\therefore$ The D.E. is exact.

The general solution can be obtained by line integration to get

$$F(x,y) = \int_0^x p(\tilde{x},0)d\tilde{x} + \int_0^y q(x,\tilde{y})d\tilde{y} = \int_0^x 2\tilde{x}\sin 0\, d\tilde{x} + \int_0^y \left(x^2\cos\tilde{y} + \sin\tilde{y}\right)d\tilde{y}$$
$$= \left[x^2\sin\tilde{y} - \cos\tilde{y}\right]_0^{\tilde{y}=y} = x^2\sin y - \cos y - 1$$

Since a constant does not matter in this context, the general solution of the D.E can be written as

$$F(x,y) = x^2\sin y - \cos y = c, c \in \Re$$

By separation of variables, we end up with a boundary-value problem in x and an equation in t. In the usual fashion, the eigenvalues are $\lambda_n = n\pi$. Since the B.C's are periodic, the associated eigenfunctions are $\sin(n\pi x)$ and $\cos(n\pi x)$. By the principle of superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \sin(n\pi x) + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \cos(n\pi x)$$

The initial conditions consist of a simple sum of the sines and cosines.

$$\Rightarrow b_0 = 1, b_5 = 1/2, a_n = 4$$

All other coefficients are zero.

Problem3.

$$f(x) \text{ is even} \Rightarrow \cos \text{ solutions only!}$$

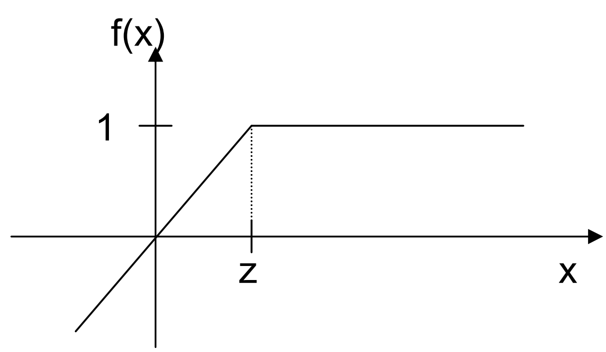
$$x^2 = b_0 + \sum_{n=1}^{\infty} b_n \cos(nx)$$

$$b_0 = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{3}\pi^2$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \cos(nx) dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$$

$$b_n = \frac{4(-1)^n}{n^2}$$

Problem4.



$$f(x) = \begin{cases} x & x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$\text{for } 0 \le z < 1, H(z) = \int_0^z x dx = \left.\frac{x^2}{2}\right|_0^z = \frac{z^2}{2}$$

$$\text{for } z \ge 1, H(z) = \int_0^1 x dx + \int_1^z dx = \left.\frac{x^2}{2}\right|_0^1 + x\Big|_1^z$$

$$H(z) = \begin{cases} z^2/2 & \text{for } 0 \le z < 1 \\ z - 1/2 & \text{for } z \ge 1 \end{cases}$$

$$1=\int_0^\infty cxe^{-x^2}dx = c\,lim_{b\to\infty}\int_0^b cxe^{-x^2}dx \qquad \left[u=x^2, du=2xdx, u(0)=0, u(b)=b^2\right]$$

$$= c\,lim_{b\to\infty}\int_0^b \frac{1}{2}e^{-u}du = c\,lim_{b\to\infty}\left[\frac{e^{-u}}{-2}\right]_0^{b^2} = c\,lim_{b\to\infty}\left\{\frac{e^{-b^2}}{-2}+\frac{1}{2}\right\} = \frac{c}{2}$$

$$\Rightarrow \frac{c}{2}=1 \Rightarrow c=2$$

Problem6.

$$\beta=\omega\sqrt{\varepsilon_0\mu_0}=\omega/c\approx\frac{2\pi f}{3\times10^8}=9.3\,rad/m$$

$$\Rightarrow \omega\approx 2.79\times10^9\,rad-s^{-1}\ \ and\ f=\omega/2\pi=444MHz$$

$$\nabla\times H=j\omega\varepsilon_0 E$$

$$H(z)=\hat{x}H_x(z)=\hat{x}0.1e^{-j9.3z}e^{-j\pi/2}\ mA/m$$

$$\Rightarrow E(z)=\frac{1}{j\omega\varepsilon_0}\left(\hat{y}\frac{\partial H_x(z)}{\partial z}\right)\approx\hat{y}\frac{0.1(-j9.3)e^{-j9.3z}e^{-j\pi/2}}{j(2.79\times10^9)(8.85\times10^{-12})}\approx\hat{y}37.7e^{-j9.3z}e^{-j\pi/2}\ mV/m$$

