NORMAL MODES

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ABSTRACT

This is one packet of notes accompanying a course *Mechanics and Electromagnetism in Accelerators,* offered as part of the U.S. Particle Accelerator School, Yale University, summer, 2002. This packet reviews (in the form of problems to be worked) some elementary mechanics, especially the normal modes of "linear" systems. Also the Laplace transform method is applied to simple harmonic motion.

Problem .1. The approximate Lagrangian for an n-dimensional system with coordinates $(q_1, q_2, \ldots, q_n)$, valid in the vicinity of a stable equilibrium point (that can be taken to be $(0, 0, \ldots, 0)$) has the form

$$L(\mathbf{q}, \dot{\mathbf{q}}) = T - V, \quad \text{where} \quad T = \frac{1}{2}\sum_{r,s=1}^{n} m_{rs}\dot{q}_r\dot{q}_s, \quad V = \frac{1}{2}\sum_{r,s=1}^{n} k_{rs}q_r q_s. \tag{1}$$

It is shown in algebra courses that a linear transformation $q_i \to y_j$ can be found such that T takes the form

$$T = \frac{1}{2}\sum_{r=1}^{n} m_r \dot{y}_r^2,$$

where, in this case each "mass" m_r is necessarily positive because T is positive definite. By judicious choice of the scale of the y_r each "mass" can be adjusted to 1;

$$T = \frac{1}{2}\sum_{=1}^{n} \dot{y}_r^2. \tag{2}$$

For these coordinates y_r the equation

$$\sum_{r=1}^{n} y_r^2 = 1 \tag{3}$$

defines a surface (to be called a hypersphere). From now on we will consider only points $\mathbf{y} = (y_1, \ldots, y_n)$ lying on this sphere. Also two points $\mathbf{u}$ and $\mathbf{v}$ will be said to be "orthogonal" if the "quadratic form" $\mathcal{I}(\mathbf{u}, \mathbf{v})$ defined by

$$\mathcal{I}(\mathbf{u}, \mathbf{v}) \equiv \sum_{r=1}^{n} u_r v_r$$

vanishes. Being linear in both arguments $\mathcal{I}(\mathbf{u}, \mathbf{v})$ is said to be "bilinear". We also define a bilinear form $\mathcal{V}(\mathbf{u}, \mathbf{v})$ by

$$\mathcal{V}(\mathbf{u}, \mathbf{v}) \equiv \sum_{r,s=1}^{n} k_{rs} u_r v_s,$$

where coefficients k_{rs} have been redefined from the values given above to correspond to the new coordinates y_r so that

$$V(\mathbf{y}) = \frac{1}{2}\,\mathcal{V}(\mathbf{y}, \mathbf{y})\,.$$

The following series of problems (adapted from Courant and Hilbert, Vol. 1, p. 37) will lead to the conclusion that a further linear transformation $y_i \to z_j$ can be found that, on the one hand, enables the equation for the sphere in Eq. (3) to retain the same form,

$$\sum_{r=1}^{n} z_r^2 = 1,$$

and, on the other, enables V to be expressible as a sum of squares with positive coefficients;

$$V = \frac{1}{2}\sum_{r=1}^{n} \kappa_r z_r^2, \quad \text{where} \quad 0 < \kappa_n \leq \kappa_{n-1} \leq \ldots \leq \kappa_1 < \infty. \tag{4}$$

Pictorially the strategy is, having deformed the scales so that surfaces of constant T are spherical and surfaces of constant V ellipsoidal, to orient the axes to make these ellipsoids erect. In the jargon of mechanics this process is known as "normal mode" analysis. The "minimax" properties of the "eigenvalues" to be found have important physical implications, but we will not go into them here.

(a) Argue, for small oscillations to be stable, that V must also be positive definite.

(b) Let $\mathbf{z}_1$ be the point on sphere (3) for which $V(\stackrel{\text{def.}}{=} \frac{1}{2}\kappa_1)$ is maximum. (If there is more than one such point pick any one arbitrarily.) Then argue that

$$0 < \kappa_1 < \infty.$$

(c) Among all points that are both on sphere (3) and orthogonal to $\mathbf{z}_1$, let $\mathbf{z}_{(2)}$ be the one for which $V(\stackrel{\text{def.}}{=} \frac{1}{2}\kappa_{(2)})$ is maximum. Continuing in this way show that a series of points $\mathbf{z}_1, \mathbf{z}_{(2)}, \ldots \mathbf{z}_{(n)}$, each maximizing V consistent with being orthogonal to its predecessors is determined, and that the sequence of values, $V(\mathbf{z}_r) = \frac{1}{2}\kappa_r, \quad r = 1, 2, \ldots, n$, is monotonically non-increasing.

