ACTION/ANGLE VARIABLES AND ADIABATIC INVARIANCE

Richard Talman

Laboratory of Nuclear Studies
Cornell University
Ithaca, NY 14853

ABSTRACT

This is one packet of notes accompanying a course *Mechanics and Electromagnetism in Accelerators,* offered as part of the U.S. Particle Accelerator School, Yale University, summer, 2002. The subject matter is *action/angle variables* and *adiabatic invariance,* the latter of which is perhaps the most essential concept of accelerator physics; at least it is the least understood of the essential concepts. This material is extracted, with little modification, from my textbook, *Geometric Mechanics.* Though there is a certain amount of introductory material some references may be unclear. Following traditional discussions of Hamiltonian mechanics the independent variable is taken to be the time t. Essentially the same equation will apply to particle trajectories in an accelerator, but with t replaced by longitudinal coordinate s.

Analytic Basis for Approximation

Once equations of motion have been found they can usually be solved by straightforward numerical methods but numerical results rarely provide much general insight and it is productive to develop analytic results to the extent possible. Since it is usually believed that the most essential "physics" is Hamiltonian, considerable effort is justified in advancing the analytic formulation to the extent possible without violating Hamiltonian requirements. One must constantly ask "is it symplectic".

In this section the method of canonical transformation will be introduced and then exercised by being applied to nonlinear oscillators. Oscillators of one kind or another are probably the most ubiquitous systems analysed using classical mechanics. Some, such as relaxation oscillators, are inherently non-sinuisoidal, but many exhibit motion that is approximately simple harmonic. Some of the sources of deviation from harmonicity are (usually weak) damping, Hooke's law violating restoring forces, and parametric drive. Hamiltonian methods, and in particular phase space representation, are especially effective at treating these systems, and adiabatic invariance, to be defined shortly, is even more important than energy conservation.

As well as the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, Hamiltonian mechanics inherits from Lagrangian mechanics the definition of momenta

$$p_i = \frac{\partial L\left(\mathbf{q}, \dot{\mathbf{q}}, t\right)}{\partial \dot{q}^i}. \tag{1}$$

The Hamiltonian itself is *defined* by

$$H\left(\mathbf{q}, \mathbf{p}, t\right) \equiv p_i q^i - L\left(\mathbf{q}, \dot{\mathbf{q}}\left(\mathbf{q}, \mathbf{p}, t\right), t\right). \tag{2}$$

(Summation over indices is implied in the term $p_i q^i$). The only aspect of this that is not entirely trivial is that, as indicated in the argument lists, velocities $\dot{\mathbf{q}}$ *have to be* eliminated in favor of the momentum variables p_i.

CANONICAL TRANSFORMATIONS

The action as a generator of canonical transformations

We have encountered the Jacobi method within the Hamilton-Jacobi theory while developing analogies between optics and mechanics. But it is possible to come upon this procedure more formally while developing the theory of "canonical transformation" which means transforming the equations in such a way that Hamilton's equations remain valid. The motivation for restricting the field of acceptable transformations in this way is provided by the large body of certain knowledge one has about Hamiltonian systems, much of it described in the previous chapter.

From a Hamiltonian system initially described by "old" coordinates $q^1, q^2, \cdots, q^n$ and "old" momenta $p_1, p_2, \cdots, p_n$ we seek appropriate transformations

$$\left(q^1, q^2, \cdots, q^n; p_1, p_2, \cdots, p_n\right) \rightarrow \left(Q^1, Q^2, \cdots, Q^n; P_1, P_2, \cdots, P_n\right), \tag{3}$$

to "new coordinates" Q^1, Q^2, $\cdots$, Q^n and "new momenta" P_1, P_2, $\cdots$, P_n.[†] (Within the Jacobi procedure these would have been known as β-parameters and α-parameters, respectively.)

Within Lagrangean mechanics we have seen the importance of variational principles in establishing the invariance to coordinate transformation of the form of the Lagrange equations. Since we have assigned ourselves essentially the same task in Hamiltonian mechanics it is appropriate to investigate Hamiltonain variational principles. This method will prove to be successful in establishing conditions that must be satisfied by the new $\mathbf{Q}$ and $\mathbf{P}$ variables.

In the context of Hamiltonian mechanics, the "action principle" asserts that the "Hamilton integral"

$$\text{H.I.} = \int_{P_1}^{P_2} \left(p_i dq^i - H\left(\mathbf{q}, \mathbf{p}, t\right) dt\right); \tag{4}$$

is stationary for the actual motion of the system. Other than starting at P_1 and ending at P_2 (and not being "pathological") the path of integration is arbitrary in the extended phase space q^i, p_i and t.

[†] It would be consistent with the more formally correct mathematical notation introduced previously to use the symbol $\widetilde{\mathbf{p}}_i$ for momentum p_i since the momenta are more properly thought of as *forms*, but this is rarely done.

H.I. has the dimensions of action and we now subject it to analysis something like that used in deriving the Lagrange equations from $\int L dt$. In particular we seek the integration path for which H.I. achieves an *extreme* value. In contrast to coordinate, velocity space where Hamilton's principle is applied in Lagrangian mechanics, consider independent smooth phase space variations ($\delta\mathbf{q}$,$\delta\mathbf{p}$) away from an arbitrary integration path through fixed end points (P_1, t_1) and (P_2, t_2). (Forgive the fact that P is being used both for momentum components and to label end points.) Evaluating the variations of its two terms individually, the condition for H.I. to achieve an extreme value is

$$0 = \int_{P_1,t_1}^{P_2,t_2} \left(\delta p_i \, dq^i + p_i d\left(\delta q^i\right) - \frac{\partial H}{\partial q^i} \delta q^i \; dt - \frac{\partial H}{\partial p_i} \delta p_i \; dt \right). \tag{5}$$

The last two terms come from $\int H dt$ just the way two terms come from $\int L dt$ in a Lagrangean derivation. Where the first two terms come from is illustrated in Fig. 1. At each point on the unvaried curve incremental displacements $\delta q(q)$ and $\delta p(q)$ locate points on the varied curve. Since the end points are fixed the deviation δp vanishes at the ends and $d(\delta q^i)$ must average to zero as well as vanishing at the ends.

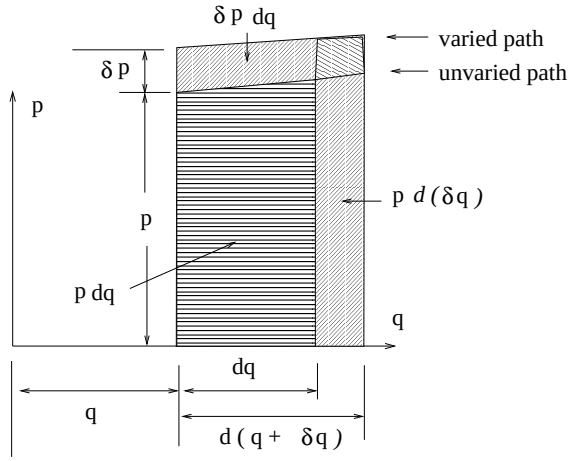


Figure 1: Areas representing terms $\delta p\, dq + p d(\delta q)$ in the Hamiltonian variational integral.

With a view toward obtaining a common multiplicative factor in the integrand, using the fact that the end points are fixed, the factor $p_i d(\delta q^i)$ can be replaced by $-\delta q^i \, dp_i$ since the difference $d(p_i \delta q^i)$ is a total differential. Then, since the variations δq^i and δp_i are

arbitrary, Hamiltons equations follow;

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{6}$$

It has therefore been proved that Hamilton's equations are implied by applying the variational principle to integral H.I. But that has not been our real purpose. Rather, as stated previously, our purpose is to derive canonical transformations. Toward that end we introduce† an arbitrary function $G(\mathbf{q}, \mathbf{Q}, t)$ of old coordinates $\mathbf{q}$ and new coordinates $\mathbf{Q}$ and alter H.I. slightly by subtracting the total derivative dG from its integrand;

$$\begin{aligned} \text{H.I.}' &= \int_{P_1}^{P_2} \left(p_i dq^i - H dt - dG\left(\mathbf{q}, \mathbf{Q}, t\right) \right) \\ &= \int_{P_1}^{P_2} \left(p_i dq^i - H dt - \frac{\partial G}{\partial q^i} dq^i - \frac{\partial G}{\partial Q_i} dQ_i - \frac{\partial G}{\partial t} dt \right). \end{aligned} \tag{7}$$

This alteration cannot change the extremal path obtained by applying the same variational principle since the integral over the added term is independent of path. We could subject H.I.$'$ to a variational calculation like that applied to I but instead we take advantage of the fact that G is arbitrary to simplify the integrand by imposing on it the condition

$$p_i = \frac{\partial G\left(\mathbf{q}, \mathbf{Q}, t\right)}{\partial q^i}. \tag{8}$$

This simplifies Eq. (7) to

$$\text{H.I.}' = \int_{P_1}^{P_2} \left(P_i dQ_i - H' dt \right), \tag{9}$$

where we have introduced the abbreviations

$$P_i = -\frac{\partial G\left(\mathbf{q}, \mathbf{Q}, t\right)}{\partial Q^i}, \quad \text{and} \quad H'\left(\mathbf{Q}, \mathbf{P}, t\right) = H + \frac{\partial G}{\partial t}. \tag{10}$$

The former equation, with Eq. (8), define the coordinate transformation and the latter equation gives the Hamiltonian in the new coordinates. The motivation for this choice of transformation is that Eq. (9) has the same form in the new variables that Eq. (4) had in the old variables. The equations of motion are therefore

$$\dot{Q}^i = \frac{\partial H'}{\partial P_i}, \quad \text{and} \quad \dot{P}_i = -\frac{\partial H'}{\partial Q^i}. \tag{11}$$

† Goldstein uses the notation $F_1(\mathbf{q}, \mathbf{Q}, t)$ for our function $G(\mathbf{q}, \mathbf{Q}, t)$.

