THE STORAGE RING AS RADIATION SOURCE

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These notes originated as the course material for my part of a Cornell course on using storage rings as synchrotron light sources, given with Sol Gruner, spring term 2000. Sol emphasized the "downstream" view, especially detection apparatus. My part gave an "upstream" view of the storage ring as radiation source. The notes were much expanded for the January 2001 USPAS School at Rice University in Houston and most of the material on undulator radiation has been developed subsequent to that school. There is substantial overlap of these notes with the famous storage ring report by Sands. The main difference is the emphasis on using the ring as source of photons rather than as a colliding beam facility. For the Yale, 2002, USPAS school, only chapters in which electromagnetic theory is prominent will be covered.

By now there is a considerable body of material describing this topic, and detailed "handbook" formulas are available for the important quantities. I refer to this material occasionally, but my intention is more to emphasize the intuitive content of the subject, even when this yields formulas having only a semi-quantitative validity. My rationale is that it is harder to grasp the essentials of the subject than to find and understand exact formulations once one has the general idea. To promote this approach I pretend that all storage ring development has been motivated by their use as sources of synchrotron light, even though early generation radiation sources were entirely parasitic, and early ring designs were driven by particle physics considerations.

In the beginning I promote the view that the evolution of internal/electron and external/photon beams are "essentially" equivalent, and are subject to the same formulas. To exploit this "economy" it is necessary to apply, to beams of photons, terms that are familiar mainly to accelerator physicists. Unfortunately this (not very deep) synthesis is sufficiently novel that I am unaware of any helpful references. A deficiency of the notes at present is that, though the main ingredients of FEL theory are described, they are not stitched together coherently. Also, though supposedly sophisticated statistical methods (such as Fokker-Planck) are explained, no examples are included (at this time.)

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Chapter 1.
Photon Beams/Electron Beams

1.1. Introduction

This course will deal with beams of particles, photons or electrons, where electron will be taken to include also positron. The kinematics of photons and electrons will be the same (unless, of course, the charge matters) since, in the highly relativistic regime, the electron rest mass can be neglected. For this reason the most fundamental features (Liouville's theorem and its generalizations, linear/nonlinear distinction, intensities, emittances, adiabatic invariance, and so on) apply equally to both sorts of beam. Unfortunately, this equivalence tends to be masked by the different terminologies that have evolved during the independent historical development of accelerator physics and optical physics. The merging of these two fields has come only comparatively recently with the development of storage rings as sources of photon beams. Though the notation in this chapter is drawn mainly from the accelerator side, *all formulas apply equally to photons and electrons.* †

By definition, a beam is a number (usually billions and billions) of particles all traveling more or less parallel. For purposes of description it is useful to start by picking some "most-central" particle as a "reference particle" and describing its "ideal" or "reference" ray or trajectory through the system. Any particular point on the reference trajectory can be located by global Cartesian coordinates (X, Y, Z) relative to some fixed origin. But, to take advantage of the essentially one dimensional placement of elements it is profitable to locate elements by arc length (called s) from the origin along the central ray. Once these coordinates are known for the central particle, all the other particles in the beam can be located by relative coordinates. For these relative coordinates it is convenient to use a reference frame aligned with the central ray. "Tangential" or "longitudinal" displacements are specified by incremental arc length $z = \Delta s$ and (x, y) serve as "transverse" coordinates,

† Since it is hard for an accelerator physicist to avoid using the word "lattice" to describe a beamline of lenses, an optical physicist will have to tolerate this usage. Another confusing practice drawn from accelerator tradition is to use the same symbol for seemingly unrelated quantities, knowing they will later be shown to be equal (in some sense, or under some circumstances). This comment applies especially to the symbols β and ϵ.

with x usually being horizontal and y vertical. The longitudinal coordinates s and z combine trivially—by simple addition. For relativistic particles, s is proportional to t ($s = vt$, $v \approx c$) and s can take over from t the role of "independent variable".†

There is nothing "small" about the reference trajectory—in a storage ring it bends through 2π, in a photon beam it may reflect through comparably large angles. On the other hand there is much that is "small" about the relative motion and, of course, that is why the division is made. Loosely speaking then, the coordinates (X, Y, Z, s) are "large" and the coordinates (x, y, z) are "small". The fundamental beam properties mentioned in the first paragraph apply primarily to these small displacements.

Much the same comments apply to velocity and momentum. The momentum components of the central ray are $(0, 0, P_0)$ and, in the absence of accelerating elements, P_0 is constant. The momentum components of a beam particle are customarily expressed as ratios

$$(p_x, p_y, \delta) = \left(\frac{P_x}{P_0}, \frac{P_y}{P_0}, \frac{P_z - P_0}{P_0}\right), \tag{1.1.1}$$

where $\delta = \Delta P/P_0$ is therefore the fractional momentum deviation from the central momentum. These definitions are traditional for particle description; for photon description the momentum is more traditionally expressed as "wave vector" $\mathbf{k} = \mathbf{p}/\hbar$ and the definitions have to be modified accordingly.

The name "Gauss" is often attached to beam transport transport systems and, with quite different meaning, to beams. In a Gaussian beam the distributions of the above-mentioned deviations, $(x, y, z, p_x, p_y, \delta)$, are governed by Gaussian probability distributions. Typically it is the spreads of these variables that are more important than the detailed distribution and the Gaussian serves primarily as a probability distribution that is big in the middle and drops off rapidly for large deviations. The standard deviation of the Gaussian is a good objective measure of beam spread that can serve as a semi-quantitative size measure in all cases and as an accurate quantitative measure in those cases where the beam distribution is truly Gaussian. We will return to Gaussian beams later.

† The practice of using longitudinal coordinate as independent variable perhaps dates from the early days of optics, before photons were even conceived of, when there was no reason to contemplate the rate of advance of anything along a ray, so the purely geometric character of the ray was emphasized.

Far more fundamental is "Gaussian" (also known as "paraxial") optics, which describes transport systems in which the trajectory deviations, both displacement and slope, away from the reference trajectory are small enough to be treated by "linearized" equations of motion. That is, the differential equations governing the evolution of $(x(s), y(s), \ldots)$ are linear in $(x, y, \ldots)$. Since these linearized equations are usefully expressed using matrices, such optical systems are said to be satisfy "matrix optics". The fundamental characteristics listed in the first paragraph are most easily described in terms of these matrices, and the "linear" distinction listed there is synonymous with "paraxial".

The propagation of a beam of particles through a sequence of beamline elements can be viewed from three different perspectives. These *particle*, *beam*, and *beamline* views will be taken up, in order, in the following sections and, after that, be reconciled. A rich source of confusion is that the same symbols (especially ϵ and β) will be used in all views, so the reader will have to trust that seemingly unrelated usages will eventually be (more or less) reconciled; this is to say that the logic of assigning the same symbol to different quantities will eventually be made clear.

1.2. One-Dimensional Transverse Propagation Equations

The most essential feature distinguishing material particles like electrons from photons is that electrons can be bent in magnetic fields. For the subject matter under discussion this distinction turns out to be "inessential", for much the same reason that plane mirrors, in spite of grossly redirecting light beams, have no essential optical effect. For unity of description we therefore discuss now only a beam traveling along a straight channel centered on the s-axis, planning later to incorporate the (small) effects due to magnetic bending. To prevent the eventual departure of even slightly divergent rays, it is necessary to have focusing elements such as quadrupoles for charged particles, fiber optics or optical lenses for visible photons, curved mirrors for x-rays, and so on. The differential equation describing such focusing is[†]

$$\frac{d^2y}{ds^2} = K(s)\, y, \tag{1.2.1}$$

[†] The symbol y, rather than, say, x, has been chosen as dependent variable, because storage rings are traditionally horizontal, so the vertical orbit equations will remain valid when, later on, we incorporate the bending magnets that are necessarily present.

where $K(s)$ is the "vertical focusing strength". Its dependence on s permits the description of systems in which the focusing strength varies along the orbit. In particular, $K(s) = 0$ describes "drift spaces" in which case Eq. (1.2.1) is trivially solved, and yields the obvious result that particles in free space travel in straight lines.

It is conventional to designate dy/ds by y'. Then, from any two solutions y_1 and y_2 of Eq. (1.2.1) one can form the "Wronskian" $y_1 y_2' - y_1' y_2$ which has the virtue of being constant, since

$$\frac{d}{ds}\left(y_1 y_2' - y_1' y_2\right) = 0. \tag{1.2.2}$$

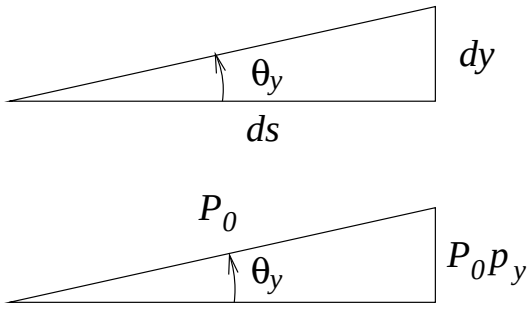

Figure 1.2.1: Relations among transverse angle, momentum, and slope.

There are three candidates for describing angular features of the beam: angle θ_y, slope y', or momentium p_y. All of these are exhibited in Fig. 1.2.1, and one sees that

$$y' \equiv \frac{dy}{ds} = \tan\theta_y = \frac{p_y}{\cos\theta_y}. \tag{1.2.3}$$

This multiple ambiguity in what constitutes the coordinate conjugate to y is something of a nuisance at large amplitudes but, fortunately, all three definitions approach equality in the small-angle limit that characterizes Gaussian optics. One knows from Hamiltonian mechanics that p_y is the safest choice but, while limiting ourselves to Gaussian optics, we will refer loosely to y' as "vertical momentum" so that we can refer to the (y, y')-plane as "vertical phase space".

Starting from any point s_0 along the beamline, one defines two special orbits, a "cosine-like" orbit $C(s, s_0)$ with unit initial amplitude and zero slope, and a "sine-like" orbit $S(s, s_0)$

with zero initial amplitude and unit slope:†

$$\begin{aligned} C(s_0, s_0) &= 1, \quad C'(s_0, s_0) = 0, \\ S(s_0, s_0) &= 0, \quad S'(s_0, s_0) = 1. \end{aligned} \tag{1.2.4}$$

Because Eq. (1.2.1) is linear and second order, *any* solution $y(s)$ and its first derivative $y'(s)$ can be expressed as that linear superposition of these two solutions that matches initial conditions:

$$\begin{aligned} y(s) &= C(s, s_0)\, y(s_0) + S(s, s_0)\, y'(s_0), \\ y'(s) &= C'(s, s_0)\, y(s_0) + S'(s, s_0)\, y'(s_0). \end{aligned} \tag{1.2.5}$$

This can be expressed in matrix form, with $\mathbf{y} = (y, y')^{\mathrm{T}}$ being a "vector in phase space":

$$\mathbf{y}(s) \equiv \begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C(s, s_0) & S(s, s_0) \\ C'(s, s_0) & S'(s, s_0) \end{pmatrix} \mathbf{y}(s_0) = \mathbf{M}(s_0, s)\, \mathbf{y}(s_0). \tag{1.2.6}$$

This serves to define $\mathbf{M}(s_0, s)$, the "vertical transfer matrix from s_0 to s". Since *any* solution of Eq. (1.2.1) can be expressed in this way, an entire beamline can be characterized by $\mathbf{M}(s_0, s)$. This matrix can be "composed" by multiplying (or "concatenating") the matrices for the successive beamline elements making up the line.

Being interested in not just one particle, but many particles making up a beam, we define a "phase space density function" $\rho(s; y, y')$ with the property that $\rho(s; y, y')dy\, dy'$ is the number of particles in the range $dy\, dy'$.

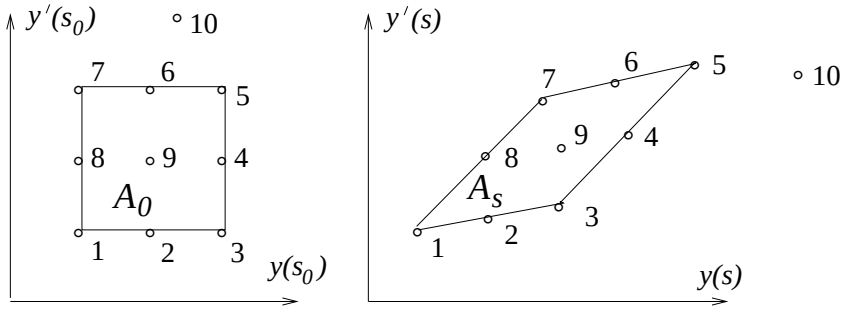

Figure 1.2.2: Phase space evolution of ten typical particles in advancing from s_0 to s along the beam line.

The evolution of a few particles in the beam in proceding from s_0 to s might resemble Fig. 1.2.2 where corresponding particles are numbered. Particles 1 through 8 outline an

† Since unity slope is manifestly *not* a small angle, these definitions only make sense after the equations have been linearized as in Eq. (1.2.1).

area A_0 that evolves into area A_s. From calculus one knows that the factor relating these these areas is the Jacobean determinant,

$$J(s_0, s) = \det \begin{vmatrix} \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial y_0'} \\ \frac{\partial y'}{\partial y_0} & \frac{\partial y'}{\partial y_0'} \end{vmatrix}. \tag{1.2.7}$$

From Eq. (1.2.6), suppressing the s_0-dependence, it can then be seen that

$$J(s) = \det |\mathbf{M}(s)| = C(s) S'(s) - S(s) C'(s) = 1. \tag{1.2.8}$$

which, being the Wronskian for the solutions $C(s)$ and $S(s)$, is known to be constant; that the value of the constant is 1 comes from evaluating $J(s_0)$, using Eqs. (1.2.4).

Particle trajectories in phase space cannot cross—this follows from the fact that instantaneous position and slope uniquely determine the subsequent motion. Hence the boundary will continue to be defined by particles 1 through 8 and a particle like 9 that is inside the box at s_0 will remain inside the box at s. Similarly, a particle like 10 will remain outside.

Putting these these things together, we obtain the result that the particle density $\rho(s)$ is, in fact, independent of s. This is Liouville's theorem, a result that can be discussed with various degrees of erudition. But it is surprisingly simple in the present context. It is most succinctly stated by the requirement

$$\det |\mathbf{M}(s)| = 1. \tag{1.2.9}$$

1.3. Transfer Matrices For Simple Elements

1.3.1. Drift Space

The most important transfer matrix is $\mathbf{M}_l$, which describes propagation through a drift space of length ℓ. Since the orbits are given by $y(s) = y_0 + y_0' s, y'(s) = y_0'$, we have

$$\mathbf{M}_l = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \tag{1.3.1}$$

In the context of this course, the basis for calling drifts "the most important element" is that the synchrotron light beamline leading from storage ring to detection apparatus is one long drift section.

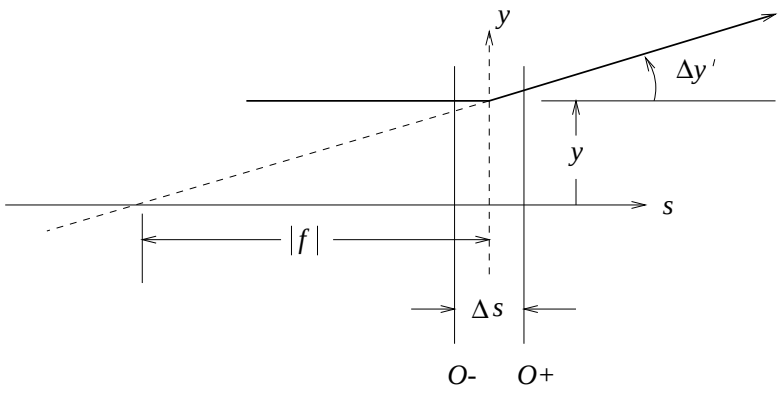

Figure 1.3.1: Focusing action of a thin lens, thickness Δs, focusing strength, K, illustrating the "focal length" $f = -(K\Delta s)^{-1} \equiv -q^{-1}$.

