M.K.S./GAUSSIAN UNIT CONVERSION

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ABSTRACT

This is one packet of notes accompanying a course *Mechanics and Electromagnetism in Accelerators*, offered as part of the U.S. Particle Accelerator School, Yale University, summer, 2002. This packet is perhaps an appendix. In it conversion factors between M.K.S. and Gaussian units are derived for the benefit of students disadvantaged by have learned E.&M. using Gaussian units.

1. Electromagnetic Equations in M.K.S. and Gaussian Units

By now most accelerator papers use S.I. units, which in electromagnetic theory are usually referred to as M.K.S. units. It causes considerable difficulty when "fundamental physics" analyses are performed using c.g.s. Gaussian units and then the derived formulas have to be put to practical, "engineering" use. For this reason this packet will be devoted to the subject. Apart from this practical consideration, discussion of the units entering various formulas gives an excuse for reviewing various formualas that all students should have encountered in the past, but may have forgotten.

Units conversions in E. & M. are noticeably more confusing than units conversions from, say, inches to centimeters, or B.T.U.'s to joules, or dynes to newtons. The formulas of mechanics that include the quantities having these units, are identical in all systems of units. For example, $F = ma$ has the same *form* in all systems of units. Here the word *form* is emphasized since the word has a quite important technical sense when applied to a physical equation. This is most significant in relativity theory; the requirement that equations have the same form in different frames of reference already encompasses an appreciable fraction of the theory. Similarly one can say that Newton's law is form-invariant to change of units. In contrast the equations of E. & M. do not have the same form in Gaussian and M.K.S. units. (They are relativistically correct, however.) Another consideration affecting units and the form of the equations is that of "rationalization". There are only two systems in common use; c.g.s., Gaussian, unrationalized, (Gaussian for short) and M.K.S. , rationalized, (M.K.S. for short.)

By and large, people who apply the formulas of E. & M. for "engineering" applications use S.I. units. This is because the practical electric units of volts for voltage, coulombs for charge, amperes for current, joules for energy, watts for power, ohms for resistance, and so on, belong to the S.I. system. Because of this, most undergraduate E. & M. texts use S.I. units. Probably, therefore, most students are more comfortable with S.I. units than with Gaussian units. The following table gives conversion factors. Many of these will be derived and/or discussed in the rest of this packet.

TABLE 1. M.K.S. / Gaussian Conversion Factors. "3" $\equiv$ 2.99792456

Quantity	Symbol	M.K.S. $\equiv$ S.I. Unit	Factor	Gaussian Unit
Length	l	m	10^2	cm
Mass	m	kg	10^3	gm
Time	t	sec	1	sec
Frequency	ν	hertz(Hz)	1	1/s
Force	F	(N)ewton	10^5	dyne
Energy(work)	W	(J)oule	10^7	erg
Power	P	(W)att	10^7	erg-sec^{-1}
Charge	q	(C)oulomb	"3" $\times\ 10^9$	statcoul
Charge density	ρ	C-m^{-3}	"3" $\times\ 10^3$	statcoul-cm^{-3}
Current	I	(A)mp	"3" $\times\ 10^9$	statamp
Current density	J	A-m^{-2}	"3" $\times\ 10^5$	statamp-cm^{-2}
Electric field	E	V-m^{-1}	10^{-4}/"3"	statvolt-cm^{-1}
Electric potential	V	(V)olt	10^{-2}/"3"	statvolt
Polarization	P	C-m^{-2}	"3" $\times\ 10^5$	dip.mom.-cm^{-3}
Electric Displacement	D	c-m^{-2}	$4\pi\times$ "3" $\times\ 10^5$	statvolt-cm^{-1}
Conductivity	σ	mho-m^{-1}	"3" $\times$ "3" $\times\ 10^9$	sec-cm^{-1}
Resistance	R	ohm(Ω)	10^{-11}/("3" $\times$ "3")	sec-cm^{-1}
Capacitance	C	(F)arad	"3" $\times$ "3" $\times\ 10^{11}$	cm
Magnetic flux	ϕ	weber(Wb)	10^8	gauss-cm^2
Magnetic induction	B	(T)esla	10^4	gauss
Magnetic field	H	A-turn-m^{-1}	$4\pi\times\ 10^{-3}$	oersted
Magnetization	M	A-m^{-1}	10^{-3}	mag.mom.-cm^{-3}
Inductance	L	(H)enry	10^{-11}/("3" $\times$ "3")	Gaussian e.s.u.