Since these are Hamilton's equations in the new variables we have achieved our goal. The function $G(\mathbf{q}, \mathbf{Q}, t)$ is known as the "generating function" of the canonical transformation defined by Eq. (8) and the first of Eqs. (10). The transformations have a kind of hybrid form (and it is an inelegance inherent to the generating function procedure) with G depending as it does on *old* coordinates and *new* momenta. Also there is still "housekeeping" to be done, expressing the new Hamiltonian H' in terms of the new variables, and there is no assurance that it will be possible to do this in closed form.

Though G could have been any function consistent with Eq. (8), if we conjecture that G is a solution of the H-J equation $H + \partial G/\partial t = 0$ we note from Eq. (10) that the new Hamiltonian is given by $H' = 0$. Nothing could be better than a vanishing Hamiltonian since, by Eqs. (11), it implies the new coordinates and momenta are constants of the motion. Stated conversely, if we had initially assigned ourselves the task of finding coordinates that were constants of the motion we would have been led to the Hamilton-Jacobi equation as the condition to be applied to generating function G.

The other equation defining the canonical transformation is the first of Eqs. (10)

$$P_i = -\frac{\partial G\left(\mathbf{q}, \mathbf{Q}, t\right)}{\partial Q^i}. \tag{12}$$

Without being quite the same, this relation resembles the Jacobi-prescription formula $\beta = \partial S/\partial \alpha$ for extracting constant of the motion β corresponding to separation constant α in a complete integral of the H-J equation. It is certainly true that if G is a complete integral and the P_i are interpreted as the separation constants in that solution then the quantities defined by Eq. (12) *are* constants of the motion. But, relative to the earlier procedure, coordinates and momenta are interchanged. The reason is that the second arguments of G have been taken to be coordinates rather than momenta.

We are therefore motivated to subtract the total differential of an arbitrary function $dS(\mathbf{q}, \mathbf{P}, t)$[†] (or rather, for reasons that will become clear immediately, the function $d(S -$

[†] Goldstein uses the notation $F_2(\mathbf{q}, \mathbf{P}, t)$ for our function $S(\mathbf{q}, \mathbf{P}, t)$. This function is also known as "Hamilton's principal function". Other generating functions, $F_3(\mathbf{p}, \mathbf{Q}, t)$ and $F_4(\mathbf{p}, \mathbf{P}, t)$ in Goldstein's notation , can also be used.

$P_iQ^i)$) from the variational integrand;

$$\begin{aligned} \text{H.I.}' &= \int_{P_1}^{P_2} \left(p_i dq^i - Hdt - \frac{\partial S}{\partial q^i} dq^i - \frac{\partial S}{\partial P_i} dP_i - \frac{\partial S}{\partial t} dt + P_i dQ^i + Q^i dP_i \right) \\ &= \int_{P_1}^{P_2} \left(P_i dQ_i - H' dt \right), \end{aligned} \tag{13}$$

where we have required

$$p_i = \frac{\partial S\left(\mathbf{q}, \mathbf{P}, t\right)}{\partial q^i}, \quad Q^i = \frac{\partial S\left(\mathbf{q}, \mathbf{P}, t\right)}{\partial P_i}, \quad \text{and} \quad H'\left(\mathbf{Q}, \mathbf{P}, t\right) = H + \frac{\partial S}{\partial t}. \tag{14}$$

(It was only with the extra subtraction of $d(P_iQ^i)$ that the required final form was obtained.) We have now reconstructed the entire Jacobi prescription. If $dS(\mathbf{q}, \mathbf{P}, t)$ is a complete integral of the H-J equation, with the P_i defined to be the α_i separation constants, then the $\beta_i \equiv Q^i$ obtained from the second of Eqs. (14) are constants of the motion.

Recapitulating: a complete integral of the H-J equation provides a generator for performing a canonical transformation to new variables for which the Hamiltonian has the simplest conceivable form—it vanishes—causing all coordinates and all momenta to be constants of the motion.

Time-independent canonical transformation

Just as the Hamilton-Jacobi equation is the short-wavelength limit of the Schrödinger equation, the time-independent H-J equation is the same limit of the time-independent Schrödinger equation. As in the quantum case, methods of treating the two cases appear to be rather different even though time independence is just a special case.

When it does not depend explicitly on time, the Hamiltonian is conserved, $H(\mathbf{q}, \mathbf{p}) = E$ and a complete integral of the H-J equation takes the form

$$S\left(\mathbf{q}, t\right) = S_0\left(\mathbf{q}, \mathbf{P}\right) - E\left(t - t_0\right), \tag{15}$$

where the independent parameters are listed as $\mathbf{P}$. The term *action*, applied to S up to this point, is commonly also used to refer to S_0. † In this case the H-J becomes

$$H\left(\mathbf{q}, \frac{\partial S_0}{\partial \mathbf{q}}\right) = E, \tag{16}$$

† Goldstein uses the notation $W\left(\mathbf{q}, \mathbf{P}\right)$ for our function $S_0(\mathbf{q}, \mathbf{P})$. This function is also known as "Hamilton's characteristic function". Landau and Lifshitz call S_0 the "abbreviated action".

and a complete integral is defined to be a solution of the form

$$S_0 = S_0\left(\mathbf{q}, \mathbf{P}\right) + \text{const.}, \tag{17}$$

with as many new parameters P_i as there are coordinates. It is important to recognize though that the energy E can itself be regarded as a Jacobi parameter, in which case the parameter set $\mathbf{P}$ is taken to include E.

In this time-independent case it is customary to use $S_0(\mathbf{q}, \mathbf{P})$ (rather than $S(\mathbf{q}, \mathbf{P}, t)$) as the canonical generating function G. By the general theory (Eq. (15)), new variables are then related to old by

$$p_i = \frac{\partial S_0}{\partial q^i}, \quad Q^i = \frac{\partial S_0}{\partial P_i}. \tag{18}$$

In particular, taking E itself as one of the new momentum, its corresponding new coordinate is

$$Q_E = \frac{\partial S_0}{\partial E}, \tag{19}$$

which is non-vanishing since the parameter set $\mathbf{P}$ includes E. Defined in this way Q_E is therefore *not constant*. The quantity whose constancy *is* assured by the Jacobi theory is

$$\frac{\partial S}{\partial E} = Q_E - t + t_0 = \text{ constant}. \tag{20}$$

This show that Q_E and time t are essentially equivalent, differing at most by the choice of what constitutes initial time. Eq. (20) is the basis of the statement that E and t are canonically conjugate variables. Continuing with the canonical transformation, the new Hamiltonian is

$$H'\left(\mathbf{Q}, \mathbf{P}, t\right) = H + \frac{\partial S_0}{\partial t} = E. \tag{21}$$

We have obtained the superficially curious result that in this simpler, time-independent, case the Hamiltonian is less simple, namely non-vanishing, than in the time-dependent case. This is due to our use of S_0 rather than S as generating function. But H' is constant, which is good enough.†

We can already test one of the Hamilton equations, namely the equation for $\dot{Q}_E$,

$$\dot{Q}_E = \frac{\partial H'}{\partial E} = 1, \tag{22}$$

† When applying the Jacobi prescription in the time-independent case one must be careful not to treat E as functionally dependent on any of the other P_i though.

in agreement with Eq. (20). For the other momenta, not including E, Hamilton's equations are

$$\dot{P}_i = 0, \quad \text{and} \quad \dot{Q}^i = \frac{\partial E}{\partial P_i} = 0. \tag{23}$$

Hence finding a complete integral of the time-independent H-J equation is tantamount to having solved the problem.