(d) Consider a point $\mathbf{z}_1 + \epsilon\boldsymbol{\zeta}$ which is assumed to lie on surface (3) but with $\boldsymbol{\zeta}$ otherwise arbitrary. Next assume this point is "close to" $\mathbf{z}_1$ in the sense that ϵ is arbitrarily small (and not necessarily positive). Since $\mathbf{z}_1$ maximizes V it follows that

$$\mathcal{V}(\mathbf{z}_1 + \epsilon\boldsymbol{\zeta}, \mathbf{z}_1 + \epsilon\boldsymbol{\zeta}) \leq 0.$$

Show therefore that

$$\mathcal{V}(\mathbf{z}_1, \boldsymbol{\zeta}) = 0.$$

This implies that

$$\mathcal{V}(\mathbf{z}_1, \mathbf{z}_r) = 0 \quad \text{for} \quad r > 1,$$

because, other than being orthogonal to $\mathbf{z}_1$, $\boldsymbol{\zeta}$ is arbitrary. Finally, extend the argument to show that

$$\mathcal{V}(\mathbf{z}_r, \mathbf{z}_s) = \kappa_r \delta_{rs},$$

where the coefficients κ_r have been shown to satisfy the monotonic conditions of Eq. (4) and δ_{rs} is the usual Kronecker-δ symbol. Taking these $\mathbf{z}_r$ as basis vectors, an arbitrary vector $\mathbf{z}$ can be expressed as

$$\mathbf{z} = \sum_{r=1}^{n} z_r \mathbf{z}_r.$$

In these new coordinates show that Eqs. (1) become

$$L(\mathbf{z}, \dot{\mathbf{z}}) = T - V, \quad T = \frac{1}{2}\sum_{r=1}^{n} \dot{z}_r^2, \quad V = \frac{1}{2}\sum_{r=1}^{n} \kappa_r z_r^2. \tag{5}$$

Write and solve the Lagrange equations for coordinates z_r.

Problem .2. Continuing the previous formula, in a more formal approach, the Lagrange equations resulting from Eq. (1) are

$$\sum_{s=1}^{n} m_{rs}\ddot{q}_s + \sum_{s=1}^{n} k_{rs} q_s = 0. \tag{6}$$

These equations can be expressed compactly in matrix form;

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = 0; \tag{7}$$

or, assuming the existence of $\mathbf{M}^{-1}$ as

$$\ddot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{K}\mathbf{q} = 0; \tag{8}$$

Seeking a solution of the form

$$q_r = A_r e^{i\omega t} \quad r = 1, 2, \ldots, n,$$

the result of substitution into Eq. (6) is

$$\left(\mathbf{M}^{-1}\mathbf{K} - \omega^2 \mathbf{1}\right)\mathbf{A} = 0. \tag{9}$$

These equations have non-trivial solutions for values of ω that cause the determinant of the coefficients to vanish;

$$|\mathbf{M}^{-1}\mathbf{K} - \omega^2 \mathbf{1}| = 0. \tag{10}$$

Correlate these "eigenvalues" with the constants κ_r defined in the previous problem.

Problem .3.

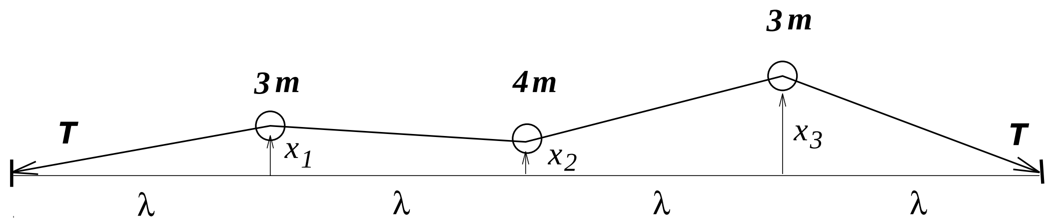

Figure 1: Three beads on a stretched string. The transverse dispacements are much exaggerated. Gravity and string mass are negligible.

Particles of mass $3m$, $4m$, and $3m$, are spaced at uniform intervals λ along a light string of total length 4λ stretched with tension $\mathcal{T}$ and rigidly fixed at both ends. To legitimize ignoring gravity the system is assumed to lie on a smooth horizontal table so the masses can oscillate only horizontally. Let the horizontal displacements be x_1, x_2, and x_3. Find the normal modes frequencies and the corresponding normal mode oscillation "shapes". Discuss the "symmetry" of the shapes, their "wavelengths", and the (monotonic) relation between frequency and number of nodes.