2. Converting Maxwell's Equations to Gaussian Units

One reliable way to obtain correct equations in Gaussian units is to look them up in a reliable source such as Jackson. Nevertheless it is worthwhile to see what is involved in deriving the equations from first principles, (where "first principles" here means "starting with M.K.S. equations".) In M.K.S. units the Maxwell equations are

$$
\begin{aligned}
\nabla \times \mathbf{E} =& -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{B} =& \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}, \\
\nabla \cdot \mathbf{E} =& \frac{\rho}{\epsilon_0}, \\
\nabla \cdot \mathbf{B} =& 0,
\end{aligned}
\tag{2.1}
$$

where all symbols are assumed to be familiar. From now on the last equation will be dropped, since its units are unimportant.

Following Purcell, to shorten formulas, we introduce the conventional number "3" = 2.99792456 so that

$$
c = \sqrt{\frac{1}{\mu_0 \epsilon_0}} = \text{“3”} \times 10^8 \, \text{m/s} \, , \tag{2.2}
$$

and "9" = "3" × "3". In terms of the *defined* quantity μ_0, one easily obtains

$$
\begin{aligned}
\frac{\mu_0}{4\pi} &= 10^{-7} \, \frac{\text{H}}{\text{m}}, \\
4\pi\epsilon_0 &= \frac{10^7}{c^2} = \frac{10^{-9}}{\text{“9”}} \, \frac{\text{F}}{\text{m}}, \\
4\pi\epsilon_0 c &= \frac{10^7}{c} = \frac{10^{-1}}{\text{“3”}} \, \frac{\text{F}}{\text{s}}.
\end{aligned}
\tag{2.3}
$$

Note that μ_0 is just a number, arbitrary chosen, and not measurable, but *with units.*† In going from M.K.S. to Gaussian units three distinct steps have to be taken. We will proceed step by step, introducing new values which leave the numerical value of each term unchanged.

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† It is certainly an inelegant feature of M.K.S. units that an arbitrary number carrying dimensions appears in the most fundamental equations. This would not be necessary if **H** were the fundamental magnetic field, but it seems to be non-controversial that **B** is the *fundamental* magnetic field.

1. Multiply Eqs. (2.1) by factors such that the dependent variables become $10^{-3}\mathbf{E}4\pi\epsilon_0 c$, $10^{-3}\mathbf{B}4\pi/\mu_0$, $10\mathbf{J}c$ and $10\rho c$. Using Eqs. (2.3) this yields

$$\begin{aligned}
\nabla\times\left(10^{-3}\mathbf{E}4\pi\epsilon_0 c\right) &= -\frac{1}{c}\frac{\partial\left(10^{-3}\mathbf{B}4\pi/\mu_0\right)}{\partial t},\\
\nabla\times\left(10^{-3}\mathbf{B}4\pi/\mu_0\right) &= \frac{1}{c}\frac{\partial\left(10^{-3}\mathbf{E}4\pi\epsilon_0 c\right)}{\partial t} + 10^{-4}\frac{4\pi}{c}\left(10\mathbf{J}c\right),\\
\nabla\cdot\left(10^{-3}\mathbf{E}4\pi\epsilon_0 c\right) &= 10^{-4}4\pi\left(10\rho c\right),
\end{aligned} \tag{2.4}$$

which begin to resemble the Maxwell equations in Gaussian units, though all quantities are still in M.K.S. units.

2. It is impossible to "justify" the next step as being logical or well-motivated; it would probably not be accepted nowadays, but it is the step that Giorgi, the inventor of M.K.S. units, took. Though charge and current do not appear in the Maxwell equation, both charge density and current density do appear. Let us change the unit of charge by a dimensionless factor equal to "3" $\times 10^9$—even though this factor begs to be interpreted as ten times the speed of light in M.K.S. units, it is, to repeat, just a dimensionless number. It is this replacement of a dimensional quantity c, a velocity, by a dimensionless number that forces the occurrence in the electromagnetic equations of an arbitrary number, μ_0, which carries dimensions.

Introducing the symbols Q_g and I_g for charge and current measured in the new units we have

$$Q_g = \text{“3”}\times 10^9 Q, \quad I_g = \text{“3”}\times 10^9 I. \tag{2.5}$$

When measured in these units, but with all other units unchanged, let us use the symbols Q', I', $\mathbf{J}'$ and ρ' for the quantities proportional to charge. Also let us introduce abbreviations for the quantities in parentheses in Eqs. (2.4)

$$\begin{aligned}
\mathbf{E}_g &= 10^{-3}\mathbf{E}4\pi\epsilon_0 c = \frac{10^{-4}}{\text{“3”}}\mathbf{E},\\
\mathbf{B}_g &= 10^{-3}\mathbf{B}4\pi/\mu_0 = 10^4\mathbf{B}.
\end{aligned} \tag{2.6}$$

(For the moment the right hand equations, obtained using Eqs. (2.3) are superfluous, as it is the abbreviations on the left that are to be used.) Then the Maxwell equations become

$$\begin{aligned}
\nabla\times\mathbf{E}_g &= -\frac{1}{c}\frac{\partial\mathbf{B}_g}{\partial t},\\
\nabla\times\mathbf{B}_g &= \frac{1}{c}\frac{\partial\mathbf{E}_g}{\partial t} + 10^{-4}\frac{4\pi}{c}\mathbf{J}',\\
\nabla\cdot\mathbf{E}_g &= 10^{-4}4\pi\rho'.
\end{aligned} \tag{2.7}$$

Except for the novel unit of charge, these equations are still in M.K.S. units. In particular the unit of distance is the meter.