1.3.2. Thin Lens

The next most important transfer matrix describes a "thin lens" where the definition of "thin" is that the thickness Δs is sufficiently small that coordinate $y(s-\Delta s/2)$, just before the lens, and $y(s+\Delta s/2)$, just after, can be taken to be equal. The lens causes a "kink" $\Delta y' = y'(s+\Delta s/2) - y'(s-\Delta s/2)$ in the orbit which, as shown in Fig. 1.3.1, will be taken as occurring at the center of the lens.

The kink can be obtained by integrating Eq. (1.2.1) from O_-, just before the lens to O_+, just after it:

$$\Delta y' = \int_{O_-}^{O_+} \frac{d}{ds}\left(\frac{dy}{ds}\right) ds = y \int_{O_-}^{O_+} K\,(s)\, ds = y\ (K\Delta s)\,. \tag{1.3.2}$$

Now, a focusing strength that changes discontinuously from 0 to K is not actually realistic. But the product $K\Delta s$, known as a "field integral", can be regarded as an abbreviation for $\int_{O_-}^{O_+} K(s)ds$ where O_- and O_+ are well outside the field region. If K is taken to be equal to K_O (the value at the center of the element) then Δs is typically slightly greater than the physical length of the element because of fringe fields. The "focal length" f of the lens, defined in Fig. 1.3.1, and the "lens strength" $q = -1/f$, are then given by

$$q = -\frac{1}{f} = \frac{\Delta y'}{y} = K\Delta s. \tag{1.3.3}$$

Building in the approximation that y is contant through the lens, the transfer matrix is then given by

$$\mathbf{M}_q = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}. \tag{1.3.4}$$

As drawn, K and q are positive, f is negative, and the lens is "defocusing".

1.3.3. Thick Lens

The condition for the thin lens formula just given to be valid ($\Delta s << |f|$) is usually well satisfied for optical systems and accelerator lenses. Even if it is not, if $K(s)$ is constant (as it usually is, according to design anyway) it is easy to integrate Eq. (1.1.1), yielding matrix elements of $\mathbf{M}$ that are no worse than sines and cosines (or hyperbolic sines and cosines, depending on the sign of K). For low energy, few element, accelerators this approach used to be considered "canonical", but for high energy accelerators the thin lens approximation is usually adequate. In any case, making use of now readily available computer power, one can always split elements longitudinally for improved accuracy. Quite apart from improving accuracy, it is handy to split elements in two, to provide output at lens centers (where aperture limits are usually determined). Even in the most extreme cases of intersection region quads, splitting by another factor of, say, three, is more than adequate, especially since the residual inaccuracy is typically less than the errors due to the neglect of other factors like fringe fields. For these reasons we will not bother to write down thick element formulas.

1.3.4. Erect Quadrupole Lens

All formulas so far apply equally to optical and charged particle lenses. Since glass lenses and spherical mirrors can be focusing in both planes, Eq. (1.3.4) applies to both transverse planes. But a quadrupole lens that focuses in one plane (unhappily) defocuses in the other. This is because the vertical bend is due (ideally) to a horizontal magnetic field $B_x = (\partial B_x/\partial y)\, y$, where the proportionality to y is what causes the lens to focus, and what justifies calling the lens linear. For substitution into Eq. (1.2.1) we therefore use

$$P_0 \frac{d\mathbf{p}}{dt} = e \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ B_x & B_y & 0 \end{vmatrix} . \tag{1.3.5}$$

Using the result $v_z = dz/dt$ this yields for the vertical deflection

$$P_0 \Delta p_y = e B_x \Delta s, \tag{1.3.6}$$

which, though approximate, is exact in the small Δs limit. Linearizing Eq. (1.2.3), this can be recast as a lens strength[†]

$$q_y = \frac{\Delta y'}{y} = K_y \Delta s \quad \text{where} \quad K_y = \frac{c\,(\partial B_x/\partial y)}{P_0 c/e}. \tag{1.3.7}$$

Because of the Maxwell equation $\partial B_x/\partial y = \partial B_y/\partial x$, this vertical focusing strength is necessarily accompanied by horizontal focusing strength of opposite sign:

$$q_x = -q_y. \tag{1.3.8}$$

To express this constraint succinctly it is convenient to write a 4×4 transfer matrix:

$$\mathbf{M}_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & q & 1 \end{pmatrix}, \tag{1.3.9}$$

which acts on the vector $(x, x'y, y')^T$. (For a glass lens the off-diagonal terms can have the same sign, for example both focusing. Clearly this is helpful for designing beamlines.)

The matrix of Eq. (1.3.9) has the desirable property of having no "cross-plane coupling" so vertical and horizontal motion can be treated independently. But if the quadrupole is rolled (around the s axis) by 45° it becomes a "skew quad". It is not hard to show that the transfer matrix is then

$$\mathbf{M}_q^S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 1 \end{pmatrix}. \tag{1.3.10}$$

Now the off-diagonal elements couple horizontal and vertical motion. In the context of synchrotron light sources, the existence of such elements limit the quality of ribbon electron beams and hence ribbon photon beams. Such beams may be especially important for apparatus that relies on Bragg scattering from crystals, and in that case the thickness of the ribbon is of pre-eminent concern. Since this is a somewhat specialized concern we will mainly simplify the discussion by assuming the absence of cross-plane coupling.

According to Eq. (1.2.9) every transfer matrix has unit determinant and one can confirm this to be the case for the matrices written so far. Any product of these matrices clearly has the same property, which is consistent with Eq. (1.2.9).[‡]

[†] The factors in Eq. (1.3.7) are grouped so that all quantities can be conveniently expressed in M.K.S. units; in particular $P_0 c/e$ is measured in volts.

[‡] The unit determinant condition is known as the "symplectic condition". For matrices larger than 2×2 this condition is necessary but not sufficient. Still, the most general symplectic matrix can be manufactured

1.4. Elliptical (in Phase Space) Beams

Because of the gigantic number of particles they contain, it is appropriate to represent entire beams by distribution functions. Beams that have elliptical shape in phase space are especially appropriate because, though the sizes, aspect ratios, and orientation vary with s, the shapes remain elliptical. The reason for this is that Eqs. (1.2.6) are linear. The sort of distribution envisaged is illustrated in Fig. 1.4.1. For example, the interior of the ellipse might be uniformly populated (typical of a proton storage ring) or particles might be uniformly distributed on the elliptical shell, or (typical of electron storage ring) the beams distributions are "Gaussian distributed", with the density function constant on elliptical curves or ellipsoidal surfaces.

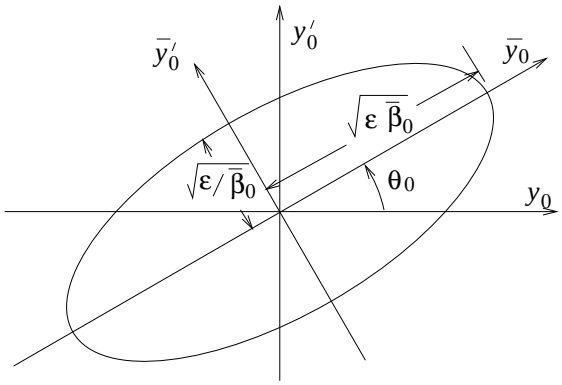

Figure 1.4.1: An elliptical beam in phase space. In the $(\overline{y}, \overline{y}')$-frame, skewed by angle θ_0, the ellipse is erect.

When expressed in coordinates aligned with the ellipse, the general equation of the beam ellipse at $s = s_0$, as shown in Fig. 1.4.1, is

$$\frac{\overline{y}_0^2}{\overline{\beta}_0} + \overline{\beta}_0\, \overline{y}_0'^{\,2} \equiv \overline{\mathbf{y}}_0^T \begin{pmatrix} 1/\overline{\beta}_0 & 0 \\ 0 & \overline{\beta}_0 \end{pmatrix} \overline{\mathbf{y}} = \epsilon_y. \tag{1.4.1}$$

However the beam evolves, its outline has this equation, with appropriate parameters $\overline{\beta}(s), \theta(s), \epsilon_y$. The aspect ratio of the ellipse is governed by $\overline{\beta}(s)$ and its area is given by

$$\text{vertical phase space area} = \pi \sqrt{\epsilon_y \overline{\beta}}\, \sqrt{\epsilon_y / \overline{\beta}} = \pi \epsilon_y. \tag{1.4.2}$$

from the matrices written so far, even in higher dimensions. See Guilleman and Sternberg, *Symplectic Techniques in Physics*, p. 23.

Since this area has been previously shown to be invariant, the parameter ϵ_y is, in fact, *invariant*, which is why it has been given no subscript 0 nor overhead bar. ϵ_y is known as the beam "emittance", though its definition is still ambiguous by an overall constant factor that depends on the nature of the beam distribution, and the relation of ϵ_y to it. For example ϵ_y could be a maximum or an r.m.s. value.

The skew coordinates $\overline{\mathbf{y}}$ are related to erect coordinates by

$$\overline{\mathbf{y}}_0 = \mathbf{R}_0 \mathbf{y}_0, \quad \text{where} \quad \mathbf{R}_0 = \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix}, \tag{1.4.3}$$

and the beam ellipse, expressed in erect coordinates, is

$$\gamma_0 y_0^2 + 2\alpha_0 y_0 y_0' + \beta_0 {y_0'}^2 \equiv \mathbf{y}_0^T \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \mathbf{y}_0 = \overline{\mathbf{y}}_0^T \mathbf{R}_0 \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \mathbf{R}_0^{-1} \overline{\mathbf{y}}_0 = \epsilon_y. \tag{1.4.4}$$

(The result $\mathbf{R}_0^{-1} = \mathbf{R}_0^T$ has been used.) The parameters α, β, γ will be called "beam-based Twiss parameters" for reasons to be explained later. Correlating (1.4.4) with Eq. (1.4.1) yields

$$\begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} = \mathbf{R}_0^{-1} \begin{pmatrix} 1/\overline{\beta}_0 & 0 \\ 0 & \overline{\beta}_0 \end{pmatrix} \mathbf{R}_0. \tag{1.4.5}$$

Since all factors on the right hand side have unit determinant, so also must the left hand side, and hence

$$\gamma_0 = \frac{1 + \alpha_0^2}{\beta_0}. \tag{1.4.6}$$

Because of Eq. (1.4.6), "completing squares" Eq. (1.4.4), the ellipse equation can be simplified to

$$\frac{y_0^2 + (\alpha_0 y_0 + \beta_0 y_0')^2}{\beta_0} = \epsilon_y. \tag{1.4.7}$$

When ϵ_y is expressed in terms of the particle coordinates in this way it is known as the "Courant-Snyder" invariant. Note that the Courant-Snyder invariant is a property of a particle while emittance is a property of a beam—an unfortunate clash of the symbol ϵ having different meanings.

The purpose in introducing the beam ellipse was that its evolution is more significant than that of any one particle. Substituting from Eq. (1.2.6) into Eq. (1.4.4), the matrix of coefficients of the form expressing ϵ_y at s is given by

$$\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} = \left(\mathbf{M}^{-1}\right)^T \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \mathbf{M}^{-1} = \begin{pmatrix} S' & -C \\ -S & C \end{pmatrix} \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}. \tag{1.4.8}$$

This can be multiplied out to be made a bit more explicit:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}. \tag{1.4.9}$$

1.5. The Beam Envelope

To interpret the meanings of the lattice functions β, α, and γ one can study the geometry of the beam ellipse, as shown in Fig. 1.5.1. Of greatest interest is the maximum vertical excursion $E(s)$ of the ellipse, since this can be compared with a scraper or other beam obstacle to see whether the particle or ray will "wipe out". A handy way of locating this point (point M in the figure) is to require that $\nabla\epsilon_y$ have no y' component. From Eq. (1.4.7) the result is $\alpha y + \beta y' = 0$, and

$$E_y(s) = \sqrt{\epsilon\beta(s)}. \tag{1.5.1}$$

The aperture "stay clear" must exceed $E_y(s)$ to avoid particle loss. It is because of the appearance of β in this formula that β is usually considered to be the most important Twiss beam parameter.

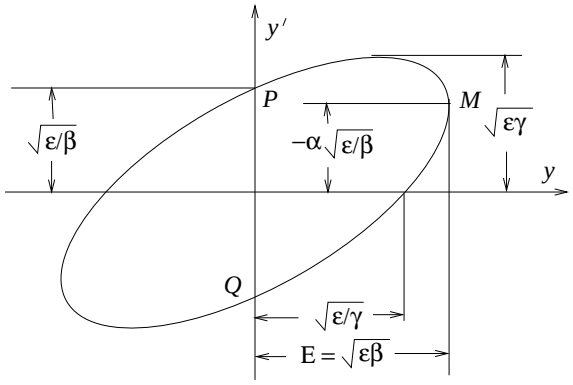

Figure 1.5.1: Correlating beam ellipse parameters with ellipse geometry.

The slope at point M is given by $y'(M) = -\alpha\sqrt{\epsilon/\beta}$. Since this point defines the "beam envelope" the particular trajectory defined by point M has the same slope as the beam envelope at s, which is given by

$$E'_y(s) \equiv dE_y/ds = \frac{\beta'}{2}\sqrt{\frac{\epsilon}{\beta}}. \tag{1.5.2}$$

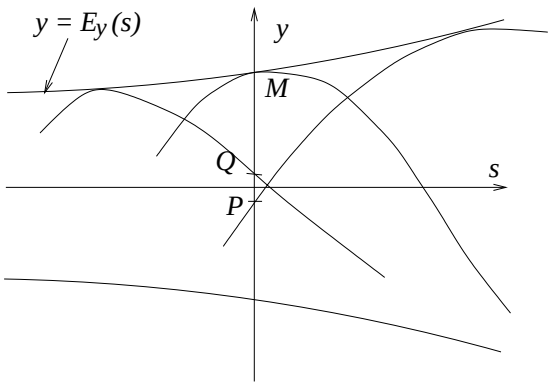

Figure 1.5.2: The beam envelope $E_y(s)$ is equal to the maximum transverse excursion y at longitudinal position s. Point labels M, P, and Q correspond to Fig. 1.5.1.

Equating y' to E'_y yields

$$\beta' = -2\alpha, \tag{1.5.3}$$

meaning that (except for factor -2) α is the slope of the curve of $\beta(s)$ plotted against s.

1.6. Gaussian Beams: Their Variances and Covariances

The joint probability distribution for a "Gaussian distributed" beam is

$$\rho\left(y, y'\right) = K \exp\left(-\frac{1}{2\epsilon_{y,b}} \mathbf{y}^T \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \mathbf{y}\right) = K \exp\left(-\frac{\epsilon_y}{2\epsilon_{y,b}}\right) \ , \tag{1.6.1}$$

where the factor K has to be chosen to fix the normalization. (Recall that the combination $\mathbf{y}^T \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \mathbf{y}$, being the Courant-Snyder invariant, has the same value everywhere in the ring.) This probability density falls to $e^{-1/2}$ when ϵ_y (as given by Eq. (1.4.7)) is equal to a specified beam emittance $\epsilon_{y,b}$. This distribution is simpler when expressed in terms of erect coordinates $\overline{\mathbf{y}} = \mathbf{R}^{-1}\mathbf{y}$, where $\mathbf{R}$ is the rotation matrix of Eq. (1.4.3), but applicable at lattice point s. Because this transformation is a simple rotation the probability distributions are related by $\overline{\rho}(\overline{y}, \overline{y}') = \rho(y, y')$, and hence

$$\overline{\rho}\left(\overline{y}, \overline{y}'\right) = \frac{1}{2\pi\epsilon_{y,b}} \exp\left(-\frac{\overline{y}^2}{2\epsilon_{y,b}\overline{\beta}} - \frac{\overline{y}'^2}{2\epsilon_{y,b}/\overline{\beta}}\right). \tag{1.6.2}$$

Here the normalization is easily done and has yielded $K = 1/(2\pi\epsilon_{y,b})$.