Already with just three degrees of freedom the eigenmode calculations are sufficiently tedious to make some efforts at simplifying the work worthwhile. In this problem, with the system symmetric about its mid-point it is clear that the modes will be either symmetric or anti-symmetric and, since the antisymmetric mode vanishes at the center point, it is characterized by a single amplitude, say $y = x_1 = -x_3$. Introducing "effective mass" and "effective strength coefficient" the kinetic energy of the mode, necessarily proportional to $\dot{y}$, can be written as $T_2 = \frac{1}{2}m_{\text{eff}}\dot{y}^2$ and the potential energy can be written as $V_2 = \frac{1}{2}k_{\text{eff}}y^2$. The frequency of this mode is then given by $\omega_2 = \sqrt{k_{\text{eff}}/m_{\text{eff}}}$ which, by dimensional analysis, has to be proportional to $\eta = \sqrt{\mathcal{T}/(m\lambda)}$. (The quantities T_2, V_2 and ω_2 have been given subscript 2 because this mode has the second highest frequency.) Factoring this expression out of Eq. (10), the dimensionless eigenvalues are the eigenfrequencies in units of η. Complete the analysis to show that the normal mode frequencies are $(\omega_1, \omega_2, \omega_3) = (1, \sqrt{2/3}, \sqrt{1/6})$, and find the corresponding normal mode "shapes".

Problem .4. Though the eigenmode/eigenvalue solution method employed in solving the previous problem is the traditional method used in classical mechanics, equations of the same form, when they arise in circuit analysis and other engineering fields, are traditionally solved using Laplace transforms—a more robust method, it seems to me. Let us continue the solution of the previous problem using this method. Individuals already familiar with this method or not wishing to become so should skip this section. Here we use the notation

$$\overline{x}\,(s) = \int_0^\infty e^{-st} x\,(t)\,dt, \tag{11}$$

as the formula giving the Laplace transform $\overline{x}(s)$, of the function of time $x(t)$. $\overline{x}(s)$ is a function of the "transform variable" s (which is a complex number with positive real part.) With this definition the Laplace transform satisfies many formulas but, for present purposes we use only

$$\overline{\frac{dx}{dt}} = s\overline{x} - x\,(0)\,, \tag{12}$$

which is easily demonstrated. Repeated application of this formula converts time derivatives into functions of s and therefore converts (linear) differential equations into (linear) algebraic equations. This will now be applied to the system described in the previous problem.

The Lagrange equations for the beaded string shown in Fig. 1 are

$$\begin{aligned} 3\ddot{x}_1 + \eta^2\,(2x_1 - x_2) &= 0, \\ 4\ddot{x}_2 + \eta^2\,(2x_2 - x_1 - x_3) &= 0, \\ 3\ddot{x}_3 + \eta^2\,(2x_3 - x_2) &= 0, \end{aligned} \tag{13}$$

Suppose the string is initially at rest but that a transverse impulse I is administered to the first mass at $t = 0$; as a consequence it acquires initial velocity $v_{10} \equiv \dot{x}(0) = I/(3m)$. Transforming all three equations and applying the initial conditions (the only non-vanishing initial quantity, v_{10}, enters via Eq. (12).)

$$\begin{aligned} \left(3s^2 + 2\eta^2\right)\overline{x}_1 - \eta^2\overline{x}_2 &= I/m, \\ -\eta^2\overline{x}_1 + \left(4s^2 + 2\eta^2\right)\overline{x}_2 - \eta^2\overline{x}_3 &= 0, \\ -\eta^2\overline{x}_2 + \left(3s^2 + 2\eta^2\right)\overline{x}_3 &= 0, \end{aligned} \tag{14}$$

Solving these equations yields

$$\begin{aligned}\overline{x}_1 =& \frac{I}{10m}\left(\frac{2/3}{s^2+\eta^2/6}+\frac{1}{s^2+\eta^2}+\frac{5/3}{s^2+2\eta^2/3}\right),\\ \overline{x}_2 =& \frac{I}{10m}\left(\frac{1}{s^2+\eta^2/6}-\frac{1}{s^2+\eta^2}\right),\\ \overline{x}_3 =& \frac{I}{10m}\left(\frac{2/3}{s^2+\eta^2/6}+\frac{1}{s^2+\eta^2}-\frac{5/3}{s^2+2\eta^2/3}\right).\end{aligned} \tag{15}$$

It can be seen, except for factors $\pm i$, that the poles (as a function of s) of the transforms of the variables, are the normal mode frequencies. This is not surprising since the determinant of the coefficients in Eq. (14) is the same as the determinant entering the normal mode solution, but with ω^2 replaced with $-s^2$. Remember then, from Cramer's rule for the solution of linear equations, that this determinant appears in the denominators of the solutions. For "inverting" Eq. (15) it is sufficient to know just one inverse Laplace transformation,

$$\mathcal{L}^{-1}\frac{1}{s-\alpha} = e^{\alpha t}, \tag{16}$$

but it is easier to look in a table of inverse transforms to find that the terms in Eq. (15) yield sinusoids that oscillate with the normal mode frequencies. Furthermore the "shapes" asked for in the previous problem can be read off directly from (15) to be (2:3:2), (1:0:1), and (1:-1:1).