3. Consider next the effect of changing length units from meters to centimeters. If a term containing the derivative operator ∇ is reckoned with the unit of distance being the centimeter it will be altered by the factor 10^{-2}; to compensate ∇ would have to be replaced by $10^2\nabla$. But in the first equation the factor $1/c$ on the right hand side will change by the same factor 10^{-2} when c is reckoned in cm/sec. Therefore the first equation is unchanged in going from meters to centimeters.

The inhomogeneous terms (i.e. terms containing ρ or $\mathbf{J}$) in the other equations require different treatment. Since the units of charge density and current density depend on the unit of distance, these terms require extra factors. Since J is current per unit area, it needs to acquire a factor 10^4 for its numerical value not to change when J is reckoned on a centimeter basis. The J term therefore needs another factor 10^4. For the same reason the ρc term in the third equation needs a factor 10^6. The resulting equations are

$$\begin{aligned}
\nabla \times \mathbf{E}_g &= -\frac{1}{c_{\mathrm{cm/s}}}\frac{\partial \mathbf{B}_g}{\partial t}, \\
\nabla \times \mathbf{B}_g &= \frac{1}{c_{\mathrm{cm/s}}}\frac{\partial \mathbf{E}_g}{\partial t} + \frac{4\pi}{c_{\mathrm{cm/s}}}\mathbf{J}_g, \qquad (2.8) \\
\nabla \cdot \mathbf{E}_g &= 4\pi\rho_g,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{J}_g &= 10^{-4}\mathbf{J}' = \text{“3”} \times 10^5 \mathbf{J}, \\
\rho_g &= 10^{-6}\rho' = \text{“3”} \times 10^3 \rho, \qquad (2.9)
\end{aligned}$$

which include the appropriate conversion factors for transforming densities from meter to centimeter length units. (Again the right hand equations should be temporarily ignored.)

Since this completes the conversion from M.K.S. to Gaussian units, the subscripts can be dropped to give the Maxwell equation in Gaussian units

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \qquad (2.10) \\
\nabla \cdot \mathbf{E} &= 4\pi\rho,
\end{aligned}$$

Also Eq. (2.5) and the right hand sides of Eqs. (2.6) and (2.9) give some of the conversion factors contained in Table 1.

3. Converting Force Equations to Gaussian Units

The top few entries in Table 1, for length, mass, time, frequency, force, work and energy, and power come from mechanics and need no further explanation. Conversions of Q and I were defined by Eq. (2.5), and their densities ρ and J can be converted using Eq. (2.9). E and B are converted using Eq. (2.6). It is appropriate to survey some of the other equations of electromagnetism to confirm that no contradictions have arisen.

The most fundamental law is Coulomb's, which reads

$$F = \frac{Qq}{4\pi\epsilon_0 r^2} = \frac{Q_g q_g}{\text{“9”} \times 10^{18}} \frac{\text{“9”}}{10^{-9}} \frac{1}{10^{-4} r_g^2} = 10^{-5} \frac{Q_g q_g}{r_g^2}. \tag{3.1}$$

Refering to the force conversion in Table 1, we obtain the (well-known) Gaussian version of Coulomb's law.

$$F = \frac{Qq}{r^2}. \tag{3.2}$$

The very definition of electric field,

$$E = \frac{F}{q} = 10^{-5} F_g \frac{\text{“3”} \times 10^9}{q_g} = \text{“3”} \times 10^4 \frac{F_g}{q_g}, \tag{3.3}$$

along with Eq. (2.6), confirms that the definition $E = F/q$ *is* form invariant. (Suppressing angular dependence) the fundamental magnetic force (the Lorentz force) is given by

$$F = qvB = \frac{q_g}{\text{“3”} \times 10^9} 10^{-2} v_g 10^{-4} B_g = 10^{-5} q_g \frac{v_g}{\text{“3”} \times 10^{10}} B_g = 10^{-5} q_g \frac{v_g}{c_g} B_g. \tag{3.4}$$