Hamilton-Jacobi treatment of 1D simple harmonic motion

Though it is nearly the most elementary conceivable system, the one dimensional simple harmonic oscillator is basic to most oscillations and provides a simple illustration of the Jacobi procedure. This formalism may initially seem a bit "heavy" for such a simple problem, but the entire theory of adiabatic invariance follows directly from it and nonlinear oscillations cannot be satisfactorily analysed without this approach. The Hamiltonian is

$$H\left(q,p\right) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2. \tag{24}$$

This yields as the (time-independent) H-J equation

$$\frac{1}{2m}\left(\frac{dS_0}{dq}\right)^2 + \frac{1}{2}m\omega_0^2 q^2 = E, \tag{25}$$

which can be solved to give

$$S_0\left(q,E\right) = m\omega_0 \int_0^q \sqrt{\frac{2E}{m\omega_0^2} - q'^2}\; dq'. \tag{26}$$

(The lower limit has been picked arbitrarily.) It will be necessary to handle the ± 1 ambiguity coming from the square root on an *ad hoc* basis; here the positive sign has been chosen. This is a *complete integral* in that it depends on E, which we now take as the first (and only) "Jacobi momentum" that would previously have been denoted by α_1 (or $-\alpha_1$). Following the Jacobi procedure we next find β_1, but which we will now call Q, or Q_E since it is to be the "new coordinate" corresponding to E. (If we were to insist on conventional terminology we would also introduce a "new momentum" $P \equiv E$.) That is, we are performing a transformation of phase space variables $(x,p) \to (Q,P)$. Since the main purpose of $S_0(q,E)$ is to be differentiated, explicit evaluation of the integral in Eq. (26) may not be necessary, but for definiteness the result is

$$S_0\left(q,E\right) = \frac{m\omega_0}{2} q\sqrt{\frac{2E}{m\omega_0^2} - q^2} + \frac{E}{\omega_0}\sin^{-1}\left(\sqrt{\frac{m\omega_0^2}{2E}}\; q\right). \tag{27}$$

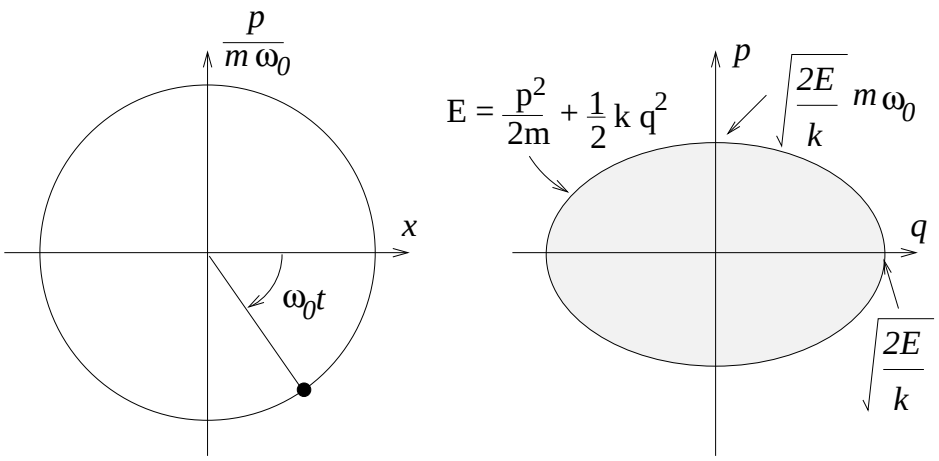


Figure 2: The phase space trajectory of simple harmonic motion is a circle traversed at constant angular velocity ω_0 if the axes are q and $p/(m\omega_0)$. The shaded area enclosed within the trajectory for one cycle of the motion in q and p *phase space* is $2\pi I$ where I is the "action".

By Jacobi's defining equation for Q_E we have

$$Q_E = \frac{\partial S_0}{\partial E} = \frac{1}{\omega_0}\int_0^q \frac{1}{\sqrt{\frac{2E}{m\omega_0^2} - q'^2}}\, dq' = \frac{1}{\omega_0}\sin^{-1}\left(\sqrt{\frac{m\omega_0^2}{2E}}\; q\right). \tag{28}$$

As previously warned, it is not obvious that Q_E is a linear function of t but from the general theory we know this to be the case;

$$Q_E = t - t_0. \tag{29}$$

Combining Eqs. (28) and (29) yields

$$q = \sqrt{\frac{2E}{m\omega_0^2}}\;\sin\;\omega_0\,(t - t_0)\,, \tag{30}$$

which begins to look familiar. The corresponding variation of p is given by

$$p = \frac{\partial S_0}{\partial q} = \pm m\omega_0\sqrt{\frac{2E}{m\omega_0^2} - q^2} = \sqrt{2mE}\;\cos\;\omega_0\,(t - t_0)\,. \tag{31}$$

Phase space plots of the motion are shown in Fig. 2. From considerations of continuity in this figure it has been necessary to restore the $\pm$ options for the square root that entered in the first place. The trajectory equation is

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2. \tag{32}$$

ACTION-ANGLE VARIABLES

The action variable of a simple harmonic oscillator

The variation of action S along a true trajectory is given by

$$dS = p_i dq^i - H dt, \quad \text{or} \quad S(P) = \int_{P_0}^{P} \left(p_i dq^i - H dt \right). \tag{33}$$

Applying this formula to the simple harmonic oscillator, since the path of integration is a true particle trajectory, $H = E$, the second term integrates to $-E(t - t_0)$. Comparing with Eq. (15), we obtain for the abbreviated action $S_0(\mathbf{q}) = \int_{P_0}^{P} p_i dq^i$, or in one dimension

$$S_0(q) = \int_{q_0}^{q} p\left(q'\right) dq'. \tag{34}$$

The word "action" has already been used to define the basic Lagrangian variational integral and as a name for the function satisfying the H-J equation, but it now now acquires yet another meaning as "$1/2\pi$ times the phase space area enclosed after one cycle". Because this quantity will be used as a dynamic variable it is called the "action variable" I of the oscillator.† For simple harmonic motion

$$I = \frac{1}{2\pi} \oint p\left(q'\right) dq' = \frac{1}{2\pi} \int\int dp\, dq = \frac{1}{2\pi} \pi \sqrt{2mE} \sqrt{\frac{2E}{m\omega_0^2}} = \frac{E}{\omega_0}. \tag{35}$$

The first form of integral here is a line integral along the phase space trajectory, the second is the area in (q, p) phase space enclosed by that curve. In quantum mechanics Planck's constant h specifies a definite area of phase space and the number of quantum states is given by $\int\int dp\, dq/h$. (Reviewing the solution of the Schrödinger equation for a particle in a box would confirm this at least approximately.) Commonly units are employed for which $\hbar = h/(2\pi) = 1$ and in those units the number of states is given by $\frac{1}{2\pi}\int\int dp\, dq$. This is a possible justification for, or at least mnemonic to remember, the factor $1/(2\pi)$ entering the conventional definition of I. This factor will also give the "right" period, namely 2π, for the motion expressed in terms of "angle variables" (to be introduced shortly).

† The terminology is certainly strained since I is usually called the "action variable", in spite of the fact that it is constant, but "variable" does not accompany "action" when describing S_0 which actually does vary. Next we will consider a situation in which I might be expected to vary, but will find (to high accuracy) that it does not. Hence the name "action non-variable" would be more appropriate. Curiously enough the word "amplitude" in physics suffers from the same ambiguity; in the relation $x = a\cos\omega t$ it is ambiguous whether the "amplitude" is x or a.

Adiabatic invariance of the action I

Consider a one dimensional system which is an "oscillator" in the sense that coordinate q returns to its starting point at some time. If the Hamiltonian is time-independent, the energy is conserved, and the momentum p returns to its initial value when q does. In this situation, the area within the phase space trajectory is closed and the action variable I just introduced is unambiguously defined.

Suppose however that the Hamiltonian $H(q,p,t)$, and hence the energy $E(t)$ have a weak dependence on time that is indicated by writing

$$E\left(t\right) = H\left(q,p,t\right) = H\left(q,p,\lambda\left(t\right)\right). \tag{36}$$

The variable λ has been introduced artificially to consolidate whatever time dependence exists into a single parameter for purposes of the following discussion. At any time t the energy $E(t)$ is *defined* to have the value it would have if $\lambda(t)$ were held constant at its current instantaneous value. Any nonconstancy of $E(t)$ reflects the time dependence of H. The prototypical example of this sort of time dependency is *parametric variation*—for example, the "spring constant" k, a "parameter" in simple harmonic motion, might vary slowly with time, $k = k(t)$. Eventually what constitutes "slow" will be made more precise but, much like short wavelength approximations previously encountered, the fractional change of frequency during one oscillation period is required to be small. Motion with λ fixed/variable will be called "unperturbed/perturbed."

During *perturbed* motion the particle energy,

$$E\left(t\right) = H\left(q,p,\lambda\left(t\right)\right), \tag{37}$$

varies, possibly increasing during some parts of the cycle and decreasing during others, and probably accumulating appreciably over many cycles. We are now interested in the systematic or averaged-over-one-cycle variation of quantities like $E(t)$ and $I(t)$.

The "time average" $\overline{f(t)}$ of a variable $f(t)$ that describes some property of a periodic oscillating system having period T is defined to be

$$\overline{f\left(t\right)} = \frac{1}{T}\int_{t}^{t+T} f\left(t'\right)dt'. \tag{38}$$

From here on we take $t = 0$.