Problem7.

$$a)\nabla\times E=-j\omega\mu_0 H\Rightarrow H=-\frac{1}{j\omega\mu_0}\left[\hat{y}\frac{\partial E_x(y,z)}{\partial z}-\hat{z}\frac{\partial E_x(y,z)}{\partial y}\right]$$

where the electric field phasor is given by

$$E(y,z)=\hat{x}E_x(y,z)=\hat{x}E_0\cos(ay)e^{-jbz}$$

Taking the partial derivatives yields

$$H(y,z)=\hat{y}\frac{bE_0}{\omega\mu_0}\cos(ay)e^{-jbz}+\hat{z}\frac{jaE_0}{\omega\mu_0}\sin(ay)e^{-jbz}$$

$$\therefore \overline{H}(y,z,t)\ can\ be\ found\ from\ the\ phasor\ H(y,z)\ as$$

$$\overline{H}(y,z,t)=Re\{H(y,z)e^{j\omega t}\}=\hat{y}\frac{bE_0}{\omega\mu_0}\cos(ay)\cos(\omega t-bz)-\hat{z}\frac{aE_0}{\omega\mu_0}\sin(ay)\sin(\omega t-bz)$$

$$b)\ substituting\ for\ H(y,z)$$

$$\nabla\times H=j\omega\varepsilon_0 E\Rightarrow E=\frac{1}{j\omega\varepsilon_0}\hat{x}\left[\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right]$$

$$\Rightarrow E(y,z) = \hat{x}\left(\frac{a^2+b^2}{\omega^2\mu_0\varepsilon_0}\right)E_0\cos(ay)e^{-jbz}$$

But this *E(y,z)* expression must be the same as

$$E(y,z) = \hat{x}E_0\cos(ay)e^{-jbz} \Rightarrow \left(\frac{a^2+b^2}{\omega^2\mu_0\varepsilon_0}\right) = 1$$

must hold so that Maxwell's equations are both satisfied. Note that

$$E(y,z) = \hat{x}E_x(y,z) \quad \left(\nabla\cdot E(y,z) = \frac{\partial E_x}{\partial x} = 0\right)$$

c) Using Euler's Formula $\cos(ay) = \frac{e^{jay}+e^{-jay}}{2}$ the electric field phasor E(y,z) can be written as

$$E(y,z) = \hat{x}E_0\left[\frac{e^{jay}+e^{-jay}}{2}\right]e^{-jbz} = \hat{x}\frac{E_0}{2}e^{j(ay-bz)} + \hat{x}\frac{E_0}{2}e^{-j(ay+bz)}$$

Now, it is clearly seen that this wave may be regarded as a combination of two uniform plane waves propagating in different directions. The direction of propagation of the two components are given by the unit vectors as

$$\hat{k}_1 = \frac{-a\hat{y}+b\hat{z}}{\sqrt{a^2+b^2}} \; and \; \hat{k}_2 = \frac{a\hat{y}+b\hat{z}}{\sqrt{a^2+b^2}}$$

<u>Problem8.</u>

a) Using plane wave (uniform) from Maxwell's equation, the corresponding H (z) can be written as

$$H(z) \cong [-\hat{y} - \hat{x}(1+j)]\left(\frac{12}{377}\right)e^{j50\pi z}\ A/m$$

b) The time-average power density carried by this wave can be found as:

$$|S_{av}| \approx \frac{1}{2}\frac{(12)^2}{377} + \frac{1}{2}\frac{(12\sqrt{2})^2}{377} \approx 0.573W - m^{-2} = 57.3\mu W - cm^{-2}$$

c) The real-time expression for the electric field vector can be written as

$$\overline{E}(z,t) = \hat{x}12\cos(\omega t + 50\pi z) - \hat{y}12\sqrt{2}\cos(\omega t + 50\pi z + \pi/4)V/m$$

$$\omega = \beta c \approx 50\pi\ rad/m \times (3\times10^8\ m/s) = 1.5\times10^{10}\ rad/s$$

This wave is elliptically polarized (LHEP wave).