When the first mass is struck at $t = 0$ all three modes are excited and they proceed to oscillate at their own natural frequencies, so the motion of each individual particle is a superposition of these frequencies. Since there is no damping the system will continue to oscillate in this superficially complicated way for ever. In practice there is always some damping and, in general, it is different for the different modes; commonly damping increases with frequency. In this case, after a while, the motion will be primarily in the lowest frequency mode; if the vibrating string emits audible sound, a an increasingly pure tone will be heard as time goes on.

Problem .5. Damped and driven simple harmonic motion. The equation of motion of mass m, subject to restoring force $-\omega_0^2 mx$, damping force $-2\lambda m\dot{x}$, and external drive force $f\cos\gamma t$ is

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 = \frac{f}{m}\cos\gamma t. \tag{17}$$

(a) Show that the general solution of this equation when $f = 0$ is

$$x(t) = ae^{-\lambda t}\cos(\omega t + \phi), \tag{18}$$

where a and ϕ depend on initial conditions and $\omega = \sqrt{\omega^2 - \lambda^2}$. This "solution of the homogeneous equation" is also known as "transient" since when it is superimposed on the "driven" or "steady state" motion caused by f it will eventually become negligible.

(b) Correlate the stability or instability of the transient solution with the sign of λ. Equivalently, after writing the solution (18) as the sum of two complex exponential terms, Laplace transform them, and correlate the stability or instability of the transient with the locations in the complex s-plane of the poles of the Laplace transform.

(c) Assuming $x(0) = \dot{x}(0) = 0$, show that Laplace transforming Eq. (17) yields

$$\overline{x}(s) = f\frac{s}{s^2+\gamma^2}\frac{1}{s^2+2\lambda s+\omega_0^2}. \tag{19}$$

This expression has four poles, each of which leads to a complex exponential term in the time response. To neglect transients we need only drop the terms for which the poles are off the imaginary axis. (By part (b) they must be in the left half-plane for stability.) To "drop" these terms it is necessary first to isolate them by partial fraction decomposition of Eq. (19). Performing these operations, show that the steady state solution of Eq. (17) is

$$x(t) = \frac{f}{m}\sqrt{\frac{1}{\left(\omega_0^2-\gamma^2\right)^2+4\lambda^2\gamma^2}}\cos(\gamma t+\delta), \tag{20}$$

where

$$\omega_0^2 - \gamma^2 - 2\lambda\gamma i = \sqrt{\left(\omega_0^2-\gamma^2\right)^2+4\lambda^2\gamma^2}\,e^{i\delta}. \tag{21}$$

(d) The response is large only for γ close to ω_0. To exploit this, defining the "small" "frequency deviation from the natural frequency"

$$\epsilon = \gamma - \omega_0, \tag{22}$$

show that $\gamma^2 - \omega^2 \approx 2\epsilon\omega$ and that the approximate response is

$$x(t) = \frac{f}{2m\omega_0}\sqrt{\frac{1}{\epsilon^2 + \lambda^2}}\cos(\gamma t + \delta)\,. \tag{23}$$

Find the value of ϵ for which the amplitude of the response is reduced from its maximum value by the factor $1/\sqrt{2}$.

Conservation of momentum and energy

It has been shown previously that the application of energy conservation in one dimensional problems permits the system evolution to be expressed in terms of a single integral—this is "reduction to quadrature". The following problem exhibits the use of momentum conservation to reduce a two dimensional problem to quadratures, or rather, because of the simplicity of the configuration in this case, to a closed-form solution.

Problem .6.

A point mass m with total energy E, starting in the left half-plane, moves in the (x, y) plane subject to potential energy function

$$U(x, y) = \begin{cases} U_1 & \text{for } x < 0; \\ U_2 & \text{for } 0 < x. \end{cases}$$

The "angle of incidence" to the interface at $x = 0$ is θ_i and the outgoing angle is θ. Specify the qualitatively different cases that are possible, depending on the relative values of the energies, and in each case find θ in terms of θ_i. Show that all results can be cast in the form of "Snell's Law" optics if one introduces a factor $\sqrt{E - U(r)}$, analogous to index of refraction.