Let us start by estimating the rate of change of E as λ varies. Since $\lambda(t)$ is assumed to vary slowly and monitonically over many cycles, its average rate of change $\overline{d\lambda/dt}$ and its instantaneous rate of change $d\lambda/dt$ differ negligibly, making it unnecessary to distinguish between these two quantities. But the variation of E will tend to be correlated with the instantaneous values of q and p so E can be expected to be above average at some times and below average at others. We seek the time-averaged value $\overline{dE/dt}$. To a lowest approximation we anticipate $\overline{dE/dt} \sim d\lambda/dt$ unless it should happen (which it won't) that dE/dt vanishes to this order of approximation.

Two features that complicate the present calculation are that the perturbed period T is in general different from the unperturbed period and that the phase space orbit is not in general closed so its enclosed area is poorly defined. To overcome these problems the integrals will be recast as integrals over one cycle of coordinate q, since q necessarily returns to its starting value, say $q = 0$. (We assume $\dot{q}(t = 0) \neq 0$.) The action variable

$$I\left(E, \lambda\right) = \frac{1}{2\pi} \oint p\left(q, E, \lambda\right) dq \tag{39}$$

is already written in this form.

From Eq. (37) the instantaneous rate of change of energy is given by

$$\frac{dE}{dt} = \frac{\partial H}{\partial \lambda} \frac{d\lambda}{dt}, \tag{40}$$

and its time average is therefore given by

$$\overline{\frac{dE}{dt}} = \frac{d\lambda}{dt} \frac{1}{T} \int_0^T \frac{\partial H}{\partial \lambda} dt. \tag{41}$$

(Because of the assumed slow, monotonic variation of $\lambda(t)$ it is legitimate to move the $\frac{d\lambda}{dt}$ factor outside the integral in this way.) To work around the dependence of T on λ we recast this expression in terms of phase space line integrals. Using Hamilton's equations we obtain

$$dq = \left.\frac{\partial H}{\partial p}\right|_{q,\lambda} dt, \quad \text{and hence} \quad T = \oint \frac{1}{\partial H/\partial p|_{q,\lambda}} dq. \tag{42}$$

Here we must respect the assumed functional form $H(q, p, t)$ and, to emphasize the point, have indicated explicitly what variables are being held constant for the partial differentiation. (To be consistent we should have similarly written $\partial H/\partial \lambda|_{q,p}$ in the integrand

of Eq. (41).) Making the same substitution (42) in the numerator, formula (41) can be written

$$\overline{\frac{dE}{dt}} = \frac{d\lambda}{dt} \oint \frac{\partial H/\partial\lambda|_{q,p}}{\partial H/\partial p|_{q,\lambda}} dq \Big/ \oint \frac{1}{\partial H/\partial p|_{q,\lambda}} dq. \tag{43}$$

Since this expression is already proportional to $d\lambda/dt$ which is the order to which we are working, it is legitimate to evaluate the two integrals using the unperturbed motion. Terms neglected by this procedure are proportional to $d\lambda/dt$ and give only contributions of order $(d\lambda/dt)^2$ to $\overline{dE/dt}$. (This is the sort of maneuver that one always resorts to in perturbation theory.)

The unperturbed motion is characterized by functional relation (37) and its "inverse"

$$E = H\left(q, p, \lambda\right), \quad \text{and} \quad p = p\left(q, \lambda, E\right), \quad \text{or} \quad E = H\left(q, p\left(q, \lambda, E\right), \lambda\right). \tag{44}$$

From now on, since λ is constant because unperturbed motion is being described, it will be unnecessary to list it among the variables being held fixed during differentiation. Differentiating the third formula with respect to E yields

$$\frac{1}{\partial H/\partial p|_q} = \left.\frac{\partial p}{\partial E}\right|_q, \tag{45}$$

which provides a more convenient form for one of the factors appearing in the integrands of Eq. (43). Differentiating the third of Eqs. (44) with respect to λ yields

$$0 = \left.\frac{\partial H}{\partial p}\right|_{q,\lambda} \left.\frac{\partial p}{\partial \lambda}\right|_{q,E} + \left.\frac{\partial H}{\partial \lambda}\right|_{q,p}, \quad \text{or} \quad \frac{\partial H/\partial\lambda|_{q,p}}{\partial H/\partial p|_{q,\lambda}} = -\left.\frac{\partial p}{\partial \lambda}\right|_{q,E}. \tag{46}$$

Finally, substituting these expressions into Eq. (43) yields

$$\overline{\frac{dE}{dt}} = -\frac{d\lambda}{dt}\,\frac{1}{T} \oint \left.\frac{\partial p}{\partial \lambda}\right|_{q,E} dq \tag{47}$$

As stated previously, the integral is to be performed over the presumed-to-be-known unperturbed motion.

We turn next to the similar calculation of $\overline{dI/dt}$. Differentiating Eq. (39) with respect to t, using Eq. (40) and the first of Eqs. (42) yields

$$\begin{aligned} \overline{\frac{dI}{dt}} &= \frac{d\lambda/dt}{2\pi} \oint \left(\left.\frac{\partial p}{\partial E}\right|_q \left.\frac{\partial H}{\partial \lambda}\right|_{q,p} + \left.\frac{\partial p}{\partial \lambda}\right|_{q,E} \right) dq \\ &= \frac{d\lambda/dt}{2\pi} \oint \frac{\partial H/\partial\lambda|_{q,p}}{\partial H/\partial p|_q}\, dq + \frac{d\lambda/dt}{2\pi} \oint \left.\frac{\partial p}{\partial \lambda}\right|_{q,E} dq. \end{aligned} \tag{48}$$

From the second of Eqs. (46) it can then be seen that

$$\overline{\frac{dI}{dt}} = 0. \tag{49}$$

Of course this is only approximate since terms of order $(d\lambda/dt)^2$ have been dropped. Even so this is one of the most important formulas in mechanics. It is usually stated as *the action variable is an adiabatic invariant.* That this is not an *exact* result might be regarded as detracting from its elegance, utility and importance. In fact the opposite is true since, as we shall see, it is often an *extremely accurate* result, with accuracy in parts per million not uncommon. This would make it perhaps unique in physics—an approximation that is as good as an exact result—except that the same thing can be said for the whole of Newtonian mechanics. It is still possible for I to vary throughout the cycle, as an example in section 6 will show, but its average is constant.

There is an important relation between action I and period T (or equivalently frequency $\omega = 2\pi/T$) of an oscillator. Differentiating the defining equation (39) for I with respect to E, and using Eq. (45) and Eq. (42) yields

$$\frac{\partial I}{\partial E} = \frac{1}{2\pi}\oint \frac{\partial p}{\partial E}\Big|_{q,\lambda} dq = \frac{1}{2\pi}\oint \frac{dq}{\partial H/\partial p}\Big|_{q,\lambda} = \frac{1}{2\pi}\oint dt = \frac{T}{2\pi} = \frac{1}{\omega}. \tag{50}$$

This formula can be checked immediately for simple harmonic motion. In Eq. (35) we had $I = E/\omega_0$ and hence

$$\frac{\partial I}{\partial E} = \frac{1}{\omega_0} = \frac{T}{2\pi}. \tag{51}$$

Recapitulating, we have considered a system with weakly time-dependent Hamiltonian H, with initial energy E_0 determined by initial conditions. Following the continuing evolution of the motion, the energy, because it is not conserved, may have evolved appreciably to a different value E. Accompanying the same evolution, other quantities such as (*a priori*) action I and oscillation period T also vary. The rates dE/dt, dI/dt, $d\lambda/dt$, *etc.* are all proportional to $d\lambda/dt$—doubling $d\lambda/dt$, doubles all rates for small $d\lambda/dt$. Since these rates are all proportional, it should be possible to find some combination that exhibits a first order cancellation and such a quantity is an "adiabatic invariant" that can be expected to vary only weakly as λ is varied. It has been shown that I itself is this adiabatic invariant.

In thermodynamics one considers "quasistatic" variations in which a system is treated as static even if it is changing slowly and this is what we have been doing here, so "quasistatic" invariant would be slightly more apt than "adiabatic", which in thermodynamics

means that the system under discussion is isolated in the sense that heat is neither added nor subtracted from the system. But the terminology is not entirely inappropriate since we are considering the effect of purely mechanical external intervention on the system under discussion.

There is an important connection between quantized variables in quantum mechanics and the adiabatic invariants of the corresponding classical system. Suppose a quantum system in a state with given quantum numbers is placed in an environment with varying parameters (such as time varying magnetic field, for example) but that the variation is never quick enough to induce a transition. Let the external parameters vary through a cycle that ends with the same values as they started with. Since the system has never changed state it is important that the physical properties of that state should have returned to their starting values—not just approximately, but exactly. This is what distinguishes an adiabatic invariant. This strongly suggests that *the dynamical variables whose quantum numbers characterize the stationary states of quantum systems have adiabatic invariants as classical analogs.* The Bohr-Somerfeld atomic theory, that slightly predated the discovery of quantum mechanics, was based on this principle. Though it became immediately obsolete, this theory was not at all *ad hoc* and hence had little in common with what passes for "the Bohr-Somerfeld model" in modern sophomore physics courses, In short, the fact that the action is an adiabatic invariant makes it no coincidence that Planck's constant is called "the quantum of action".