The elements of the "variance-covariance matrix" are the expectation values, or moments, $< y^2 >, < yy' >, < y'^2 >$:

$$\begin{aligned}\begin{pmatrix} < y^2 > & < yy' > \\ < yy' > & < y'^2 > \end{pmatrix} &= < \mathbf{y}\mathbf{y}^T > = < \mathbf{R}^{-1}\overline{\mathbf{y}\mathbf{y}}^T\mathbf{R} > = \mathbf{R}^{-1}\begin{pmatrix} < \overline{y}^2 > & 0 \\ 0 & < \overline{y}'^2 > \end{pmatrix}\mathbf{R} \\ &= \mathbf{R}^{-1}\begin{pmatrix} \epsilon_{y,b}\overline{\beta} & 0 \\ 0 & \epsilon_{y,b}/\overline{\beta} \end{pmatrix}\mathbf{R}.\end{aligned} \tag{1.6.3}$$

The expression on the right hand side can be obtained from Eq. (1.4.5):

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}^{-1} = \mathbf{R}^{-1}\begin{pmatrix} \overline{\beta} & 0 \\ 0 & 1/\overline{\beta} \end{pmatrix}\mathbf{R}. \tag{1.6.4}$$

Finally then, we obtain

$$\begin{pmatrix} < y^2 > & < yy' > \\ < yy' > & < y'^2 > \end{pmatrix} = \begin{pmatrix} \epsilon_{y,b}\beta & -\epsilon_{y,b}\alpha \\ -\epsilon_{y,b}\alpha & \epsilon_{y,b}\gamma \end{pmatrix}. \tag{1.6.5}$$

For the special case $\alpha = 0$ this can be easily checked.

It has been shown then that (except for factor $\epsilon_{y,b}$) the matrix of coefficients in the quadratic form for ϵ_y is the inverse of the variance-covariance matrix. Commonly (for example in the program TRANSPORT) the variance-covariance matrix is designated as $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \begin{pmatrix} < y^2 > & < yy' > \\ < yy' > & < y'^2 > \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \epsilon_{y,b}\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}. \tag{1.6.6}$$

Because of this relation, the known evolution of the beam Twiss parameters (see the first of Eqs. (1.4.8)) can be used to obtain the evolution of σ:

$$\boldsymbol{\sigma} = \mathbf{M}^T\boldsymbol{\sigma}_0\mathbf{M}. \tag{1.6.7}$$

Note that it is $\mathbf{M}$ appearing in this equation, whereas it was $\mathbf{M}^{-1}$ that appeared in Eq. (1.4.8).

1.7. Pseudoharmonic Trajectory Description

It has been seen in Eq. (1.5.1) that the beam envelope scales proportional to $\sqrt{\beta}$. One conjectures therefore, that individual trajectories will scale the same way and be describable in the form

$$y\left(s\right)=a\sqrt{\beta\left(s\right)}\cos\left(\psi-\psi_0\right), \tag{1.7.1}$$

where ψ depends on s and a is a constant amplitude. This form has to satisfy Eq. (1.2.1), and the $\cos(\psi-\psi_0)$ "ansatz" is based on the known solution when $K(s)$ is, in fact, constant. This is the basis for naming the trajectory description "pseudoharmonic". As β appears in this equation, it has no apparent "beam" attribute and it will be referred to as a "lattice function". Differentiating Eq. (1.7.1) we get

$$y'\left(s\right)=-a\sqrt{\beta\left(s\right)}\psi'\sin\left(\psi-\psi_0\right)+\frac{a\beta'}{2\sqrt{\beta}}\cos\left(\psi-\psi_0\right). \tag{1.7.2}$$

Substituting into Eq. (1.2.1) we can demand that the coefficients of sin and cos terms vanish independently, since that is the only way of maintaining equality for all values of ψ_0. This leads to the equations

$$\begin{aligned}&\beta\,\psi''+\beta'\psi'=0,\\&2\beta\,\beta''-\beta'^2-4\beta^2\psi'^2+4\beta^2K\left(s\right)=0.\end{aligned} \tag{1.7.3}$$

From the first equation it follows that $\beta\,\psi'$ is constant. To obtain the conventional description we pick this constant to be 1 and obtain

$$\psi'=\frac{1}{\beta},\quad\text{or}\quad\psi\left(s\right)=\psi\left(s_0\right)+\int_{s_0}^{s}\frac{ds'}{\beta\left(s'\right)}. \tag{1.7.4}$$

Since the argument of a harmonic wave is $2\pi s/$wavelength this permits us to visualize $2\pi\beta(s)$ as a "local wavelength". Substituting into the second of Eqs. (1.7.3), we obtain

$$\beta''=2\beta\,K+2\frac{1+\beta'^2/4}{\beta}. \tag{1.7.5}$$

This is usually considered to be the fundamental defining relationship for the lattice β-function. A "first integral" can be obtained by combining Eqs. (1.7.1) and (1.7.2) to give

$$\frac{y^2}{\beta}+\beta\left(y'-\frac{\beta'}{2\beta}y\right)^2=a^2. \tag{1.7.6}$$

According to Eq. (1.5.3) we have $\beta' = -2\alpha$, so this result confirms the constancy of the Courant-Snyder invariant, (Eq. (1.4.7)) and a^2 can be identified with what was previously called $\epsilon_{y,b}$. There is an essential difference however. In Eq. (1.4.7) the constancy referred to all points on the beam ellipse. Here it refers to all points s on the ray or trajectory. Fortunately, as the symbols have been introduced, one can now afford to forget this distinction.

1.8. Transfer Matrix Parameterization

The transfer matrix from s_0 to s can by obtained by substituting pseudoharmonic trajectory formulas into Eq. (1.2.6). The sine-like and cosine-like solutions are

$$\begin{aligned} C(s, s_0) &= \sqrt{\beta/\beta_0}\,(\cos(\psi - \psi_0) + \alpha_0 \sin(\psi - \psi_0)) \\ S(s, s_0) &= \sqrt{\beta\beta_0}\,\sin(\psi - \psi_0)\,. \end{aligned} \tag{1.8.1}$$

As a result the transfer matrix is

$$\mathbf{M}(s_0, s) = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}\,(\cos(\psi - \psi_0) + \alpha_0 \sin(\psi - \psi_0)) & \sqrt{\beta_0\beta}\,\sin(\psi - \psi_0) \\ \frac{-(1+\alpha_0\alpha)\sin(\psi-\psi_0)+(\alpha_0-\alpha)\cos(\psi-\psi_0)}{\sqrt{\beta_0\beta}} & \sqrt{\frac{\beta_0}{\beta}}\,(\cos(\psi - \psi_0) - \alpha \sin(\psi - \psi_0)) \end{pmatrix}. \tag{1.8.2}$$

Since these parameters have arisen from describing the "lattice" of focusing elements, one can call them "lattice Twiss parameters". The evolution of beams or particles with distance s along the beam line, can be represented by specifying s-dependent parameters, $\beta(s), \alpha(s), \psi(s), \ldots$, instead of in the form of Eq. (1.2.6).

In the special (and important) case $\beta = \beta_0$, $\alpha = \alpha_0$, Eq. (1.8.2) reduces to

$$\mathbf{M} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}, \quad \text{where} \quad \gamma = \frac{1+\alpha^2}{\beta}. \tag{1.8.3}$$

This is known as the standard Twiss parameterization of a 2×2 matrix with unit determinant. One should confirm that this form has the correct number of free parameters and the correct determinant.† This form will turn out to be most useful for describing *periodic* focusing arrays, for example the "once-around" transfer matrix for a complete storage ring.

† Any valid transfer matrix can be expressed as in Eq. (1.8.3), but it is a mistake to suppose that the parameters β, α, μ, would bear any simple relation with the parameters β, α, ψ appearing in Eq. (1.8.2), except in the special case mentioned at the start of the paragraph.

1.9. Reconciliation of Beam and Lattice Parameters

As yet no clean connection has been demonstrated between beam-based Twiss parameters and lattice-based Twiss parameters, even though the same symbols have been used. Furthermore, the lattice properties are fixed, once and for all, while the beam properties can vary wildly, depending upon injection tuning or mistuning. It is clear, therefore, that there can be no simple equality between beam parameters and lattice parameters. Nevertheless there is an extremely close connection that will be investigated next.

1.9.1. Beam Evolution Through a Drift Section

Suppose the region from s_0 to s is purely a drift section. According to Eq. (1.3.1) and the first of Eqs. (1.4.8), the beam evolution is given by

$$\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \tag{1.9.1}$$

Completing the multiplication, one finds

$$\begin{aligned} \beta &= \beta_0 - 2\alpha_0\, s + \gamma_0\, s^2, \\ \alpha &= \alpha_0 - \gamma_0\, s, \\ \gamma &= \gamma_0. \end{aligned} \tag{1.9.2}$$

One finds, upon substituting $\beta(s)$ given here into Eq. (1.7.5), that the beam-based $\beta(s)$ satisfies the lattice-based differential equation, at least in drift sections. If the two versions of β are equal at one point in the drift section, they will remain equal throughout the section.

An important special case of Eqs. (1.9.2) describes the evolution of β away from a "beam waist" where $\alpha = \alpha^* = 0$ so that $\gamma^* = 1/\beta^*$ and $\beta = \beta^*$ has an extreme value, which we take to be a minimum. Substituting into the first of Eqs. (1.9.2) we obtain

$$\beta\,(z) = \beta^* + \frac{z^2}{\beta^*}. \tag{1.9.3}$$

A "waist length" can be defined by

$$z^{(P)}_{\text{waist}} = \beta^*; \tag{1.9.4}$$

it is the length after which the beam transverse-dimension-squared has doubled.† Because the transverse beam size is proportional to $\sqrt{\beta}$, the variation of the beam half-width is given by $w(z) = w^* \sqrt{1 + (z/\beta^*)^2}$, as shown in Fig. 1.9.1.

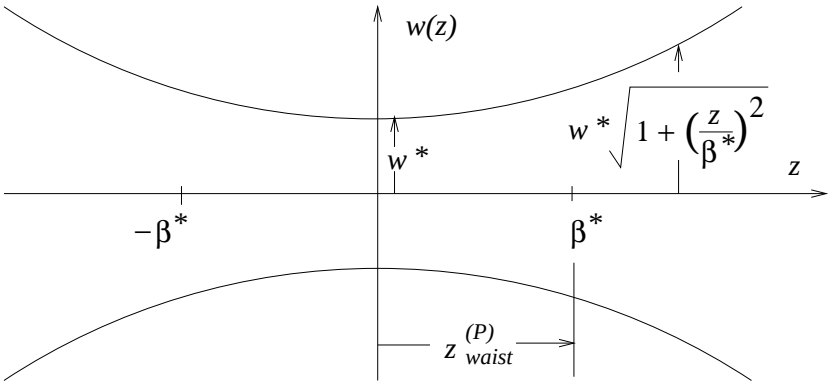

Figure 1.9.1: Beam half-width w variation near a beam waist.

1.9.2. Beam Evolution Through a Thin Lens

Consider next the beam evolution in passing through a thin lens. According to Eq. (1.3.4) and the first of Eqs. (1.4.8), the beam evolution is given by

$$\begin{pmatrix} \gamma_+ & \alpha_+ \\ \alpha_+ & \beta_+ \end{pmatrix} = \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_- & \alpha_- \\ \alpha_- & \beta_- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \tag{1.9.5}$$

Completing the multiplication, one finds

$$\begin{aligned} \beta_+ &= \beta_-, \\ \alpha_+ &= \alpha_- - \beta\, q, \\ \gamma_+ &= \gamma_- - 2\alpha_- q + \beta\, q^2. \end{aligned} \tag{1.9.6}$$

Problem. Equations (1.9.2) showed that evolution of *beam*-based (α, β, γ) parameters through a drift section is governed by the same formulas as is the evolution of *particle*-based (α, β, γ) parameters. Using Eq. (1.7.5), show that the same statement can be made for evolution through a thin lens. That is, use Eq. (1.7.5) to calculate the "kink" in the β-function caused by a thin lens, and show that the result agrees with Eqs. (1.9.6).

† The superscript P on $z^{(P)}_{\rm waist}$ refers to the implicit assumption that particles (as contrasted to the waves that will be introduced shortly) are being analysed.

For lines made up only of drifts and thin lenses, it has therefore been shown that beam-based and lattice-based Twiss functions, if once equal, remain equal. Since, as stated earlier, all transfer matrices can be constructed from drifts and thin lenses, the result is true for any general (uncoupled) beamline.

1.10. Photon Beam Features "Inherited From" the Electron Beam

It has been stated repeatedly that all formulas so far apply to either electrons or photons. To analyse the storage ring as a source of synchrotron radiation, athough we initially emphasize electrons, we are ultimately more interested in the photons. It is the high "intensity" available from the storage ring that justifies using such large and expensive apparatus and justifies our continuing effort to quantify and increase intensity. The word "intensity" has been placed in quotation marks to warn that it will be used loosely to stand for any one of the numerous measures going by such names as brightness, brilliance, current, illuminance, intensity, inverse emittance, luminance, (spectral) radiance, and so on. To the extent these terms are not equivalent, they refine different essentials of the radiation such as angular, spatial, or frequency spread, or power content. We will return to some of these things in detail later on. (Of course intensity will also retain its unambiguous, quantitative, relationship to power flux and the Poynting vector.)

For now we note that the photon beam "inherits", from the electron beam, limitations on all these measures. The main measures of electron intensities are total number N of circling particles and I, the total beam current. They are related by

$$I = eNf = eN\frac{c}{\mathcal{C}} \quad \left(\stackrel{\text{e.g.}}{=} 0.1\,\text{A} \right), \tag{1.10.1}$$

where f is the revolution frequency and $\mathcal{C}$ is the circumference. A prototypical differential measure of electron intensity is $I/\epsilon_{y,b}$ where "emittance" $\epsilon_{y,b}$ was introduced in Eq. (1.6.1). There is a (not particularly well-known nor well-developed) accelerator formalism that identifies emittance with entropy.† One thing for sure though is that emittance has a strong tendency to increase. This could probably be put on a firm, second law of thermodynamics,

† I learned this from ex-CLEO-postdoc, Garry Trahern, but the only reference I know is *Emittance, Entropy, and Information*, Lawson, LaPostalle and Gluckstern, Particle Accelerators, **5**, 61 (1973).

foundation. The emittance of a particle beam decreases only because of stochastic cooling (which does not violate the second law) or because entropy is carried off by radiation.

Even if the electrons form a perfect pencil beam (no spatial or angular spread) and are perfectly monochromatic (no momentum spread) the radiated photons have a continuous frequency spread (usually scaled to the so-called "critical frequency" ω_c.) and a finite angular spread. The characteristic radiation angle is

$$\theta_{\gamma,typ.} = \frac{1}{\gamma} = \frac{mc^2}{E} \quad \left(\stackrel{\text{e.g.}}{=} \frac{0.5\,\text{MeV}}{5\,\text{GeV}} = 10^{-4}. \right) \tag{1.10.2}$$

This is a very small angle. Is it negligible? That is, is this spread small compared to the angular spread of electrons in the storage ring? In one sense it is certainly small, since the electron's horizontal angle ranges through 2π.† On the other hand, the typical vertical electron angle (from Eq. (1.6.5)) is

$$\theta_{y,typ.} = \sqrt{\frac{\epsilon_{b,y}}{\beta}} \quad \left(\stackrel{\text{e.g.}}{=} \sqrt{\frac{10^{-8}\,\text{m-rad}}{1\,\text{m}}} = 10^{-4}. \right) \tag{1.10.3}$$

(Unfortunately) according to these estimates, $\theta_{\gamma,typ.}$ and $\theta_{e,typ.}$ are comparable in magnitude, so neither can be neglected. Of course the factors entering Eq. (1.10.3) are, to some extent, subject to adjustment. For example, β can be easily increased by 10 or 100, so the electron angle can be made negligible. But one knows (from Eq. (1.6.5)) that a proportional increase in spatial spread is inevitable.