One hates to replace a dimensionless (arbitrarily assigned) number by a dimensional (measurable) quantity, as has just been done, but in Gaussian units the magnetic force equation is therefore

$$F = q\frac{v}{c}B, \tag{3.5}$$

As the experimental value of the speed of light changes the ratio between units changes. It was therefore not correct to say that the factor “3” is arbitrarily assigned. Rather this factor has to be taken from the best known value of the speed of light (with power of 10 such that the value is close to 3). The fact that the Lorentz force equation is *not* form invariant is (at least philosophically) a troublesome aspect of the change of units. For a fully relativistic particle the Gaussian unit relation $F = qB$ is independent of the speed of light while the M.K.S. unit relation $F = qcB$ depends on the speed of light. This makes it very confusing to decide which equations are “fundamental” and which are “derived”.

4. Integral Relations

Gauss's law is the integral equation equivalent of the third Maxwell equation. Also it is equivalent to Coulomb's law. By now we have learned how to convert both of these, so conversion is straightforward. The M.K.S. version is

$$\int_{\mathcal{A}} \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho dV, \tag{4.1}$$

The left side gives the electric flux through closed surface $\mathcal{A}$, and the right side is the charge contained in the volume $\mathcal{V}$ bounded by $\mathcal{A}$, divided by ϵ_0. The Gaussian version is

$$\int_{\mathcal{A}} \mathbf{E} \cdot d\mathbf{A} = 4\pi \int_{\mathcal{V}} \rho. \tag{4.2}$$

The main purpose of "rationalization" was to achieve the dubious virtue of moving 4π from Gauss's law to Coulomb's law; it is the same 4π that enters the formula for the area of a sphere. In optics and microwaves, where there are no free charges, and where Maxwell's equations are everywhere, this is a good bargain. Perhaps this accounts for the universal adoption of M.K.S. units (in the 1930's and 1940's I believe) by radio physicists and engineers.

Ampère's Law. In M.K.S. units Ampère's Law in free space is

$$\oint_{\mathcal{C}} \mathbf{H} \cdot \mathbf{dl} = \oint_{\mathcal{C}} \frac{\mathbf{B}}{\mu_0} \cdot \mathbf{dl} = I; \tag{4.3}$$

the line integral of $\mathbf{H}$ along closed curve $\mathcal{C}$ is equal to the current I "linked" by $\mathcal{C}$. The usual manipulation yields the Gaussian unit version,

$$\oint_{\mathcal{C}} \mathbf{B} \cdot \mathbf{dl} = \frac{4\pi}{c} I. \tag{4.4}$$

Vector Potential. By definition, in both M.K.S. and Gaussian units, the vector potential $\mathbf{A}$ is related to magnetic field $\mathbf{B}$ by

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{4.5}$$

Free Space Energy Density. The usual manipulations, starting with the M.K.S. version of energy densities

$$u_E = \frac{1}{2}\epsilon_0 E^2, \quad u_M = \frac{1}{2\mu_0} B^2, \tag{4.6}$$

yield the Gaussian versions

$$u_E = \frac{1}{8\pi} E^2, \quad u_M = \frac{1}{8\pi} B^2. \tag{4.7}$$

5. Energy and Momentum of Atomic Particles

It is conventional and convenient to measure the energies of atomic particles in electron-volts, a unit that belongs neither to the M.K.S. nor to the Gaussian system. At the same time, it is convenient, at least in relativistic situations, to express particle masses not as masses but as rest energies; e.g. for electrons and protons

$$\begin{aligned} m_e c^2 &= 0.511003 MeV \\ m_p c^2 &= 938.28 MeV. \end{aligned} \tag{5.1}$$

Energies, masses and momenta are related by

$$\mathcal{E}^2 = p^2c^2 + m^2c^4. \tag{5.2}$$

For connecting these quantities with electromagnetic quantities, it is convenient to group these quantities as $\mathcal{E}/e$, pc/e, and mc^2/e, all of which are measured in volts, and can hence be regarded as belonging to the S.I. system. Thus, for example, to obtain the bending radius ρ, of a particle of momentum p, in a magnetic field B, the formula to use in M.K.S. units reads

$$\rho\,[\mathrm{m}] = \frac{(pc/e)\,[\mathrm{V}]}{c\,[\mathrm{m/s}]\,B\,[\mathrm{T}]}. \tag{5.3}$$

Some accelerator physicists prefer to rearrange this equation to work with a kind of "particle momentum" called the $B\rho$ value,

$$B\,[\mathrm{T}]\,\rho\,[\mathrm{m}] = (p/e)\,[\mathrm{S.I.}\ ]\,, \tag{5.4}$$

where the right hand side is worked out in S.I. units initially, and then subsequent relations are expressed in terms of $B\rho$.