Action/angle conjugate variables

Because of its adiabatic invriance, the action variable I is an especially appropriate choice as parameter in applying the Jacobi procedure to a system with slowly varying parameters. We continue to focus on oscillating systems. Recalling the earlier discussion of time-independent canonical transformations, we introduce the *abbreviated action*

$$S_0\left(q, I, \lambda\right) = \int_0^q p\left(q', I, \lambda\right) dq'. \tag{52}$$

Until further notice λ will be taken as constant but it will be carried along explicitly in preparation for allowing it to vary later on. Since λ is constant, both E and I are constant, and either can be taken as the Jacobi "momentum" parameter; previously we have taken

E, now we take I, which is why the arguments of S_0 have been given as (q, I, λ). Since holding E fixed and holding I fixed are equivalent,

$$\left.\frac{\partial S_0}{\partial q}\right|_{E,\lambda} = \left.\frac{\partial S_0}{\partial q}\right|_{I,\lambda}. \tag{53}$$

Being a function of q through the upper limit of its defining equation, $S_0(q, I, \lambda)$ increases by $2\pi I$ as q completes one cycle of oscillation since, as in Eq. (39),

$$I\left(E, \lambda\right) = \frac{1}{2\pi} \oint p\left(q, E\left(I\right), \lambda\right) dq. \tag{54}$$

Using $S_0(q, I, \lambda)$ defined by Eq. (52) as the generator of a canonical transformation Eqs. (18) become

$$p = \frac{\partial S_0\left(q, I, \lambda\right)}{\partial q}, \quad \varphi = \frac{\partial S_0\left(q, I, \lambda\right)}{\partial I}. \tag{55}$$

where φ, the *new coordinate* conjugate to *new momentum* I, is called an "angle variable". For the procedure presently under discussion to be useful it is necessary for these equations to be reduced to explicit transformation equations $(q, p) \rightarrow (I, \varphi)$, such as Eqs. (62) of the next section. By Eq. (21) the new Hamiltonian is equal to the energy (expressed as a function of I)

$$H'\left(I, \varphi, \lambda\right) = E\left(I, \lambda\right), \tag{56}$$

and Hamilton's equations are

$$\dot{I} = -\frac{\partial H'}{\partial \varphi} = 0, \quad \text{and} \quad \dot{\varphi} = \frac{\partial E\left(I, \lambda\right)}{\partial I} = \omega\left(I, \lambda\right), \tag{57}$$

where Eq. (50) has been used, and the symbol $\omega(I, \lambda)$ has been introduced to stand for the oscillator frequency. Integrating the second equation yields

$$\varphi = \omega\left(I, \lambda\right)\left(t - t_0\right). \tag{58}$$

This is the basis for the name "angle" given to φ. It is an angle that advances through 2π as the oscillator advances through one period.

In these $(q, p) \rightarrow (\varphi, I)$ transformation formulas, λ has appeared simply as a fixed parameter. One way to exploit the concept of adiabatic invariance is now to permit λ to depend on time in a formula such as the second of Eqs. (57), $\dot{\varphi} = \omega(I, \lambda(t))$. This formula, giving the angular frequency of the oscillator when λ is constant, will continue to be valid

with the value of I remaining constant, even if λ varies arbitrarily, as long as its variation over one cycle is negligible when the frequency is being observed.

A more powerful way of proceeding is to recognize that it is legitimate to continue using Eqs. (55) as transformation equations, even if λ varies, provided λ is replaced by $\lambda(t)$ *everywhere* it appears. The generating function is then $S_0(q, I, \lambda(t))$ and φ will still be called the "angle variable", conjugate to I. Using Eq. (9), and taking account of the fact that the *old* Hamiltonian is now time-dependent, the *new* Hamiltonian is

$$H'(\varphi, I, t) = H + \frac{\partial S_0}{\partial t} = E(I, \lambda(t)) + \left.\frac{\partial S_0}{\partial \lambda}\right|_{q,I} \dot{\lambda}. \tag{59}$$

The *new* Hamilton equations are

$$\begin{aligned} \dot{I} &= -\frac{\partial}{\partial \varphi}\left(\left.\frac{\partial S_0}{\partial \lambda}\right|_{q,I}\right)\dot{\lambda}, \\ \dot{\varphi} &= \frac{\partial E(I,\lambda)}{\partial I} + \frac{\partial}{\partial I}\left(\left.\frac{\partial S_0}{\partial \lambda}\right|_{q,I}\right)\dot{\lambda}, \end{aligned} \tag{60}$$

Since no approximations have been made these are exact equations of motion provided the function S_0 has been derived without approximation.

Parametrically-driven simple harmonic motion

Generalizing simple harmonic motion by allowing the spring constant $k(t)$ to be time-dependent, the Hamiltonian is[†]

$$H(q, p, t) = \frac{p^2}{2m} + \frac{1}{2} m\lambda^2(t)\, q^2. \tag{61}$$

Though time-dependent, this Hamiltonian represents a *linear* oscillator because the frequency is independent of amplitude. The time-independent transformations corresponding to Eqs. (55) can be adapted from Eq. (30) by substituting $\omega_0 = \lambda$, $E = I\omega_0 = I\lambda$, and $\omega_0(t - t_0) = \varphi$;

$$\begin{aligned} q(I, \varphi) &= \sqrt{\frac{2E}{m\lambda^2}}\ \sin\varphi = \sqrt{\frac{2I}{m\lambda}}\ \sin\varphi, \\ p(I, \varphi) &= \sqrt{2Im\lambda}\ \cos\varphi. \end{aligned} \tag{62}$$

[†] In accelerator physics, while describing the motion of a particle, the independent variable, instead of time t, is taken to be a longitudinal spatial variable s. Note that, with t replaced by s, Eq. (61) describes one dimensional betatron motion. Its solution must necessarily reduce to the Courant-Snyder formalism. This line of reasoning is pursued in the problems.

The abbreviated action is given by

$$S_0\left(q,I,\lambda\right)=\int^{q}p'dq'=2I\int^{\sin^{-1}\left(q\sqrt{\frac{m\lambda}{2I}}\right)}\cos^2\varphi'd\varphi'. \tag{63}$$

The dependence on q is through its presence in the upper limit. This dependence can be re-arranged as

$$\lambda=\frac{2I}{q^2m}\,\sin^2\varphi. \tag{64}$$

This can be used to calculate the quantity

$$\left.\frac{\partial S_0}{\partial\lambda}\right|_{q,I}=2I\cos^2\varphi\left.\frac{1}{\partial\lambda/\partial\varphi}\right|_{q,I}=\frac{I}{2\lambda}\sin 2\varphi. \tag{65}$$

which can then be substituted into Eqs. (60);

$$\begin{aligned}\dot{I}&=-\frac{\partial}{\partial\varphi}\left(\frac{I}{2\lambda}\sin 2\varphi\right)\dot{\lambda}=-I\cos 2\varphi\,\frac{\dot{\lambda}}{\lambda},\\ \dot{\varphi}&=\omega\left(I,\lambda\right)+\frac{\partial}{\partial I}\left(\frac{I}{2\lambda}\sin 2\varphi\right)\dot{\lambda}=\lambda+\sin 2\varphi\,\frac{\dot{\lambda}}{\lambda}.\end{aligned} \tag{66}$$

Here the frequency $\omega(I,\lambda)$ has been calculated as if λ were time-independent; that is $\omega(I,\lambda)=\lambda$. Since in this case the slowly varying parameter has been chosen as $\lambda=\omega$ one can simply replace λ by ω in Eqs. (60), eliminating the artificially introduced λ.

The first equation shows that dI/dt is not identically zero, but the fact that $\cos 2\varphi$ averages to zero shows that the equation implies that dI/dt averages to zero to the extent that I *is* constant over one cycle and can therefore be taken outside the averaging. Though this statement may seem a bit circular—if I is constant then I is constant—it shows why I is approximately constant and can be the starting point of an estimate of the accuracy to which this is true.

The new Hamiltonian is obtained from Eqs. (59) and (65),

$$H'\left(\varphi,I,t\right)=\;E\left(I,\omega\left(t\right)\right)+\left.\frac{\partial S_0}{\partial\omega}\right|_{q,I}\dot{\omega}=I\omega\left(t\right)+\frac{I}{2}\sin 2\varphi\,\frac{\dot{\omega}}{\omega}, \tag{67}$$

where the time dependence is expressed as the dependence on time (but not amplitude) of the "natural frequency" $\omega(t)$. The *linearity* of the oscillator is here reflected by the fact that H' depends linearly on I. Problems below illustrate how this can be exploited to complete the solution in this circumstance. Eq. (67) can be used to check Eqs. (66) by

substituting into Hamilton's equations though that is not different from what has already been done.