The ultimate photon distribution is a convolution of the electron distribution with the radiation pattern. Based on the above estimates, the horizontal angle of the photon relative to its radiating electron can sometimes be neglected, but vertical distributions are influenced both by the electron distribution and the distribution of vertical photon angles relative to the electron.

† The estimates in this section will have to be reconsidered when insertion devices (wigglers or undulators) are introduced. For now we just consider the pure synchrotron radiation.

1.11. Light: Waves or Particles?

Another issue to be considered, at least qualitatively, is whether it is valid to consider synchrotron radiation as made up of discrete photons (as we have been doing.) When we consider undulators we will find that wave features are essential, but we will now see that treating the synchrotron radiation from the regular arc bending magnets as particles is justified for sufficiently hard X-rays.

For light of frequency ν, wavelength $\lambda \overset{\text{e.g.}}{=} 10^3 \mathring{A}$; (being near ultraviolet, this is toward the long wavelength end of the range of wavelengths we are likely to be interested in) the wave number k is given by

$$k = \frac{2\pi\nu}{c} = \frac{2\pi}{\lambda} \quad \left(\overset{\text{e.g.}}{=} 2\pi \times 10^7 \, 1/\text{m} \right). \tag{1.11.1}$$

It is possible to form, say at $z = 0$, a "circular point image" of radius w_0 using plane waves of this wavelength, all traveling more or less parallel to the z axis. But the width w of such a wave necessarily increases for $z \neq 0$. By symmetry the spreading is $\pm z$-symmetric, so the leading deviation from w_0 is quadratic in z. It is shown on p. 268 of *Optical Coherence and Quantum Optics*, Mandel and Wolf, that the spreading of a cylindrically symmetric beam is given by†

$$w^2(z) = w_0^2 + \frac{z^2}{k^2 w_0^2/4}. \tag{1.11.2}$$

In a purely particle picture discussed up to this point the beam spreading is given by Eq. (1.9.3). With w being proportional to $\sqrt{\beta}$ these dependencies are strikingly similar. We can now define a "wave-based, waist length" $z^{(W)}_{\text{waist}}$

$$z^{(W)}_{\text{waist}} = \frac{k w_0^2}{2}, \tag{1.11.3}$$

as the length over which the beam transverse-dimension-squared doubles according to wave theory. This can be compared to $z^{(P)}_{\text{waist}}$ as given by Eq. (1.9.4). Since the beam waist is determined by the electron beam we set $w_0^2 = \epsilon_{y,b}\beta^*$ and obtain

$$\frac{z^{(W)}_{\text{waist}}}{z^{(P)}_{\text{waist}}} = \frac{k\epsilon_{y,b}}{2} \quad \left(\overset{\text{e.g.}}{=} \pi \times 10^7 \times 10^{-8} \right). \tag{1.11.4}$$

† Essentially the same formula is derived in the chapter of these notes devoted to free electron lasers. The spreading can perhaps also be calculated by using only the Heisenberg uncertainty principle, since the transversely localized beam spot has to have a matching angular spread.

For the particle picture to be justified this ratio should be larger than 1. For the example numbers that have been used (near ultraviolet) the condition is not quite satisfied (in the vertical dimension) but for shorter wavelengths it becomes progressively more valid. On the other hand emittances smaller than the value $\epsilon_{y,b} = 10^{-8}\,\mathrm{m}$ assumed here can certainly be achieved, and this makes the condition harder to meet.

In the jargon of synchrotron light sources, the radiated beam is said to be "diffraction dominated" when the ratio in Eq. (1.11.4) is *not* large compared to 1. This is more likely at long than at short wavelengths.

1.12. Measures of Intensity, Flux, Brilliance and Brightness

An X-ray beam consists of the synchrotron radiation (inevitably restricted by a fixed aperture) some five or more meters from an ID (insertion device) in the storage ring. The least ambiguous property of such a beam is its *power*. Also, since the radiation comes in the form of photons, one can count the photons coming down the line one-by-one. Any apparatus making use of this beam has a "nominal" or "central" wavelength λ_{nom}, and in most cases it is only the photons close to this wavelength that are useful. For purposes of comparing the relative usefulness of different X-ray sources, this bandwidth is conventionally taken to be $\lambda_{\mathrm{nom}}/1000$. "Flux" is defined to be the number of photons per second within this band.† The more nearly parallel these photons are, the more useful they are, and one defines "brightness" to be (number of photons within 0.1% of nominal)/solid angle/time. However, defined in this way, brightness is not very useful, because it is not an *invariant* measure; i.e. it is altered by optical elements (lenses and drift spaces) along the beamline. To obtain an invariant measure, one defines the "brilliance" to be (number of photons within 0.1% of nominal)/area/solid angle/time. By Liouville's theorem, this quantity is preserved by arbitrary paraxial optical systems.‡ Brilliance is therefore much

† In every area of physics other than synchrotron light, the term "flux" has units of "something" per area. But the definition given here seems to be the agreed-upon usage.

‡ The distinction between brightness and brilliance is is much like the distinction, in beam physics, between r.m.s. transverse beam "spot size" σ_x and transverse emittance ϵ_x, which are related by $\sigma_x(s) = \sqrt{\epsilon_x\,\beta_x(s)}$. Because the lattice function $\beta_x(s)$ depends on s, "spot size" $\sigma_x(s)$ also depends on s. But the ratio $\epsilon_x = \sigma_x^2(s)/\beta_x(s)$ is invariant; i.e. independent of s.

Table 1.12.1: Beam Intensity Measures

parameter	unit	invariant?
power	Watt	yes
flux	photons/0.1%BW/s	yes
brightness	photons/0.1%BW/(mrad)^2/s	no
brilliance	photons/0.1%BW/(mrad)^2/(mm)^2/s	yes
cleanliness	photons/0.1%BW/(mrad)^2/(mm)^2/Joule	yes

preferable to brightness, but neither can distill into a single number all relevant spectral features.¶

There is another measure that has no name, and is rarely spoken of except implicitly, though it should be, since it is important in the design of ID's, and is what makes undulators attractive. It measures the fraction of the power that is actually useful. For an apparatus that happens to make use of only those photons within the nominal bandwidth (0.1%) and the nominal spot-size×solid-angle product (which is 1 mrad^{2}mm^2) this measure would be brilliance/power. For low flux beams this measure may be inessential because monochromators can filter out the useless photons effectively, but for high flux beams, power expense may be appreciable and power handling difficult. When "raw power" considerations become important, a design that produces mainly useful photons is obviously desirable. An ideal ID would have enough flexibility to meet this goal. I will refer to this brilliance/power ratio as "cleanliness", since it quantifies the mess that may have to be cleaned up. Also I adopt "cleanliness is next to godliness" as one design criterion.

We will only make some estimates that illustrate the considerations involved. A prototypical synchrotron light beamline is shown in Fig. 1.12.1. For various practical reasons it is difficult to make the length L of the beamline really short. The separation distance from the accelerator, $D \approx L^2/(2R)$, is at least a few (say 2) meters, so

$$L \approx \sqrt{2RD} \quad \left(\stackrel{\text{e.g.}}{=} \sqrt{2 \times 100 \times 2} \approx 20\,\text{m.} \right) \tag{1.12.1}$$

¶ The fact that some people interchange the meanings of "brightness" and "brilliance" makes the distinction more confusing. Following Sol Gruner's recommendation, and APS Report ANL-88-9, I believe that the usage here is more "correct" and urge that it be adopted universally.

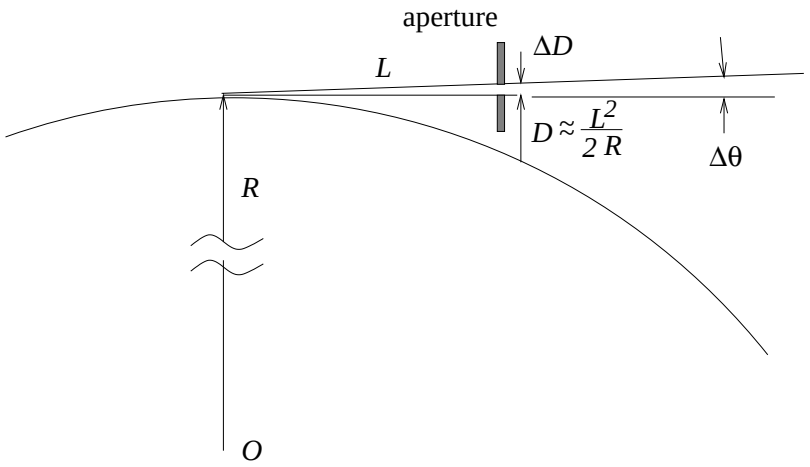

Figure 1.12.1: Prototypical synchrotron light beamline.

A result to be derived in a next chapter (Eq. (2.2.6)) is that the total energy radiated, per electron, per turn, is given by

$$U_0^{\rm rad} = C_\gamma \, \frac{E^4}{R} \quad \left(\stackrel{\rm e.g.}{=} 1\,{\rm MeV} \right), \tag{1.12.2}$$

where $C_\gamma = 0.885 \times 10^{-4}\,{\rm m/Gev}^3$ and E is the electron energy. This formula gives $U_0^{\rm rad}$ in GeV units. A rather small fraction ($\Delta\theta/(2\pi)$) of all the radiated energy passes through the aperture ΔD. Combining these results with Eqs. (1.10.1), the total power through the aperture is

$$\text{power} \;= \frac{I}{e} C_\gamma \, \frac{E^4}{R} \, \frac{\Delta D}{L} \, \frac{1}{2\pi} \quad \left(\stackrel{\rm e.g.}{=} 10^6 {\rm eV/e} \times \frac{10^{-2}}{20} \, \frac{1}{2\pi} = 80\, I \, \frac{\text{watts}}{\text{(horizontal)cm}}. \right) \tag{1.12.3}$$

Another result to be derived later is that the total number of synchrotron radiation photons emitted, per revolution, by a single electron in a uniform ring, is given by

$$\mathcal{N}_0 = \frac{ecC_\gamma E^3 B}{0.3079 \gamma u_c} = 0.0662\,\gamma, \tag{1.12.4}$$

where the "critical energy" u_c, ($\stackrel{\rm e.g.}{=} 3\,{\rm keV}$) will be defined later. One need only accept (for now) the remarkable facts that this number is independent of magnetic field B and that the dependence on E is simple proportionality to γ.† Combining this with Eqs. (1.10.1) and (1.12.4), the total number of photons passing through the aperture per second is

$$\mathcal{N} = \frac{I}{e}\, 0.0662\, \gamma \, \frac{\Delta D}{2\pi L} \quad \left(\stackrel{\rm e.g.}{=} \frac{1}{1.6 \times 10^{-19}{\rm C}}\, 662\, I \, \frac{\Delta D}{L} \, \frac{1}{2\pi} = 0.66 \times 10^{21}\, I \, \frac{\Delta D}{L} \, \frac{\text{photons}}{\text{s}}. \right) \tag{1.12.5}$$

† The fact that the number $\mathcal{N}_0$ can be defined *at all* is because the spectrum is *not* divergent at long wavelengths—there is no infrared divergence.

By convention the "flux" $\mathcal{F}$, counts only photons in an energy range $\Delta u = 10^{-3} u_{\rm oper.}$ where $u_{\rm oper.}$ is the operative photon energy. This is the appropriate figure of merit for apparatus that responds only to a narrow band of energies. Let us measure photon energy u in units of the "critical energy" u_c. (i.e. $u_c = 1$.) The energy spectrum is described by a (normalized to 1) probability distribution $P_u(u)$ such that $P_u(u)du$ is the probability a particular photon energy u lies in the range u to $u + du$. The "number spectrum" is then given by

$$\frac{d\mathcal{N}}{du} = \mathcal{N}\, P_u\,(u) \ . \tag{1.12.6}$$

For purposes of estimation, let us take $u_{\rm oper.} = u_c$. With u measured in units of u_c (as we are assuming) it happens that $P_u(1) \approx 0.1$.‡ Then $\mathcal{F}$ acquires a factor $P_u(1)\, 10^{-3} \approx 10^{-4}$, compared to Eq. (1.12.5):

$$\mathcal{F} = 0.66 \times 10^{17}\, I\, \frac{\Delta D}{L}\, \frac{\text{photons}/s}{0.1\%BW} \ . \tag{1.12.7}$$

One is also interested in intensities that are differential in solid angle and source area—though the rates given so far are already proportional to ΔD. One complication is that, at the plane of the aperture, the horizontal angle is correlated with the horizontal position. This is like the correlation of rays emerging from a point source. This correlation is especially significant if $\Delta\theta >> 1/\gamma$, which is commonly the case. In principle this correlation can be removed by sending the beam through a cylindrical, horizontally-focusing lens. In the interest of obtaining a simple formula, let us assume this has been done (whether or not it is practical).† This compensates for the longitudinally spread out source and validates treating all the photons as being radiated from the same, constant s, plane. Neglecting the photon angle relative to the electron, the photon beam simply inherits the emittances of the electron beam. According to Eq. (1.4.2) the product of transverse phase space areas is

$$\Delta x \Delta x' \Delta y \Delta y' = \pi^2 \epsilon_{x,b} \epsilon_{y,b}. \tag{1.12.8}$$

‡ Since this approximate value becomes implicitly contained in all subsequent formulas, one should check the actual numerical value of $P_u(u_c)$ before using the formulas for accurate work.

† Lenses are certainly practical for the visible part of the spectrum, but not necessarily for ultraviolet and X-rays. Unfortunately the approximations that have been made are somewhat more valid for the hard photons than for the soft. So the formulas have only semi-quantitative validity, for one reason or another, at all wavelengths.

(Recall that this assumes somewhat unrealistic, uniform-within-ellipse, vanishing-outside, phase space distributions.) A six-fold differential flux is then given by

$$\frac{d^6 N_\gamma}{dt\, dx\, dx'\, dy\, dy' du} = \frac{d^5 N_\gamma}{dt\, dx\, dx'\, dy\, dy'} P_u(u)\,. \tag{1.12.9}$$

and the brilliance is given by

$$\mathcal{B}(u_c) = \frac{d^6 N_\gamma}{dt\, dx\, dx'\, dy\, dy'\, du}\, 10^{-4} \approx \frac{I}{e}\, 0.0662\, \gamma\, \frac{\Delta D}{2\pi L}\, \frac{10^{-4}}{\pi^2 \epsilon_{x,b} \epsilon_{y,b}}. \tag{1.12.10}$$

A virtue of formula (1.12.7), for flux $\mathcal{F}$, is that it contains only unambiguous factors. The formula for $\mathcal{B}$ is not quite so unambiguous, though the extra (emittance) factors are advertised storage ring parameters, known at the percent level. For CESR, a typical value is $\epsilon_{x,b} \approx 2 \times 10^{-7}$ m-rad. The factor $\epsilon_{y,b} \tilde{<} \epsilon_{x,b}/100$, is somewhat less well known (and, in colliding beam operation, varies with time). But, if necessary, $\epsilon_{y,b}$ can be known to several percent accuracy. It must be remembered though, that there are different conventions governing the definition of emittance. For example, the CESR values just mentioned are r.m.s. values, not necessarily quite right for immediate substitution into Eq. (1.12.10).

The brilliance parameter $\mathcal{B}$ is used primarily as a figure of merit specifying wiggler and undulator beams in "later generation" synchrotron light sources that have ultra-small emittances. Customarily the units of the phase space factor are taken as $\mathrm{mm}^2\,\mathrm{mr}^2$, which reduces the numerical value by 10^{12} compared with M.K.S. units. Even so, since beam dimensions in dedicated synchrotron light sources are typically small compared to a mm, the numerical values of $\mathcal{B}$ can exceed the numerical value of $\mathcal{F}$ by several orders of magnitude. Unfortunately the photon spreads for these beams contribute importanly to the brilliance. One must therefore be very careful in using the brilliance as a figure of merit. Technically, the brilliance is the value of the derivative shown in Eq. (1.12.10) evaluated at the origin. Since the distribution is highly peaked, Eq. (1.12.10) is unreliable for high brilliance beams.