The angle φ has appeared in these equations only in the forms $\sin\varphi$, $\cos\varphi$, $\sin 2\varphi$, $\cos 2\varphi$. This is not an accident since, though the abbreviated action is augmented by $2\pi I$ every period, with this subtracted it is necessarily a periodic function of φ. The accumulating part does not contribute to $\left.\frac{\partial S_0}{\partial \lambda}\right|_{q,I}$ because I is held constant. It follows that H' is a periodic function of φ with period 2π and can therefore be expanded in a Fourier series with period 2π in variable φ. For the particular system under study this Fourier series has a single term, $\sin 2\varphi$.

Problem .1. Eq. (62) gives a transformation $(q,p) \to (I,\varphi)$. Derive the inverse transformation $(I,\varphi) \to (q,p)$.

Problem .2. Consider a one dimensional oscillator for which the Hamiltonian expressed in action-angle variables is

$$H = \omega I + \epsilon I \cos^2\varphi.$$

where ω and ϵ are constants (with ϵ not allowed to be arbitrarily large). From Hamilton's equations express the time dependence $\varphi(t)$ as an indefinite integral and perform the integration. Then express $I(t)$ as an indefinite integral.

Problem .3. For the system with Hamiltonian given by $H(q,p,t) = \frac{p^2}{2m} + \frac{1}{2}m\lambda^2(t)q^2$ as in Eq. (61), consider the transformation $(q,p) \to (Q,P)$ given by

$$\begin{aligned} Q &= -\tan^{-1}\left(\frac{r}{q}\left(\frac{rp}{m} - q\dot{r}\right)\right), \\ P &= \frac{m}{2}\left(\frac{q^2}{r^2} + \left(\frac{rp}{m} - q\dot{r}\right)^2\right), \end{aligned} \tag{68}$$

where $r(t)$ will be specified more precisely in a later problem but is, for now, an arbitrary function of time. Show that this transformation is symplectic. Note that this same Hamiltonian applies to the transverse motion of a particle in an accelerator. In that context the particle is said to be executing "betatron oscillations". The solution to the problem is therefore equivalent to the so-called "Courant-Snyder formalism" describing betatron motion in a ring containing "linear" focusing elements, especially quadrupoles or "gradient magnets".

Problem .4. For the same system, in preparation for finding the generating function $G(q, Q, t)$ defined in Eqs. (8) and (10), re-arrange the transformation equations of the previous problem into the form $P = P(q, Q, t)$ and $p = p(q, Q, t)$. Then find $G(q, Q, t)$ such that

$$p = \frac{\partial G}{\partial q}, \quad P = -\frac{\partial G}{\partial Q}. \tag{69}$$

Problem .5. In preparation for finding the new Hamiltonian $H'(Q, P, t)$ and expressing it (as is obligatory) explicitly in terms of Q and P, invert the same transformation equations into the form $q = q(Q, P, t)$ and $p = p(Q, P, t)$. Then find $H'(Q, P, t)$ and simplify it by assuming that $r(t)$ satisfies the equation

$$\ddot{r} + \lambda^2(t)\, r - r^{-3} = 0. \tag{70}$$

Then show that Q is ignorable and hence that P is conserved.

Problem .6. Assuming that the system studied in the previous series of problems is oscillatory, find its action variable and relate it to the action variable E/ω of simple harmonic motion.

EXAMPLES OF ADIABATIC INVARIANCE

Variable length pendulum

Consider the variable-length pendulum shown in Fig. 3. Tension T holds the string which passes over a frictionless peg, the length of the string below the peg being $l(t)$. Assuming small amplitude motion the "oscillatory energy" of the system E_{osc} is defined so that the potential energy (with pendulum hanging straight down) plus kinetic energy of the system is $-mgl(t) + E_{\text{osc}}$. With fixed l,

$$E_{\text{osc}} = \frac{1}{2} mgl\theta_{\text{max}}^2, \tag{71}$$

If the pendulum is not swinging, E_{osc} continues to vanish when the length is varied slowly enough that the vertical kinetic energy can be neglected. We assume the length changes slowly enough that $\dot{l}^2$ and $\ddot{l}$ can be neglected throughout.

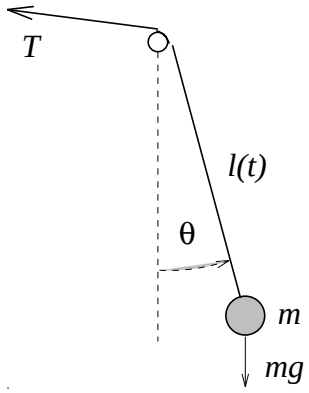


Figure 3: Variable-length pendulum. The fractional change of length during one oscillation period is less than a few percent.

The equation of motion is

$$\ddot{\theta} + \frac{\dot{l}\dot{\theta}}{l} + \frac{g}{l}\sin\theta = 0. \tag{72}$$

For "unperturbed" motion the second term is neglected, and the (small amplitude) action is given by

$$I = \sqrt{\frac{l}{g}}\, E_{\rm osc}. \tag{73}$$

Change dl in the pendulum length causes change $d\theta_{\rm max}$ in maximum angular amplitude. The only real complication in the problem is that the ratio of these quantities depends on θ. The instantaneous string tension is given by $mg\cos\theta + ml\dot{\theta}^2 - m\ddot{l}$, but we will neglect the last term. The energy change $dE_{\rm osc}$ for length change dl is equal to the work done $-T dl$ by the external agent acting on the system less the change in potential energy;

$$dE_{\rm osc} = -\left(mg\cos\theta + ml\dot{\theta}^2\right)\, dl + mg\, dl. \tag{74}$$

Continuing to assume small oscillation amplitudes,

$$\frac{dE_{\rm osc}}{dl} = \frac{1}{2}mg\theta^2 - ml\dot{\theta}^2. \tag{75}$$

The right hand side can be estimated by averaging over a complete cycle of the unperturbed motion and for that motion

$$\overline{\theta^2} = \frac{1}{2}\theta_{\rm max}^2 \quad \text{and} \quad \overline{\dot{\theta}^2} = \frac{1}{2}\frac{g}{l}\theta_{\rm max}^2. \tag{76}$$

As a result, using Eq. (71), we have

$$\overline{\frac{dE_{\rm osc}}{dl}} = -\frac{E_{\rm osc}}{2l}. \tag{77}$$

Then from Eq. (73)

$$\overline{\frac{dI}{dl}} = \sqrt{\frac{\ell}{g}}\,\overline{\frac{dE_{\rm osc}}{dl}} + \frac{1}{2}\,\frac{E_{\rm osc}}{\sqrt{gl}} = 0. \tag{78}$$

Here we have treated both l and $E_{\rm osc}$ as constant and moved them outside the averages. The result is that I is conserved, in agreement with the general theory.

Charged particle in magnetic field

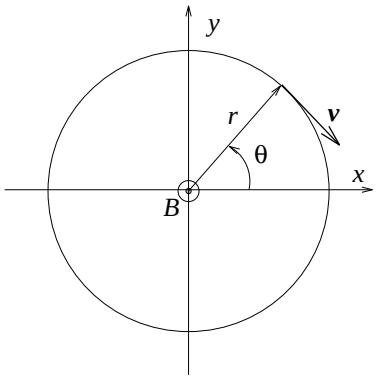


Figure 4: A charged particle moves in a slowly varying, uniform magnetic field.

Consider a charged particle moving in a uniform magnetic field $B(t)$ which varies slowly enough that the Faraday law electric field can be neglected, and also so that the adiabatic condition is satisfied. With coordinate system defined in Fig. 4 the vector potential of such a field is

$$A_x = -\frac{1}{2}yB, \quad A_y = \frac{1}{2}xB, \quad A_z = 0, \tag{79}$$

since

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = B\hat{\mathbf{z}}. \tag{80}$$

Introducing cylindrical coordinates, the (non-relativistic) Lagrangian is

$$\begin{aligned} L =& \frac{1}{2}mv^2 + e\mathbf{A}\cdot\mathbf{v} \\ =& \frac{1}{2}mv^2 + \frac{eB}{2}\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)\cdot\left(\frac{y}{r}\hat{\mathbf{x}} - \frac{x}{r}\hat{\mathbf{y}}\right)\left(-r\dot{\theta}\right) \\ =& \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + \frac{1}{2}eB\left(t\right)r^2\dot{\theta}. \end{aligned} \tag{81}$$