The external factor $\Delta D/(2\pi L)$, characterizing the external acceptance angle is, of course, well known, but certain idealizations have gone into Eq. (1.12.7), that can cause it to break down, especially at long wavelengths. Also the brilliance given by Eq. (1.12.10) must still be corrected for the actual operating energy:

$$\mathcal{B}(u) = \mathcal{B}(u_c)\, \frac{u_{\mathrm{oper.}}\, P_u(u_{\mathrm{oper.}})}{u_c\, P_u(u_c)}\,. \tag{1.12.11}$$

(As mentioned in a footnote, for accurate work, the actual value of $P_u(u_c)$ has to be checked.)

Also it is important to appreciate that only radiation from regular arc bends has been considered. Obviously wigglers and undulators have to be considered separately.

Chapter 2.
Synchrotron Radiation: Wave Properties

An electron traveling in a uniform magnetic field emits synchrotron radiation. In this chapter we will assume the orbit is a continuous circle, but the formulas can be easily adapted to shorter segments (as long as they are not *too* short.) The so-called "arc magnets" of an ordinary electron storage ring amply satisfy this requirement, but some insertion devices (and proton accelerators) may require special treatment.

Though the theoretical formulas describing synchrotron radiation are well within the difficulty level of texts like Jackson, *Classical Electrodynamics*, their accurate evaluation is rather difficult. In this chapter the most important ideas and formulas will be introduced, and plausible approximation schemes contemplated, but accurate evaluation will not yet be attempted.

The text I recommend (and follow) for everything up to, but not yet including, synchrotron radiation, is Griffiths, *Introduction to Electrodynamics*. For detailed synchrotron radiation calculations there is, of course, Jackson. Other good references are Albert Hoffmann, *Characteristics of Synchrotron Radiation*, in *Synchrotron Radiation and Free Electron Lasers*, CERN 98-04, and Julian Schwinger, *On Radiation by Electrons in a Betatron*, 1945, (transcribed by Miguel Furman), LBNL-39088/CBP Note-1996.

The most subtle physics associated with synchrotron radiation is the concept of retarded time. Once that has been grasped all that is left is mathematical manipulation. Unfortunately these manipulations bring in Macdonald functions and Airy integrals and other complications too fierce to mention. This mathematical complexity tends to obscure the physics. A major purpose of this chapter is to attempt approximations for which only elementary mathematical functions are adequate. Most of these approximations are well justified, but two are rather poorly controlled. The first of these is a "physics" assumption. Speaking very loosely, it will entail an uncertainty at the 30% level. For some purposes an error of this magnitude might be tolerable but still it is uncomfortably large. Fortunately we will later be able to repair this approximation. The second poorly justified approximation is of a purely mathematical nature. It will introduce an uncontrolled uncertainty at perhaps the 10% level.

The basis for these error estimates is that an exact formulation (due originally to Schwinger) is known. This being the case, one might well ask, "why not just study the exact theory?" The reason, as already stated, is that the exact theory is mathematically repulsive and does not provide much useful intuition. A consequence of this is that one must later approach new devices, such as undulators or free electron lasers, without much intuitive guidance. It is hoped that the approximate formulation presented here will improve intuition concerning where the dominant contributions and polarizations come from and into such subtleties as coherence and diffraction.

Even if these justifications are all just rationalizations, following the formulas in these notes can serve for "back of envelope" calculations and the deviations can serve as warm-up to the herculean task of following an exact treatment.

2.1. Electric Field Calculation

The basic formulas on which synchrotron radiation is based are the so-called "retarded potentials":

$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\int\int\frac{\rho(\mathbf{r}',t_r)}{\mathcal{R}}\,d^3\mathbf{r}',\quad \mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\int\int\frac{\rho(\mathbf{r}',t_r)\;\mathbf{v}(\mathbf{r}',t_r)}{\mathcal{R}}\,d^3\mathbf{r}', \tag{2.1.1}$$

where $\mathbf{r}'$ and $\mathbf{r}$ are know as "source point, Q" and "field point, P" respectively, as shown in Fig. 2.1.1. The integrals run over all charge-containing volumes. Formulas essentially like these should be familiar from elementary E.&M. The present formulas differ from electro- and magnetostatics only in the seemingly inconsequential respect of allowing for the time of signal propagation from $\mathbf{r}'$ to $\mathbf{r}$. That is, to obtain the fields at P at time t the integrands are to be evaluated at a "retarded time" t_r such that

$$t=t_r+\frac{\mathcal{R}}{c}\ . \tag{2.1.2}$$

(If the charge density ρ is time-independent, this modification has no effect, and Eqs. (2.1.1) reduce to the Coulomb and Biot-Savart laws.)

This retardation modification makes an important difference only when charges are moving at close to the speed of light, which is of course the case in a synchrotron light

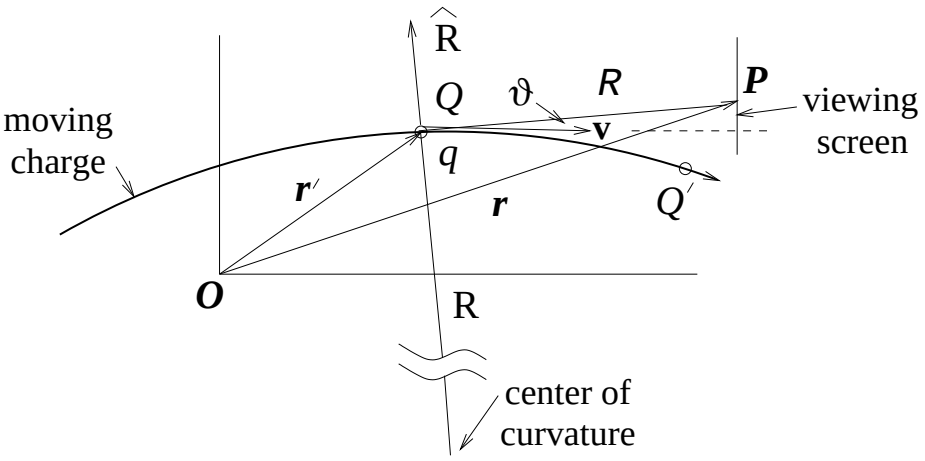

Figure 2.1.1: Symbol definitions: $\mathbf{r}$ = radius vector of "field point" P; $\mathbf{r}'$ = radius vector of "source" point Q; $\mathcal{R}$ = distance from source point to field point. R is the local bending radius and $\hat{R}$ is the outward normal. When light from Q gets to P the particle is already at Q'.

source. There is a kind of "sonic boom" phenomenon in which, if a source particle approaches the field point P at high speed, signals from an appreciable path interval "pile up" at P.†

The electric and magnetic fields are obtained from the potentials by†

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{2.1.3}$$

It is important to relate retarded time interval dt_r to corresponding observation time interval dt. Consider the case shown in Fig. 2.1.1 of the single charge q moving with speed $\mathbf{v}$, and with observer P not moving. Differentiating Eq. (2.1.2), and using the relation $\mathbf{v}(t_r) = d\mathbf{r}'/dt_r$, yields

$$dt = dt_r \left(1 + \frac{1}{c}\frac{d\mathcal{R}}{dt_r}\right) = dt_r \left(1 - \widehat{\mathcal{R}} \cdot \frac{\mathbf{v}\,(t_r)}{c}\right). \tag{2.1.4}$$

It is because the factor in parenthesis on the right hand side can be extremely small that dt_r can greatly exceed dt. This situation occurs only when $v \approx c$ and $\mathbf{v}$ is directed approximately toward P.

† Because the particle speed v cannot exceed c, light from two points on the particle orbit cannot arrive at P simultaneously, so synchrotron light is not *exactly* analogous to a sonic boom. But with the speed of light being six orders of magnitude higher than the speed of sound, a gigantic "magnification" is nevertheless possible. A more nearly exact analog of a sonic boom is Cherenkov radiation.

† To correlate formulas in this paper with formulas in papers by Albert Hoffmann, one must appreciate that his $\mu_0\mathbf{A}$ is the same as our $\mathbf{A}$.

Because of the implicit relation between t and t_r, it is extremely tedious to evaluate the derivatives in Eq. (2.1.3). The result of doing it, and retaining only "radiation" terms, is that electric and magnetic fields at P are given by

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0 c}\,\frac{1}{\mathcal{R}}\left[\frac{\widehat{\mathcal{R}}\times\left(\left(\widehat{\mathcal{R}}-\frac{\mathbf{v}}{c}\right)\times\frac{\dot{\mathbf{v}}}{c}\right)}{\left(1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}\right)^3}\right]_{\text{ret.}} \\ \mathbf{B}(\mathbf{r},t) &= \frac{1}{c}\,\widehat{\mathcal{R}}\times\mathbf{E}(\mathbf{r},t)\,.\end{aligned} \tag{2.1.5}$$

The terms that have been dropped are unimportant for the observer at P since they describe the fields "attached" to the charge; they are appreciable only near the point Q' where q has arrived at time t. It is a sufficient approximation to take the factor $1/\mathcal{R}$ outside the retarded time calculation since $\mathcal{R}$ varies little over the relevant range. All formulas so far have been known for about a century.

All quantities in Eq. (2.1.5) are to be evaluated at the retarded time. It is important to specify the meaning of $\dot{\mathbf{v}}$ unambiguously. For circular (radius R) motion at constant speed v, its magnitude is $|d\mathbf{v}/dt| = v^2/R$, which is constant, and it points toward the instantaneous center of curvature. While thinking about this point let me introduce an alternate expression for $\mathbf{E}$ that will be needed later on:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \frac{q/\mathcal{R}}{4\pi\epsilon_0 c}\left[\frac{1}{1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}}\,\frac{d}{dt}\left(\frac{\widehat{\mathcal{R}}\times\left(\widehat{\mathcal{R}}\times\frac{\mathbf{v}}{c}\right)}{1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}}\right)\right]_{\text{ret.}} \\ &= \frac{q/\mathcal{R}}{4\pi\epsilon_0 c}\left[\frac{1}{1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}}\,\frac{d}{dt}\left(\frac{-\mathbf{v}_\perp/c}{1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}}\right)\right]_{\text{ret.}},\end{aligned} \tag{2.1.6}$$

where $\mathbf{v}_\perp$ is the component of $\mathbf{v}$ normal to $\widehat{\mathcal{R}}$. This surprisingly simple result can be proved by straightforward differentiation, (as before) treating $\widehat{\mathcal{R}}$ as constant. The derivative d/dt acts on a function in which $\mathbf{v}$ is the only variable and, when d/dt does act on $\mathbf{v}$ it generates $\dot{\mathbf{v}}$, whose meaning has already been specified.

A rather reckless approximation (the first of those mentioned in the introduction) simplifies the final factor of $\mathbf{E}$ in Eq. (2.1.5):

$$\widehat{\mathcal{R}}\times\left(\left(\widehat{\mathcal{R}}-\frac{\mathbf{v}}{c}\right)\times\frac{\dot{\mathbf{v}}}{c}\right) = \left(\widehat{\mathcal{R}}-\frac{\mathbf{v}}{c}\right)\left(\widehat{\mathcal{R}}\cdot\frac{\dot{\mathbf{v}}}{c}\right) - \frac{\dot{\mathbf{v}}}{c}\left(\widehat{\mathcal{R}}\cdot\left(\widehat{\mathcal{R}}-\frac{\mathbf{v}}{c}\right)\right) \approx -\frac{\dot{\mathbf{v}}}{c}\left(1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}\right), \tag{2.1.7}$$

The motivation behind this simplification is that $\dot{\mathbf{v}}$ is approximately orthogonal to $\widehat{\mathcal{R}}$. What makes it reckless is that the other factors in this equation all cancel in lowest approximation, and $\dot{\mathbf{v}}$ is orthogonal to $\widehat{\mathcal{R}}$ only at the center of the radiation pattern. Using $c^2 = \mu_0\epsilon_0$, approximation (2.1.7) leads to

$$\mathbf{E}\,(\mathbf{r},t) \approx -\frac{\mu_0}{4\pi}\,\frac{q}{\mathcal{R}}\left[\frac{\dot{\mathbf{v}}}{\left(1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}\right)^2}\right]_{\text{ret.}} . \tag{2.1.8}$$

Comparison with Eqs. (2.1.1) and (2.1.3) shows that this contribution to $\mathbf{E}$ comes from the term $-\partial\mathbf{A}/\partial t$—one of the factors $(1-\widehat{\mathcal{R}}\cdot\mathbf{v}/c)^{-1}$ comes from the previously mentioned piling up[†], and the other comes from the d/dt operation in Eq. (2.1.3).

From the second of Eqs. (2.1.1) one can visualize $\mathbf{A}_P$ as resulting from the simple translation of $\mathbf{v}$ from Q to P. Then $\dot{\mathbf{A}}_P$ is determined from the pure rotation of $\mathbf{v}$ as the particle follows its circular path and this is what yields the radial electric field $\mathbf{E}_P$. In this approximation the polarization is such that the electric field, being parallel to $\dot{\mathbf{v}}$, is purely horizontal (assuming the bend plane is horizontal). The magnetic field, given by the second of Eqs. (2.1.3), is vertical.

If the charged particle q is traveling in a circle of radius R we have $\dot{\mathbf{v}} = -\hat{\mathbf{R}}\,c^2/R$ and

$$\mathbf{E}\,(\mathbf{r},t) \approx \frac{q}{4\pi\epsilon_0}\,\frac{1}{\mathcal{R}R}\left[\frac{\hat{\mathbf{R}}}{\left(1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}\right)^2}\right]_{\text{ret.}} . \tag{2.1.9}$$

The range of validity of approximation (2.1.7) remains to be investigated,[‡] but this formula is certainly easier to use and remember than the exact formula (2.1.5). (It should lead to a conceptually simplified discussion of wigglers and undulators.)

From the previous chapter we know that typical values are $R = 100\,\text{m}$, $\mathcal{R} \approx 20\,\text{m}$. Except for the factor $(1-\widehat{\mathcal{R}}\cdot\mathbf{v}/c)^{-2}$, the magnitude of the field given by Eq. (2.1.9) would be that of a point charge q at (the very great) distance $\sqrt{\mathcal{R}R}$. But the denominator factor

[†] Griffiths (for example) shows that the integrals in Eq. (2.1.1) acquire a factor $(1-\widehat{\mathcal{R}}\cdot\mathbf{v}/c)^{-1}$ because of an effective elongation of charged volumes by this factor with no change in charge density ρ.

[‡] Remember that the range of validity of approximation has not been investigated, so the reader is advised to be skeptical of seemingly over-simple results, and to refer to exact calculations for more accurate formulas. The approximation made in Eq. (2.1.7) retains only the dominant contribution to horizontal polarization, and keeps no vertical polarization component. For reasons I do not find particularly defensible, Hoffmann refers to horizontal polarization as "σ-mode" and vertical polarization as "π-mode".

is

$$1 - \widehat{\mathcal{R}} \cdot \frac{\mathbf{v}}{c} = 1 - \cos\vartheta \sqrt{1 - \frac{1}{\gamma^2}} \approx \frac{\vartheta^2}{2} + \frac{1}{2\gamma^2}. \tag{2.1.10}$$

where ϑ is the angle between $\mathbf{v}$ and $\widehat{\mathcal{R}}$, and $\gamma = E/mc^2$. The maximum enhancement factor is therefore $4\gamma^4 \overset{\text{typ.}}{=} 4 \times 10^{16}$. This factor can convert $\mathbf{E}$ as given by Eq. (2.1.9) into an extremely large electric field. The same formula shows that the electric field has fallen to one quarter of its maximum value already when $\vartheta = \vartheta_{\text{typ.}} = 1/\gamma \overset{\text{typ.}}{=} 10^{-4}$. This falloff occurs both horizontally and vertically. The radiation pattern is often compared to the headlight of a locomotive rounding a curve. This is illustrated in Fig. 2.1.2.

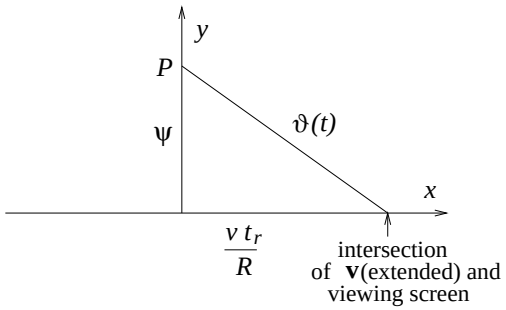

Figure 2.1.2: Passage of the radiation maximum across the viewing screen through observation point P.

If one arranges the time origins $t = 0$ and $t_r = 0$ to correspond then, from Eq. (2.1.4), one has $t = t_r(1 - \widehat{\mathcal{R}} \cdot \mathbf{v}/c) + \ldots$, where the dots stand for higher powers of t_r. We can understand the relationship between "observer time" t and and retarded time t_r by accounting for delays in the "electron's signal". A "nominal signal" is emitted toward P at the instant the electron passes the point whose tangent intersects the viewing screen at P's horizontal position $x = 0$. A signal emitted at a later retarded time gets to P almost at the same observer time because the electron is going almost at the speed of light and almost in the optimal direction. There are however three delays relative to this. Because the electron is a bit slow, a delay

$$t_r \left(1 - \frac{v}{c}\right) \approx \frac{t_r}{2\gamma^2}, \tag{2.1.11}$$

is introduced. There is also a delay due to the slightly non-optimal vertical path taken by the electron and another due to the slightly longer horizontal path taken by the electron

before it emits its signal. These delays are calculated in Fig. 2.1.3. Combining these deviations and taking both time origins so the signal arrive at the screen origin at $t = t_r = 0$ we get

$$t \approx t_r \left(\frac{1}{2\gamma^2} + \frac{\psi^2}{2} + \frac{c^2 t_r^2}{6R^2} \right) \stackrel{?}{\approx} t_r \, \frac{1 + \gamma^2 \psi^2}{2\gamma^2}. \tag{2.1.12}$$

Unfortunately, without the approximation indicated, this is a cubic relation between t and t_r. The electron's motion is described simply in terms of t_r but becomes extremely complicated when expressed in terms of t.

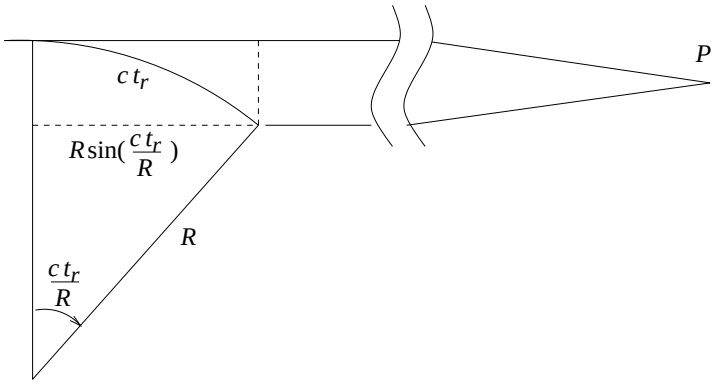

Figure 2.1.3: Top view of geometry for calculation of retarded time. Following the curved path introduces delay $t_r - (R/c)\sin(ct_r/R) \approx c^2\, t_r^3/(6R^2)$. There is also a delay $\approx t_r\psi^2/2$, because the electron stays in the median plane while a signal from the origin takes a more direct path.

A feeble estimate of the importance of the nonlinear term can be based on Fig. 2.3.1 (which is copied from Hoffmann.) It amounts to repeating the retarded time calculation for a typical retarded time, where "typical" refers to the fact that the radiation is largely contained in a cone of angular radius $1/\gamma$ about the electron direction. Setting to zero the radiation outside this cone, the start of the radiation pulse comes from A and the end comes from A'. The difference of arrival times of these signals is equal to the time difference Δt between electrons taking a curved path and photons taking a straight path;

$$\Delta t = \frac{2R}{\gamma c} \left(\frac{c}{v} - \gamma \sin\left(\frac{1}{\gamma}\right) \right) \approx \frac{2R}{\gamma c} \left(1 + \frac{1}{2\gamma^2} - 1 + \frac{1}{6\gamma^2} \right) = \frac{4}{3} \frac{R}{\gamma^3 c} \, . \tag{2.1.13}$$

One sees that this time is less than the revolution time R/c, by a factor of order $\gamma^3 \sim 10^{12}$. Expressed in terms of frequencies, there will be frequency components at the observation point that are greater than the electron's revolution frequency by a factor of order γ^3.

Including a typical vertical angle $\psi_{\text{typ.}} = 1/\gamma$ would change the coefficient $4/3 \to 7/3$. One sees therefore, that only about 1/7 of the retardation is ascribable to the nonlinear (path length) term. Of course an error scaling like t_r^2 becomes fractionally less at smaller times and fractionally greater at larger times. Since, if anything, $1/\gamma$ is an over-estimate of what is typical, one can be optimistic that dropping the t_r^2 term in Eq. (2.1.12) will introduce only a small error, at least as regards dominant behavior. This is not to say that behavior of "tails" will still be faithfully described. Later, especially when discussing the transition from "undulator regime" to "wiggler regime," the correct treatment of this term will be essential.

To make the formula for $\mathbf{E}(\mathbf{r}, t)$ explicitly dependent on the observation time t, it is necessary to evaluate the denominator factor Eq. (2.1.10), which depends on the angle ϑ, as shown in Fig. 2.1.2. Since the horizontal angle is given as a function of t_r, it is necessary to employ relation (2.1.12). I will use the approximate version, substituting into Eq. (2.1.9) to obtain

$$\mathbf{E}\left(\mathbf{r}, t\right) \approx \frac{q}{4\pi\epsilon_0}\,\frac{4\gamma^4}{\mathcal{R}R}\left(1 + \gamma^2\,\psi^2 + \left(\frac{2c\,\gamma^3/R}{1+\gamma^2\,\psi^2}\right)^2 t^2\right)^{-2}\hat{\mathbf{R}}. \tag{2.1.14}$$

Clearly, even apart from this approximation, this formula is valid only if the magnet can be treated as "long":

$$L > \frac{2R}{\gamma}\,. \tag{2.1.15}$$

If this condition is satisfied, the full radiation cone of angle $\pm 1/\gamma$ will sweep past an observer at P. Otherwise, the radiation pattern will be limited to a shorter time interval. Inequality (2.1.15) is *not* satisfied in an undulator.

2.2. Total Power Radiated and its Angular Distribution

The Poynting vector observed at point P is

$$\mathbf{S} = \frac{1}{\mu_0}\,\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0 c}\,\mathbf{E}\times\left(\widehat{\mathcal{R}}\times\mathbf{E}\right) = \frac{E^2}{\mu_0 c}\,\widehat{\mathcal{R}}\,, \tag{2.2.1}$$

where $\mathbf{B}$ has been obtained from Eq. (2.1.5) and $\mathbf{E}$ is known to be perpendicular to $\widehat{\mathcal{R}}$ (in the radiation field.) The power passing through transverse area $(\mathcal{R}\,d\psi)(\mathcal{R}\,d\theta_h)$ in the observation plane is given by $|\mathbf{S}|\mathcal{R}^2\,d\Omega$, where θ_h is a horizontal angle away from the local

center (where the maximum falls at $t = 0$) and $d\Omega = d\psi\, d\theta_h$. From the single charge q this flux is strongly peaked, both in t and in vertical angle ψ. The total energy dU passing through a band of width $\mathcal{R}\, d\theta_h$ is given by

$$dU = (\mathcal{R}\, d\theta_h) \int_{-\infty}^{\infty} dt \int_{-\pi/2}^{\pi/2} (\mathcal{R} d\psi)\, \frac{E^2}{\mu_0 c}. \tag{2.2.2}$$

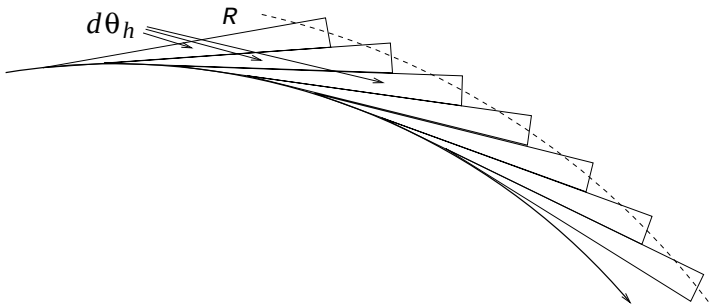

Figure 2.2.1: Angular ranges segmented for purposes of calculating total energy carried away from ring by a single passage of the moving charge q. To avoid worrying about "double counting" and "leakage" through tangential edges, flux can be calculated through the dashed circle (actually cylinder.) But the area and obliquity factors cancel in (2.2.3).

The total energy carried away from the ring can be calculated as suggested in Fig. 2.2.1. By taking the limit $d\theta_h \to 0$ any dependence of the integral on horizontal angle is suppressed, yet the overall result can be obtained by setting $d\theta_h = 2\pi$. Also, the fall-off with increasing ψ is so fast that these limits can be taken to ∞. The total energy carried away after one revolution of charge q is therefore given by

$$U_0 \approx \frac{2\pi}{\mu_0 c} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\psi\ \mathcal{R}^2\, E^2 = \frac{2q^2 c\gamma^8}{\pi\epsilon_0 R^2} \int_{-\infty}^{\infty} d\psi \int_{-\infty}^{\infty} \frac{dt}{\left(1 + \gamma^2\,\psi^2 + \left(\frac{2c\gamma^3/R}{1+\gamma^2\,\psi^2}\right)^2 t^2\right)^4}. \tag{2.2.3}$$

It is reassuring that this result is independent of $\mathcal{R}$ which was chosen arbitrarily.

To complete the evaluation we use the integrals

$$\int_{-\infty}^{\infty} \frac{dx}{\left(a^2 + b^2 x^2\right)^4} = \frac{5\pi}{16 a^7 b}\ , \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{dx}{\left(1 + a^2 x^2\right)^{5/2}} = \frac{4}{3a}\ , \tag{2.2.4}$$

and obtain

$$U_0 \approx \frac{5}{16}\, \frac{q^2\gamma^5}{\epsilon_0 R} \int_{-\infty}^{\infty} \frac{d\psi}{\left(1 + \gamma^2\,\psi^2\right)^{5/2}} = \frac{5}{12}\, \frac{q^2\gamma^4}{\epsilon_0 R}\ . \tag{2.2.5}$$

According to M. Sands, *The Physics of Electron Storage Rings,* in International School of Physics, "Enrico Fermi", Academic Press, 1971, the exact value (in GeV) of U_0 is

$$U_0 = \frac{C_\gamma\, E^4}{R}, \quad \text{where} \quad C_\gamma = \frac{4\pi}{3}\frac{r_e}{(m_e c^2)^3} = 0.885 \times 10^{-4}\,\mathrm{m(GeV)}^{-3}\ , \qquad r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2} = 2.81784 \times 10^{-15}\,\mathrm{m}. \tag{2.2.6}$$

Our estimate (2.2.5) differs from the exact value in the ratio $\frac{5/12}{1/3} = 1.25$. i.e. the estimate is 25% on the high side.

Curiously enough, had we gone even further by dropping not just the t_r^2 term but also the $\gamma^2\psi^2$ term from the denominator of the coefficient of t^2 in Eq. (2.2.3), the value of U_0 would have come out exactly correct. This has to be regarded as completely fortuitous since, for example, vertical polarization components (included in (2.2.6)) have been completely suppressed in our approximation. Still, since the t_r^2 correction has previously been estimated to be considerably less important than the ψ^2 correction, the suggestion is that neglect of t_r^2 terms will cause error only at the 10% level.

To improve result (2.2.5) it is necessary to refine some earlier approximations. The terms dropped from Eq. (2.1.7) yield both horizontal and vertical polarization. Following Hoffmann, we work with (x, y, z) components, using t_r as independent variable,

$$\begin{pmatrix} \mathbf{r}' \\ \mathbf{v} \\ \dot{\mathbf{v}} \\ \widehat{\mathcal{R}} \\ \widehat{\mathcal{R}} - \mathbf{v}/c \end{pmatrix} = \begin{pmatrix} R\,(1-\cos\,(vt_r/R)) & 0 & R\sin\,(vt_r/R) \\ v\sin\,(vt_r/R) & 0 & v\cos\,(vt_r/R) \\ \left(v^2/R\right)\cos\,(vt_r/R) & 0 & -\left(v^2/R\right)\sin\,(vt_r/R) \\ 0 & \sin\psi & \cos\psi \\ -\,(v/c)\sin\,(vt_r/R) & \sin\psi & \cos\psi - (v/c)\cos\,(vt_r/R) \end{pmatrix}. \tag{2.2.7}$$

(Components arrayed horizontally here.) With components now arrayed vertically,

$$\widehat{\mathcal{R}} \times \left(\left(\widehat{\mathcal{R}} - \frac{\mathbf{v}}{c}\right) \times \frac{\dot{\mathbf{v}}}{c}\right) = \frac{v^2}{cR}\begin{pmatrix} -\cos\,(vt_r/R) + (v/c)\cos\psi \\ \cos\psi\,\sin\psi\,\sin\,(vt_r/R) \\ \sin^2\psi\,\sin\,(vt_r/R) \end{pmatrix}. \tag{2.2.8}$$

Using $v/c \approx 1/(2\gamma^2)$ and small angle approximations this becomes

$$\widehat{\mathcal{R}} \times \left(\left(\widehat{\mathcal{R}} - \frac{\mathbf{v}}{c}\right) \times \frac{\dot{\mathbf{v}}}{c}\right) = \frac{c}{2\gamma^2 R}\begin{pmatrix} -\left(1 + \gamma^2\,\psi^2 - (\gamma c t_r/R)^2\right) \\ \gamma^2\psi c t_r/R \\ 0 \end{pmatrix}. \tag{2.2.9}$$

Problem. This formula can be used for a more accurate determination of the total power radiated. Note that the x-component is, except for the sign of the last term, the same

as the denominator factor of Eq. (2.1.14). After adding twice this term, the numerator cancels against one of the denominator factors, and reduces to our approximate formula. In this way, find a formula that gives the overestimate. Also find the power radiated in vertically polarized radiation. Complete the evaluation of these integrals and compare with Eq. (2.2.6). One hopes that this will result in a tolerably accurate determination that avoids the formidable formulas in more exact treatments. This could lead to a serious investigation into the error that results from dropping the cubic term in Eq. (2.1.12).

Reverting to our approximate formulation, the (vertical) angular distribution of the radius can be read off from Eq. (2.2.5):

$$\frac{dU_0}{d\psi} \approx \frac{5\pi}{4}\,\frac{q^2}{4\pi\epsilon_0 R}\,\frac{\gamma^5}{(1+\gamma^2\,\psi^2)^{5/2}}. \tag{2.2.10}$$

Since the horizontal distribution is uniform the power radiated per solid angle is given by

$$\frac{dU_0}{d\Omega} \approx \frac{5}{8}\,\frac{q^2}{4\pi\epsilon_0 R}\,\frac{\gamma^5}{(1+\gamma^2\,\psi^2)^{5/2}}. \tag{2.2.11}$$

2.3. Spectral Power Density of the Radiation

Since we have a closed form expression for the electric field, it is straightforward to find its Fourier transform, and from that the spectrum of the radiation. Before doing this we can obtain a rough estimate by referring to Fig. 2.3.1 and Eq. (2.1.13). The radiation pulse arriving at the observer can be regarded as a roughly square pulse of length

$$\Delta t = \frac{4}{3}\,\frac{R}{\gamma^3 c}\ . \tag{2.3.1}$$

A typical frequency is therefore approximately $1/(2\Delta t)$:

$$\omega_c = \frac{3}{2}\,\frac{c\gamma^3}{R}\ . \tag{2.3.2}$$

This sort of argument does not fix the numerical factor, but we have matched the conventionally defined "critical frequency" ω_c, and corresponding "critical photon energy" u_c, by picking the factor to be 3/2. Expressed as a photon energy

$$u_c = \hbar\,\omega_c = \frac{3}{2}\,\frac{hc}{2\pi R/\gamma^3} = \frac{3}{2}\,\frac{1.2406\times10^{-6}\,\text{eV-m}}{2\pi R/\gamma^3}\ . \tag{2.3.3}$$

For CESR this works out to a few keV.