Since this is independent of θ, the conjugate momentum,[†]

$$P_\theta = mr^2\dot{\theta} + \frac{1}{2}eB\,(t)\,r^2, \tag{82}$$

is conserved. With B fixed, and the instantaneous center of rotation chosen as origin, a condition on the unperturbed motion is obtained by equating the centripetal force to the magnetic force;

$$m\dot{\theta} = -eB, \tag{83}$$

with the result that

$$P_\theta = \frac{1}{2}mr^2\dot{\theta}, \tag{84}$$

and the action variable is

$$I_\theta = \frac{1}{2\pi}\oint P_\theta d\theta = P_\theta. \tag{85}$$

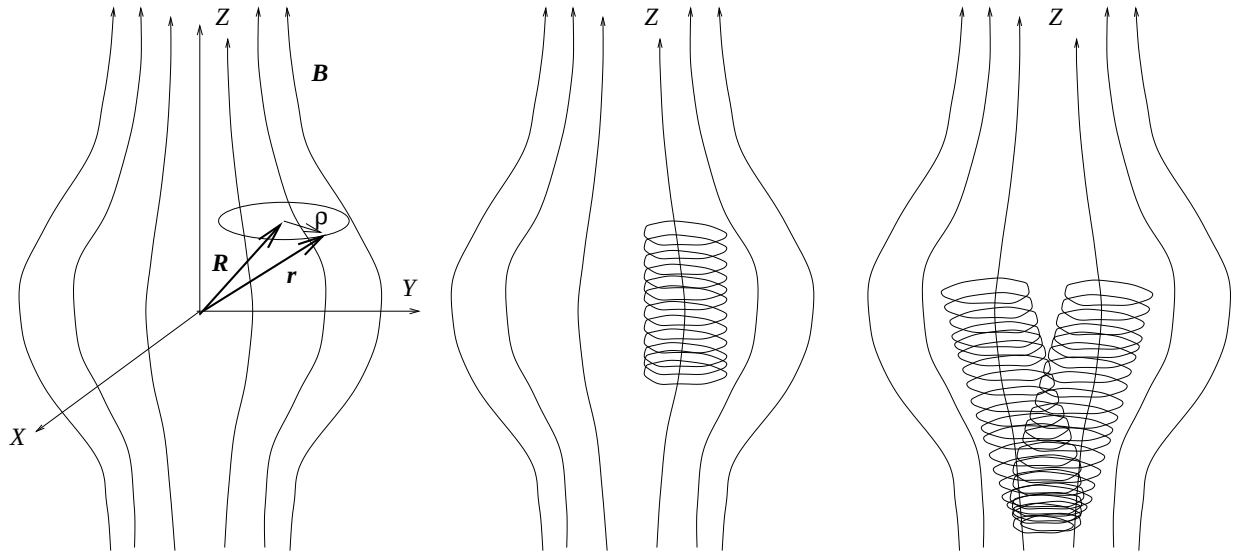
It is useful to express I_θ in terms of quantities that are independent of the origin using Eq. (83),

$$I_\theta = \frac{1}{2}m\left(r\dot{\theta}\right)^2\frac{1}{\dot{\theta}} = -\frac{m^2}{2e}\frac{v_\perp^2\,(t)}{B\,(t)}. \tag{86}$$

where $v_\perp$ is the component of particle velocity normal to the magnetic field.

Recapitulating, $v_\perp^2/B$ is an adiabatic invariant. The important result is not that P_θ is conserved when B is constant, which we already knew, but that it is conserved even when B varies (slowly enough) with time. Furthermore, since the change in B is to be evaluated at the particle's nominal position, changes in B can be due either to changes in time of the external sources of B or to spatial variation of B in conjunction with displacement of the moving particle's center of rotation (for example parallel to $\mathbf{B}$). P_θ is one of the important invariants controlling the trapping of charged particles in a magnetic "bottle". This is pursued in the next section.

[†] Recall that (upper case) P stands for conjugate momentum which differs from (lower case) p which is the mechanical momentum.



(a) gyration (b) plus longitudinal drift (c) plus perpendicular drift

Figure 5: (a) Charged particle gyrating in a non-uniform magnetic field. Its longitudinal and azimuthal motion is exhibited in (b) and (c). The reduction in radius of gyration near the end of the trap is also shown.

Charged particle in a magnetic trap

A particle of charge e moves in a time-independent, axially-symmetric magnetic field $\mathbf{B}(\mathbf{R})$. Symbolizing the component of particle velocity normal to $\mathbf{B}$ by w, the approximate particle motion follows a circle of radius ρ with angular rotation frequency ω_c (known as the "cyclotron frequency"). These quantities are given by

$$\rho = \frac{mw}{eB}, \quad \text{and} \quad \omega_c = 2\pi \frac{w}{2\pi\rho} = \frac{eB}{m}, \tag{87}$$

with the latter being independent of the speed of the particle. The field is assumed to be non-uniform but *not too* nonlinear. This is expressed by the condition

$$\rho \frac{|\nabla \mathbf{B}|}{B} << 1. \tag{88}$$

This condition assures that formulas derived in the previous section are applicable and the particle "gyrates" in an almost circular orbit. This is also a kind of adiabatic condition in that the particle retraces pretty much the same trajectory turn after turn. The system is then known as a "magnetic trap"; the sort of magnetic field envisaged is illustrated in Fig. 5 which also shows typical particle orbits.

But in general the particle also has a component of velocity parallel to $\mathbf{B}$ so the center of the circle (henceforth to be known as the "guiding center") also travels along $\mathbf{B}$. This motion is said to be "longitudinal". There will also be an even slower drift of the guiding center "perpendicular" to $\mathbf{B}$. This is due to the fact that condition (88) is not exactly satisfied and the radius of gyration is least in regions where $B = |\mathbf{B}|$ is greatest.

To describe these motions we introduce the radius vectors shown in Fig. 5(a).

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}. \tag{89}$$

The corresponding three velocities $\mathbf{v} = d\mathbf{r}/dt$, $\mathbf{u} = d\mathbf{R}/dt$, and $\mathbf{w} = d\boldsymbol{\rho}/dt$ satisfy

$$\mathbf{v} = \mathbf{u} + \mathbf{w}. \tag{90}$$

Presumeably $\mathbf{R}$ and $|\boldsymbol{\rho}|$ are slowly-varying compared to $\boldsymbol{\rho}$ which gyrates rapidly.

Since particles with large longitudinal velocities can escape out the ends of the bottle (as we shall see) the ones that have not escaped have transverse velocity at least comparable with their longitudinal velocity and it is clear from condition (88) that the transverse guiding center drift velocity is small compared to the gyration velocity. These conditions can be expressed as

$$\mathbf{v}_{\|} = \mathbf{u}_{\|}, \quad \text{and} \quad \mathbf{u}_{\perp} << \mathbf{w}, \quad \text{and hence} \quad \mathbf{v}_{\perp} \approx \mathbf{w}. \tag{91}$$

General strategy. To start one will ignore the slow motion of the guiding center in analysing the gyration. (This part of the problem has already been analysed in section 8 but we will repeat the derivation using the current notation and approximations.) Having once calculated the adiabatic invariant μ for this gyration it will subsequently be possible to ignore the gyration (or rather to represent it entirely by μ) in following the guiding center. This accomplishes a kind of "averaging over the fast motion". It will then turn out that the motion of the guiding center itself can be similarly treated on two time scales. There is an oscillatory motion of the guiding center parallel to the z-axis in which the azimuthal motion is so slow that it can be ignored. This motion is characterized by adiabatic invariant $I_{\|}(\mu)$. As mentioned already, its only dependence on gyration is through μ. Finally there is a slow azimuthal drift $I_{\perp}(\mu, I_{\|})$ that depends on gyration and longitudinal drift only through their adiabatic invariants. In this way, at each stage there is a natural time scale

defined by the period of oscillation and this oscillation is described by equations of motion that neglect changes occuring on longer time scales and average over effects that change on shorter time scales.

Gyration. In relativistic mechanics the components of the canonical momentum are given by

$$\begin{aligned} \mathbf{P}_\perp &= m\mathbf{w} + e\mathbf{A}_\perp, \\ \mathbf{P}_\parallel &= m\mathbf{u}_\parallel + e\mathbf{A}_\parallel, \end{aligned} \tag{92}$$

where approximations (91) have been used. The non-relativistic Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} = \frac{\left(\mathbf{P}_\perp - e\mathbf{A}_\perp^2\right)}{2m} + \frac{\left(\mathbf{P}_\parallel - e\mathbf{A}_\parallel^2\right)}{2m}; \tag{93}$$

this is the mechanical energy expressed in terms of appropriate variables. There is no contribution from a scalar potential since there is no electric field.