Proceeding more quantitatively, the Fourier transform of $\mathbf{E}$ is defined by

$$\widetilde{\mathbf{E}}\left(\omega\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t}\, \mathbf{E}\left(t\right)\, dt \ . \tag{2.3.4}$$

Its horizontal (dominant) component can be approximated from Eq. (2.1.14), using the fact that the integrand is an even function of t:

$$\begin{aligned}
\frac{\widetilde{E}_x\left(\omega,\psi\right)}{\frac{q}{4\pi\epsilon_0}\frac{4\gamma^4}{\mathcal{R}R}} &\approx \sqrt{\frac{2}{\pi}}\, \frac{1}{\left(1+\gamma^2\,\psi^2\right)^2} \left(\frac{\left(1+\gamma^2\,\psi^2\right)^{3/2}}{2c\,\gamma^3/R}\right)^4 \int_0^{\infty} dt \cos\omega t \left(\left(\frac{\left(1+\gamma^2\,\psi^2\right)^{3/2}}{2c\,\gamma^3/R}\right)^2 + t^2\right)^{-2} \\
&= \sqrt{\frac{2}{\pi}}\, \frac{\left(3\omega_\psi^{-1}/4\right)^4}{\left(1+\gamma^2\,\psi^2\right)^2} \int_0^{\infty} dt\, \frac{\cos\omega t}{\left(\left(3\omega_\psi^{-1}/4\right)^2 + t^2\right)^2} \quad \text{where} \quad \omega_\psi = \frac{3c\,\gamma^3/R}{2\left(1+\gamma^2\,\psi^2\right)^{3/2}} \\
&= \sqrt{\frac{9\pi}{128}}\, \frac{1}{\left(1+\gamma^2\,\psi^2\right)^2}\, \frac{1}{\omega_0}\, \left(1+(3/4)\ |\omega|/\omega_\psi\right) e^{-(3/4)\,|\omega|/\omega_\psi} \ .
\end{aligned} \tag{2.3.5}$$

Note that this is purely real. ω_ψ is a kind of "generalized critical frequency" such that $\omega_0 = \omega_c$.

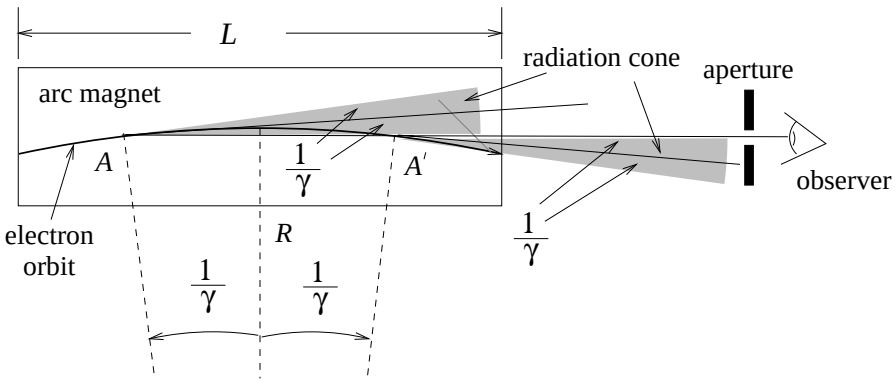

Figure 2.3.1: Figure used to estimate spectral content of synchrotron radiation from arc magnets. The angle $1/\gamma$ is greatly exaggerated to make the figure legible.

To obtain the photon energy spectrum we must calculate the power spectrum of the radiation. This can be obtained by expressing $E_x(t)$ in terms of $E_x(\omega)$ by Fourier inversion:

$$E_x\left(t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{E}_x\left(\omega\right) e^{-i\omega t}\, d\omega \ , \tag{2.3.6}$$

and then substituting this into Eq. (2.2.3) to give

$$\begin{aligned} U_0 &\approx \frac{\mathcal{R}^2}{2\pi\mu_0 c}\int d\Omega \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{E}_x(\omega)\,\widetilde{E}_x\left(\omega'\right)\,d\omega d\omega' \int_{-\infty}^{\infty} e^{-i\left(\omega+\omega'\right)t}\,dt \\ &= \frac{\mathcal{R}^2}{\mu_0 c}\int d\Omega \int_{-\infty}^{\infty} \widetilde{E}_x(\omega)\,\widetilde{E}_x(-\omega)\,d\omega\ . \end{aligned} \tag{2.3.7}$$

In the last line the identity

$$\int_{-\infty}^{\infty} e^{iat}\,dt = 2\pi\,\delta(a)\ , \tag{2.3.8}$$

has been used. The integrand of Eq. (2.3.7) can be interpreted as the energy per unit frequency range and per unit solid angle corresponding to one revolution:

$$\frac{dU_0}{d\Omega d\omega} \approx \frac{q^2}{4\pi\epsilon_0 R}\frac{a_\psi^2 c}{R}\frac{\gamma^8}{\left(1+\gamma^2\,\psi^2\right)^4}\left(1+(3/4)\,\omega/\omega_\psi\right)^2 e^{-(3/2)\,\omega/\omega_\psi}, \quad \omega \geq 0\ . \tag{2.3.9}$$

One factor of 2 has entered because ω is here restricted to be positive. The (most readily measurable) quantity $dU_0/d\omega$, is obtained by integrating over $d\Omega$:

$$\frac{dU_0(\omega)}{d\omega} \approx \frac{q^2 c\gamma^8}{2\epsilon_0 R^2}\int_0^{\infty}\left(\frac{\left(3\omega_\psi^{-1}/4\right)\left(1+(3/4)\,\omega/\omega_\psi\right)}{\left(1+\gamma^2\,\psi^2\right)^2}\right)^2 e^{-(3/2)\,\omega/\omega_\psi}\,d\psi\ . \tag{2.3.10}$$

Because ω_ψ is itself a function of ψ this is a rather complicated integral. It can be expanded in series or approximated for high and low values of ω but it may as well just be evaluated numerically. Unfortunately this makes it little more convenient than the "exact" expression that we have been avoiding.

It will turn out that a more nearly exact formula is somewhat simpler than the approximate formula just derived. Returning to Eq. (2.3.4), and substituting from Eq. (2.1.6) we obtain

$$\frac{\widetilde{\mathbf{E}}(\omega,\psi)}{\frac{q}{4\pi\epsilon_0 c\mathcal{R}}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\left[\frac{1}{1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}}\frac{d}{dt}\left(\frac{-\mathbf{v}_\perp/c}{1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}}\right)\right]_{\text{ret.}} dt\ . \tag{2.3.11}$$

Changing the variable of integration from t to t_r (using Eqs. (2.1.4) and (2.1.12)) yields

$$\frac{\widetilde{\mathbf{E}}(\omega,\psi)}{\frac{q}{4\pi\epsilon_0 c\mathcal{R}}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t_r\left(\frac{1}{2\gamma^2}+\frac{\psi^2}{2}+\frac{c^2t_r^2}{6R^2}\right)}\frac{d}{dt_r}\left(\frac{-\mathbf{v}_\perp/c}{1-\widehat{\mathcal{R}}\cdot\frac{\mathbf{v}}{c}}\right)dt_r\ . \tag{2.3.12}$$

Since we are now treating t_r as the parameter describing the electron's orbit (as in Eqs. (2.2.7)) the retarded time designation has been removed and d/dt has been replaced by d/dt_r.

We can now take advantage of the form of the integrand to integrate by parts (using Eq. (2.1.4)):

$$\begin{aligned}\frac{\widetilde{\mathbf{E}}\left(\omega,\psi\right)}{\frac{q}{4\pi\epsilon_0 c\mathcal{R}}} &= \frac{\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t_r\left(\frac{1}{2\gamma^2}+\frac{\psi^2}{2}+\frac{c^2t_r^2}{6R^2}\right)}\,\frac{-\mathbf{v}_\perp}{c}\,dt_r \\ &= \frac{\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t_r\left(\frac{1}{2\gamma^2}+\frac{\psi^2}{2}+\frac{c^2t_r^2}{6R^2}\right)}\,\left(-\sin\left(\frac{vt_r}{R}\right)\,\hat{\mathbf{x}}+\psi\,\cos\left(\frac{vt_r}{R}\right)\,\hat{\mathbf{y}}\right)\,dt_r \\ &\approx \frac{\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t_r\left(\frac{1}{2\gamma^2}+\frac{\psi^2}{2}+\frac{c^2t_r^2}{6R^2}\right)}\,\left(-\frac{ct_r}{R}\,\hat{\mathbf{x}}+\psi\,\hat{\mathbf{y}}\right)\,dt_r\ .\end{aligned} \tag{2.3.13}$$

(Compare Eq. (14.79) of Jackson.) Here the approximations $\sin\psi \approx \psi$ and $\sin^2\psi = 0$ require no comment, but we have also used $\sin(vt_r/R) \to ct_r/R$ and $\cos(vt_r/R) \to 1$. What makes this valid is that the exponential factor oscillates so rapidly that contributions to the integral are negligible outside the region of validity of these approximations.

Since the x-integrand is an odd function of t_r, $\widetilde{E}_x(\omega)$ is pure real. But the y-integrand is an even function of t_r, so $\widetilde{E}_y(\omega)$ is pure imaginary. As a result x and y components are 90° out of phase. Should their amplitudes be equal, this would be circular polarization. In general it is elliptical polarization, with the major and minor axes erect.

The exact evaluation of the integrals in Eq. (2.3.13) bring in the feared Airy functions.

Problem. It has been "argued" previously that simply dropping the retardation term proportional to t_r^3 introduces only a small error. Investigate this by evaluating the integrals in Eq. (2.3.13) with the term $c^2t_r^2/(6R^2)$ dropped. Perform numerical comparisons with exact values given, for example, by Hoffmann.

One can calculate the energy content of x and y fields independently. Having again calculated the Fourier transform $\widetilde{\mathbf{E}}(\omega)$ (now in the so-called "exact" treatment) we substitute it into a generalized form of Eq. (2.3.7)

$$U_0 = U_{0,x} + U_{0,y} = \frac{\mathcal{R}^2}{\mu_0 c}\int d\Omega \int_{-\infty}^{\infty}\left[\widetilde{E}_x\left(\omega\right)\widetilde{E}_x\left(-\omega\right)+\widetilde{E}_y\left(\omega\right)\widetilde{E}_y\left(-\omega\right)\right]\,d\omega\ . \tag{2.3.14}$$

We know that the radiation comes in the form of photons of energy $u = \hbar\omega$. Designating the number of photons in the energy range u to $u + du$ as $n_u(u)du$ we have

$$n_u\left(u\right) = \frac{1}{u}\frac{dU_0}{du} = \frac{1}{u\hbar}\frac{dU_0}{d\omega}\ . \tag{2.3.15}$$

2.4. Radiation From Multiple Charges

Actual electron rings contain (a very large number) N of electrons and the observed radiation is some superposition of the radiation from each electron. One can idealize the N electrons as being regularly arrayed around the ring or as forming one "superelectron" of charge Ne. In the former case one obtains essentially a steady ring of current which, from magnetostatics, we know emits no radiation. In the latter case the formulas up to this point need only be modified by the replacement $e \to Ne$ with the consequence that $U_0 \to N^2 U_0$. With N being a number of order 10^{12}, we are therefore uncertain, so far, by 24 orders of magnitude as to how much radiation actually occurs.

It was Schwinger who first gave what is now considered to be a correct treatment of this aspect of the radiation. To a first approximation the electrons radiate incoherently and the total radiation is obtained by multiplying all formulas to this point by N.† During the 1950's this quesion was somewhat contraversial and, if I am not mistaken, Schwinger's calculation was first confirmed experimentally by Corson at Cornell's first electron synchrotron. Corson's experiment consisted of measuring the energy loss of electrons by measuring the rate at which they "spiral in" because the RF accelerating field is turned off.

Suppose that the position of the j'th (of N) particles is positioned in the ring such that the center of its radiation pulse arrives at P at time Δt_j. The total energy can be calculated using Eq. (2.3.7), but with $\widetilde{E}_x$ summed over all particles. (We work only with the x-component, but the y-component calculation is the same.) Since the Fourier transform of the j field acquires a factor $e^{i\omega \Delta t_j}$, the energy radiated by all particles is given by

$$U_N = \frac{\mathcal{R}^2}{\mu_0 c} \int d\Omega \int_{-\infty}^{\infty} \widetilde{E}_x(\omega)\, \widetilde{E}_x(-\omega)\, d\omega \sum_{j,k=1}^{N} e^{i\omega(\Delta t_j - \Delta t_k)} \ . \tag{2.4.1}$$

Since there are N^2 terms in the final factor, its value can, in principle, range over the vast (previously mentioned) range from 0 to N^2. Since this factor can be written

$$\sum_{j,k=1}^{N} e^{i\omega(\Delta t_j - \Delta t_k)} = N + \sum_{j \neq k} e^{i\omega(\Delta t_j - \Delta t_k)} \ , \tag{2.4.2}$$

† At the same time Schwinger calculated the suppression of extremely long wavelengths due to the conductive vacuum chamber walls.

we have

$$\begin{aligned} U_N &= N\,U_0 + \frac{\mathcal{R}^2}{\mu_0 c}\int d\Omega \int_{-\infty}^{\infty} \widetilde{E}_x\left(\omega\right)\,\widetilde{E}_x\left(-\omega\right)\,d\omega \sum_{j\neq k}^{N} e^{i\omega\left(\Delta t_j - \Delta t_k\right)} \\ &= N\,U_0 + \frac{\mathcal{R}^2}{\mu_0 c}\int d\Omega \int_{0}^{\infty} \widetilde{E}_x^2\left(\omega\right)\,d\omega\, 4 \sum_{j>k}^{N} \cos\omega\left(\Delta t_j - \Delta t_k\right)\ . \end{aligned} \tag{2.4.3}$$

If the arrival times Δt_j are randomly distributed then the final factor averages to zero. The situation is not quite this simple though because (a) the particles are, in fact, bunched in time and (b) the factor $4\sum_{j>k}^{N}\cos\omega(\Delta t_j - \Delta t_k)$ has r.m.s. fluctuation $4\sqrt{N(N-1)/2}\,\sqrt{\overline{\cos^2}} \approx 2N$. Integration over ω causes the latter fluctuations to cancel, but the beam bunching cannot be neglected at low frequency. However "low" here implies photon wavelengths comparable with the electron bunch length. For purposes of high energy synchrotron light sources (with bunch lengths of order 1 cm) such photons are in the far infrared and can be regarded as negligible.

Chapter 3.
The Simplest Possible Storage Ring

3.1. Motivation

According to Eq. (1.71), the brilliance at critical wavelength λ_c of the radiation from a storage ring, for aperture ΔD at distance L, is given (approximately) by

$$\mathcal{B} = \frac{I}{e} 0.0662\gamma \frac{\Delta D}{2\pi L} \frac{10^{-4}}{\pi^2 \epsilon_x \epsilon_y} \ . \tag{3.1.1}$$

The emittance factors in the denominator quantify the influence on the brilliance of the spatial and angular distributions of the circulating electrons.