The gyration can be analysed as the superposition of sinuisoidal oscillations in two mutually perpendicular directions in the transverse plane. For adiabatic invariant I_g we can take their average

$$I_g = \frac{1}{4\pi}\oint \left(P_{\perp x}dx + P_{\perp y}dy\right) = \frac{1}{4\pi}\oint \mathbf{P}_\perp \cdot \mathbf{dl}_\perp, \tag{94}$$

where $\mathbf{dl}_\perp$ is incremental tangential displacement in the (x, y) plane, "right handed" with the (x, y, z) axes being right handed. It is therefore directed opposite to the direction of gyration as shown in Fig. 4 since $\mathbf{B}$ is directed along the (local) positive z-axis. Using Eq. (92) we have

$$I_g = \frac{1}{4\pi}\oint m\mathbf{w}\cdot\mathbf{dl}_\perp + \frac{e}{4\pi}\oint \mathbf{A}\cdot\mathbf{dl}_\perp = -\frac{mw\rho}{2} + \frac{e}{4\pi}\int \mathbf{B}\cdot\hat{\mathbf{z}}\, dS, \tag{95}$$

where dS is an incremental area in the plane of gyration. The first term is negative because the gyration is directed opposite to $\mathbf{dl}_\perp$. The second term (in particular its positive sign) has been obtained using Stokes's theorem and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Using Eq. (87) we get

$$I_g = -\frac{eB\rho^2}{4}. \tag{96}$$

This agrees with Eq. (86). I_g can be compared to the "magnetic moment" $\mu = \frac{e^2}{2m}B\rho^2$ of the orbit (which is equal to the average circulating current $e\omega_c/(2\pi)$ multiplied by the

orbit area $\pi\rho^2$.) Except for a constant factor, μ and I_g are identical so we can take μ as the adiabatic invariant from here on. If we regard $\boldsymbol{\mu}$ as a vector perpendicular to the plane of gyration then

$$\boldsymbol{\mu} \cdot \mathbf{B} < 0. \tag{97}$$

We also note that the kinetic energy of motion in the perpendicular plane is given by

$$E_\perp = \frac{1}{2} m w^2 = -\boldsymbol{\mu} \cdot \mathbf{B} = \mu\, B. \tag{98}$$

Longitudinal drift of the guiding center. Because of its longitudinal velocity the particle will drift along the local field line. Since the field is non-uniform this will lead it into a region where B is different. Because the factor $B\rho^2$ remains constant we have $\rho \sim B^{-1/2}$ and (by Eq. (87)) $w \sim B^{1/2}$. Superficially this seems contradictory since the speed of a particle cannot change in a pure magnetic field. It *has to be* that energy is transferred to or from motion in the longitudinal direction. We will first analyse the longitudinal motion on the basis of energy conservation and later analyse it in terms of the equations of motion.

The total particle energy is given by

$$E = \mu B\left(\mathbf{R}\right) + \frac{1}{2} m\, u_\parallel^2. \tag{99}$$

Since the first term depends only on position $\mathbf{R}$ it can be interpreted as potential energy. It is larger at either end of the trap than in the middle. Since both E and μ are conserved, this equation can be solved for the longitudinal velocity

$$u_\parallel = \pm\sqrt{\frac{2}{m}\left(E - \mu B\left(\mathbf{R}\right)\right)}. \tag{100}$$

In a uniform field $u_\parallel$ would be constant, but in a spatially variable field it varies slowly. As the particle drifts toward the end of the trap the B field becomes stronger and $u_\parallel$ becomes less. At some value $Z_{\rm tp}$ the right hand side of Eq. (100) vanishes. This is therefore a "turning point" of the motion and the guiding center is turned back to drift toward the center and then the other end. Perpetual longitudinal oscillation follows, but the motion may be far from simple harmonic, depending as it does on the detailed shape of $\mathbf{B}(\mathbf{R})$—for example B can be essentially constant over a long central region and then become rapidly larger over a short end region.

In any case an adiabatic invariant $I_\parallel$ for this motion can be calculated (on-axis) by

$$I_\parallel = \frac{1}{2\pi}\oint \mathbf{P}_\parallel \cdot \hat{\mathbf{Z}}\, dZ = \frac{m}{2\pi}\oint u_\parallel dZ, \tag{101}$$

where, by symmetry (as in Eq. (79)) A_z vanishes on-axis. Then the period of oscillation can be calculated using Eq. (50);

$$T_\parallel = 2\pi \frac{\partial I_\parallel}{\partial E_\parallel}. \tag{102}$$

Problem .7. For the long uniform field magnetic trap with short end regions mentioned in the text, use Eq. (102) to calculate the period of longitudinal oscillation $T_\parallel$ and show that the result is the same as one would obtain from elementary kinematic considerations.

Equation of motion of the guiding center. We have still to study the transverse drift of the guiding center and in the process will corroborate the longitudinal motion inferred purely from energy considerations in the previous paragraph. The equation of motion of the particle is

$$m\frac{d\,(\mathbf{u}+\mathbf{w})}{dt} = e\,(\mathbf{u}+\mathbf{w}) \times (\mathbf{B}|_0 + [(\boldsymbol{\rho}\cdot\boldsymbol{\nabla})\,\mathbf{B}]_0)\,, \tag{103}$$

which approximates the magnetic field by its value $\mathbf{B}|_0$ at the guiding center plus the first term in a Taylor expansion evaluated at the same point. We wish to average this equation over one period of the (rapid) gyration which is described relative to local axes by

$$\begin{aligned} \rho_x &= w\cos\theta, \quad \rho_y = w\sin\theta, \\ w_x &= w\sin\theta, \quad w_y = -w\cos\theta. \end{aligned} \tag{104}$$

When Eq. (103) is averaged with all other factors held fixed, the result is

$$m\frac{d\mathbf{u}}{dt} = e\mathbf{u}\times\mathbf{B} + e\langle(\mathbf{w}\times\boldsymbol{\rho}\cdot\boldsymbol{\nabla})\,\mathbf{B}\rangle. \tag{105}$$

Terms with an odd number of factors of $\boldsymbol{\rho}$ and $\mathbf{w}$ have averaged out to zero. The second term evaluates to

$$\begin{aligned} e\langle(\mathbf{w}\times\boldsymbol{\rho}\cdot\boldsymbol{\nabla})\,\mathbf{B}\rangle &= e\langle\det\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ w_x & w_y & 0 \\ \rho_x\frac{\partial B_x}{\partial x}+\rho_y\frac{\partial B_x}{\partial y} & \rho_x\frac{\partial B_y}{\partial x}+\rho_y\frac{\partial B_y}{\partial y} & \rho_x\frac{\partial B_z}{\partial x}+\rho_y\frac{\partial B_z}{\partial y} \end{vmatrix}\rangle \\ &= -\frac{ew\rho}{2}\left(\hat{\mathbf{x}}\frac{\partial B_z}{\partial x}+\hat{\mathbf{y}}\frac{\partial B_z}{\partial y}\right) = -\frac{ew\rho}{2}\boldsymbol{\nabla}\mathbf{B_z}, \\ &= -\boldsymbol{\nabla}\,(\mu\,B)\,, \end{aligned} \tag{106}$$

where $\boldsymbol{\nabla}\cdot\mathbf{B}=0$, and $B_z\approx B$ have been used. The equation of motion is therefore

$$m\frac{d\mathbf{u}}{dt}=e\mathbf{u}\times\mathbf{B}-\boldsymbol{\nabla}(\mu B). \tag{107}$$

When applied to the longitudinal motion of the guiding center, the final term can be seen to be consistent with our earlier interpretation of μB as a potential energy. Furthermore, the only influence of gyration is through the parameter μ.

The magnitude of the magnetic field presumably falls with increasing R. this causes the gyration to be not quite circular, with its radius increased by $\Delta\rho$ when the field is reduced by ΔB;

$$\frac{\Delta\rho}{\rho}=-\frac{\Delta B}{B}. \tag{108}$$

Along with the cyclotron frequency $\frac{\omega_c}{2\pi}$ this can be used to estimate the ration of the transverse drift velocity $u_\perp$ to w;

$$\frac{u_\perp}{w}\approx\frac{(\omega_c/2\pi)\,\Delta\rho}{w}\approx\frac{(\omega_c/2\pi)\,(\partial B/\partial r)\,\rho^2}{wB}=\frac{1}{2\pi}\,\frac{\rho}{B}\,\frac{\partial B}{\partial r}=\frac{1}{2\pi}\,\frac{\rho}{R_{\rm typ}}, \tag{109}$$

where $R_{\rm typ}$ is a length of the order of the transverse dimensions of the apparatus. Since typical values of the cyclotron radius are much less than this, and since $u_\parallel$ and w have comparable magnitudes, our estimate shows that

$$u_\perp << w, \quad \text{and} \quad u_\perp << u_\parallel. \tag{110}$$

There will nevertheless be a systematic azimuthal motion of the guiding center on a circle of some radius $R_\perp$ centered on the axis. Let the angular frequency of this motion be $\omega_\perp$. We then have

$$\omega_\perp << \omega_\parallel << \omega_c. \tag{111}$$

By a calculation just like that by which I_g was calculated an adiabatic invariant can also be obtained for this perpendicular drift;

$$I_\perp=-\frac{mu_\perp}{2}+\frac{eBR_\perp^2}{4}=\frac{eR_\perp^2B}{4}\left(1-\frac{2\omega_\perp}{\omega_c}\right). \tag{112}$$

In practical situations the second term is negligible and we conclude that the third adiabatic invariant $I_\perp$ is proportional to the magnetic flux linked by the guiding center as it makes a complete azimuthal circuit.

ADIABATIC INVARIANCE APPLIED TO THE PROTON LINAC

Sections 3.5 and 3.6 of G. Loew and R. Talman, *Lectures on the ELementary Principles of Linacs,* contained in *Physics of High Energy Particle Accelerators,* AIP Conference Proceedings 105, M. Month Editor, 1983, describe the application of adiabatic invariance to the longitudinal of (non-relativistic) protons in an Alvarez Linac.