The purpose of this chapter will be to calculate these factors, as well as other features of the distributions of circulating electrons. As mentioned before, our approach is to "rewrite history" by perpetrating the fiction that all developments in storage rings were motivated by the desire to improve their performance as synchrotron radiation sources. From Eq. (3.1.1) one sees that this amounts to reducing the emittance factors. As it happens, accelerator development has, in fact, actually been driven largely by the desire to reduce these emittances. One reason for this has been to reduce magnet costs by reducing the need for transverse aperture. Another reason for reducing emittances has been to reduce beam spot sizes and hence increase luminosity in colliding beam operation. For these reasons, the evolution in storage ring design to be described here is more or less consistent with the true chronology.

Implicit in Eq. (3.1.1) is the assumption that the spread of photon emission angles is negligible. For early accelerators this assumption was abundantly true. For later generation synchrotron light sources, designed for ultra-low emittances, it will be necessary to take better account of the radiation patterns on the brilliance.

The sequence of accelerators to be descibed consists of: uniform field (cyclotron geometry), weak focusing, combined function, separated function, and special purpose, low emittance dedicated light sources. There are two major sub-topics. First one must define and calculate parameters such as "tunes" Q_x and Q_y and beta-functions β_x and β_y. The latter have already been introduced in chapter 1, where they were used to characterize

beam distributions. The second major sub-topic is to determine the actual values of emittances ϵ_x and ϵ_y. On the one hand, according to Eq. (3.1.1), ϵ_x and ϵ_y determine the brilliance but, on the other hand, it turns out that they are themselves determined by the properties of the synchrotron radiation.

To make a long story short, all emittances are determined by a competition between damping and stochastic exitation, both of which are due to the radiation. One strongly damping effect can be inferred from Eq. (2.2.6), which can be modified slightly to describe the energy radiated per turn by a particle whose momentum (or energy) is offset by fractional amount δ:

$$U_0(\delta) = \frac{C_\gamma \left(E(1+\delta)\right)^4}{R(1+\delta)} \ . \tag{3.1.2}$$

This formula shows that a particle whose energy is temporarily slightly above nominal will radiate somewhat more than the nominal amount of energy. This will pull its energy closer to the nominal energy; this constitutes "damping". But there is a countervailing effect. The energy is radiated in the form of photons of random energy and at random rate. These "quantum fluctuations" introduce "stochasticity" into the motion which increases the spread of energies. Competition between these effects leads to an equilibrium energy spread and (as it turns out) bunch length. In this, as other areas of physics, competitions such as this are subject to "fluctuation/dissipation theorems"

What has been outlined in this paragraph is the "longitudinal equilibrium", but we will also have to calculate the horizontal and vertical equilibria.

3.2. The Uniform Field Ring

The leading requirement of a (more or less) circular accelerator is that the particles be bent through 360°, say in a circle of radius R ($\overset{\text{e.g.}}{=} 100\,\text{m}$.) Assume a uniform magnetic field B_0, directed along the (vertical) y-axis. The non-relativistic equation governing the motion is

$$\frac{mv^2}{R} = evB_0, \quad \text{or, for } v \approx c, \quad \frac{pc}{R} = ecB_0 \ . \tag{3.2.1}$$

The latter form (which is relativistically valid) will be all that is used in what follows. It can be expressed in alternative forms:

$$\frac{d\mathbf{p}}{dt} = e\mathbf{v}\times\mathbf{B}, \quad \text{or} \quad B_0 = \frac{pc/e}{cR} \left(\stackrel{\text{e.g.}}{=} \frac{5\times 10^9\,\text{V}}{2\times 10^8\,\text{m/s}\times 10^2\,\text{m}} = 0.16\,\text{Tesla} = 1.6\,\text{kGauss.} \right) \tag{3.2.2}$$

A particle with pc/e given exactly by cB_0R and exactly on the center line of the vacuum chamber, will be called the "central" or "reference" particle, and will be said to follow the "reference trajectory".

3.3. Horizontal Deviation

In practice the particles in an accelerator follow trajectories that deviate from the reference trajectory, as shown, for example, in Fig. 3.3.1. Simple geometry shows that a particle deviating by angle $\Delta x'$ at point O will again pass through O after one revolution. Letting the radial offset x be defined by $r = R + x$, one sees from the figure that the transverse displacement is given (at least approximately) by

$$x\,(s) = R\Delta x'\,\sin\frac{s}{R}\ . \tag{3.3.1}$$

Since the point O could have been taken anywhere in the ring this dependence could have been any linear combination of $\sin(s/R)$ and $\cos(s/R)$, and these are the only forms satisfying the linearized, paraxial focusing equation (1.2.1). Based on Eq. (3.3.1), this equation, known as the "betatron equation", must specialize to

$$\frac{d^2x}{ds^2} \equiv x'' = -\frac{1}{R^2}x\ . \tag{3.3.2}$$

By construction this is satisfied by $x(s)$ as given in Eq. (3.3.1). This shows there is an effective horizontal focusing force caused by the geometry of circular motion. We will find eventually that this is a *very weak* focusing. This is just as well, since in the first packet of notes it was stated that the focusing associated with the bending of a charged particle in a magnet is negligible. Here is where we repairing this neglect to improve the treatment by accounting for this weak focusing effect.

One defines the "tune" Q_x to be the number of betatron oscillations per particle revolution. Since, the motion exhibited in Fig. 3.3.1 has exactly one complete betatron

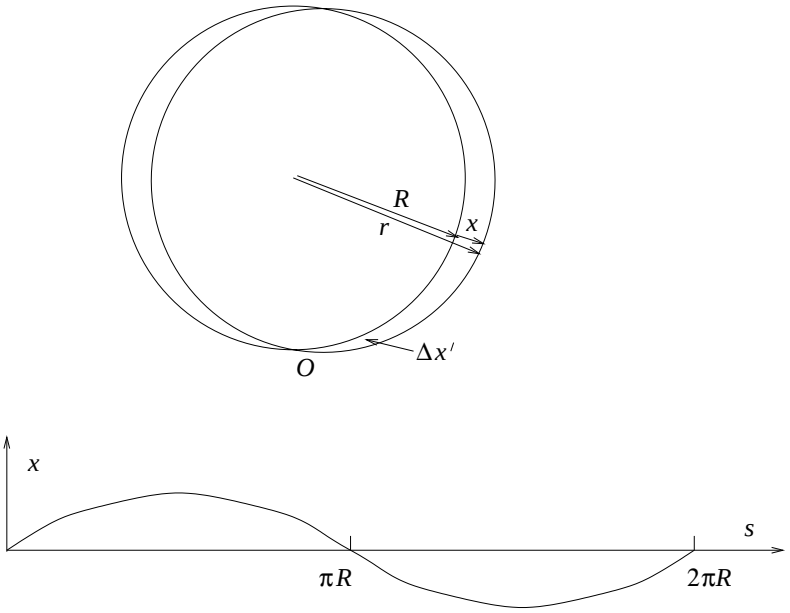

Figure 3.3.1: Orbit of particle with non-vanishing horizontal slope offset $\Delta x'$ at the origin.

oscillation per revolution, $Q_x = 1$. Another way of expressing the motion is the so-called pseudoharmonic description of Eqs. (1.7.1) and (1.7.2):

$$x\,(s) = a\,\cos\psi_x\,(s)\ , \tag{3.3.3}$$

where

$$\psi_x\,(s) = \int_0^s \frac{ds'}{\beta_x\,(s')}, \quad \text{or} \quad \frac{d\psi_x}{ds} = \frac{1}{\beta_x}\ . \tag{3.3.4}$$

For these equations to be consistent with Eq. (3.3.1) requires

$$\psi_x\,(s) = \frac{s}{R}, \quad \text{and hence} \quad \beta_x = R\ . \tag{3.3.5}$$

Finally, the tune is given by

$$Q_x = \frac{\psi_x\,(s = 2\pi R)}{2\pi} = 1\ . \tag{3.3.6}$$

This confirms what is obvious from the figure. We will find eventually that $Q_x = 1$ is an undesirable tune, both because it is an integer and because it is too small, and that $\beta_x = R$ is an undesirably large β-function, but they are what we have to work with for the time being.

3.4. Vertical Deviation

Unfortunately, the particle motion just described will be unstable vertically, because any nonvanishing component of vertical velocity will cause a particle to drift inexorably in the vertical (y) direction until it strikes a magnet pole and is lost. To overcome this problem one can shape the magnet poles as shown in Fig. 3.4.1.

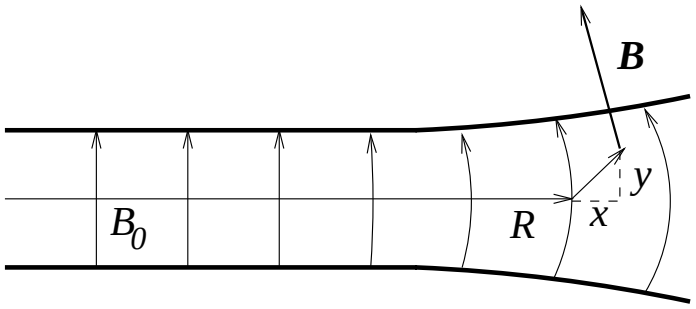

Figure 3.4.1: Magnet poles shaped to produce a horizontal component of magnet field B_x to prevent particles from drifting vertically. This is a "weak focusing" magnet. Since there are never any particles in the left two thirds of the field region shown, the magnetic field need not, in fact, be present there.

The field nonuniformity is conventionally parameterized by introducing "field index" n according to which

$$B_x = -\frac{B_0\, n}{R}\, y. \tag{3.4.1}$$

In free space, according to Maxwell's equations, one has $\nabla \times \mathbf{B} = 0$. We will ignore the gentle curvature of the magnet as s varies, in order to be able to treat the magnetic field as if it depends only on x and y. It then follows that $\partial B_y/\partial x = \partial B_x/\partial y$, which enables us to approximate the spatial dependence of vertical field component:

$$B_y = B_0 \left(1 - \frac{n}{R}\, x\right) \ . \tag{3.4.2}$$

Now the force equation (3.2.2) becomes

$$\frac{d\mathbf{p}}{dt} = e\, B_0 \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{s}} \\ 0 & 0 & c \\ -\frac{n\,y}{R} & 1 - \frac{n\,x}{R} & 0 \end{vmatrix} \tag{3.4.3}$$

Using $ds = cdt$ and $p_y = y'p$, this yields, for the vertical equation of motion,

$$y'' = -\frac{B_0\, c\, n}{(pc/e)\ R}\, y = -\frac{n}{R^2}\, y \ . \tag{3.4.4}$$

Provided the field index satisfies $n > 0$, this exhibits the vertical restoring force that the magnet was designed to provide.

3.5. Simultaneous Horizontal and Vertical Stability

Providing vertical stability has not come for free. With the field modified as shown in Fig. 3.4.1 it is clear that a particle for which $x > 0$ finds itself in a somewhat reduced magnetic field, which is the opposite of what would bend it back toward the reference orbit; in other words, some horizontal defocusing has been introduced. The effect of this force can described by modifying Eq. (3.3.2), using the x-component of Eq. (3.4.3):

$$x'' = -\frac{1-n}{R^2}\,x \ . \tag{3.5.1}$$

Provided the field index satisfies $1 - n > 0$, there is net horizontal focusing. Since it is necessary for both horizontal and vertical motion to be stable, we must have

$$0 < n < 1 \ . \tag{3.5.2}$$

We can now extract tunes and β-functions for both planes, starting from the betatron phases required to satisfy Eqs. (3.4.4) and (3.5.1):

$$\psi_x = \sqrt{\frac{1-n}{R^2}}\,s \ , \quad \psi_y = \sqrt{\frac{n}{R^2}}\,s \ . \tag{3.5.3}$$

Again employing the second of Eqs. (3.3.3), we therefore have

$$\beta_x = \frac{R}{\sqrt{1-n}} \ , \quad \beta_y = \frac{R}{\sqrt{n}} \ . \tag{3.5.4}$$

Since both $|n|$ and $|1-n|$ are less than 1, both β-functions are greater that R. Finally, proceeding as in Eq. (3.3.6),

$$Q_x = \sqrt{1-n} \ , \quad Q_y = \sqrt{n} \ , \quad \text{and hence} \quad Q_x^2 + Q_y^2 = 1 \ . \tag{3.5.5}$$

This final condition makes it clear that the focusing in both planes must be relatively weak.

3.6. Dispersion

Just as no particle lies precisely on the reference orbit, no particle has precisely the nominal momentum. The fractional momentum offset $\delta = \Delta p/p$ was defined in Eq. (1.1.1). A possible orbit for such a particle is a circle uniformly outside the reference circle of radius R, with radius given by $R + \eta\delta$. This is the defining relation for the "dispersion function" $\eta(s)$, which, in general, depends on s. But in all cases treated so far, η is independent of s.

For a uniform magnetic field, applying Eq. (3.2.1) to this "off-momentum" orbit yields

$$R + \eta\delta = \frac{pc\ (1+\delta)}{ecB_0}\ , \quad \text{or} \quad \eta = R\ . \tag{3.6.1}$$

For a weak focusing magnet, the same equation can be applied, but it is necessary to take account of the reduced magnet field strength for $\Delta x > 0$:

$$R\left(1 + \frac{\Delta x}{R}\right) = \frac{pc\ (1+\delta)}{ecB_0\left(1 - n\,\frac{\Delta x}{R}\right)}\ . \tag{3.6.2}$$

Cross-multiplying the factor $(1 - n\Delta x/R)$, Taylor expanding both sides of this equation, setting $\Delta x = \eta\delta$, and equating first order terms yields, for the dispersion of a weak focusing ring,

$$\eta = \frac{R}{1-n} \overset{\text{also}}{=} \frac{R}{Q_x^2}\ . \tag{3.6.3}$$

(The final relation notes, in passing, a standard "rule of thumb" relation between dispersion and tune, which happens to be exact for the weak focusing lattice.) By modern standards this is a *gigantic* dispersion, but it was what earlier accelerator users had to deal with. For CESR, with $R \approx 100\,\text{m}$, if one had to deal with $n = 0.5$ the dispersion would be $\eta = 200\,\text{m}$. Then, for an off-momentum value of 10^{-3} (which is typical) the off-momentum orbit would be displaced by $200 \times 10^{-3} = 20\,\text{cm}$. Being roughly twice the full width of vacuum chamber, this would clearly be unacceptable.

3.7. Momentum Compaction

Because of dispersion, the orbit circumference $C(p)$ depends on p. A parameter that enters discussion of off-momentum orbits, and can be quite confusing, is the so-called "momentum compaction", which is obtained from $C(p)$ using the relation,

$$\alpha = \frac{p}{C\,(p)}\,\frac{dC}{dp}\ . \tag{3.7.1}$$

α is equal to the fractional increase in circumference $\Delta C/C$ divided by the fractional increase in momentum $\delta = \Delta p/p$. This quantity is important for analysing longitudinal stability, which depends on the time of arrival of the electron at the RF accelerating cavity. For an off-momentum particle, this time depends on its excess time of flight relative to the reference particle. For non-relativistic particles the velocity deficit, relative to the reference velocity, contributes to this deficit, but for electrons (which are relativistic), the entire deficit is due to the excess circumference, which is what α quantifies;

$$\frac{\Delta C}{C} = \alpha\delta\ . \tag{3.7.2}$$

For a uniform field magnet, with $R \sim p$ one obtains immediately from Eq. (3.7.1) that $\alpha = 1$, and the name "momentum compaction" derives from this. That is, momentum compaction less than 1 implies that the circumference increases less rapidly than linearly with p, so the transverse density of trajectories is "compactified" relative to what would be true in a uniform magnetic field. For a weak focusing magnet, copying η from Eq. (3.6.3),

$$\alpha = \frac{p_0}{2\pi R}\,\frac{d}{dp}\left(R + \eta\,\frac{p-p_0}{p_0}\right) = \frac{1}{1-n} \overset{\text{also}}{=} \frac{1}{Q_x^2}\ . \tag{3.7.3}$$

It might be thought that the value of α would always be of order 1, but modern lattices are designed to have $\alpha << 1$. In fact, the final equation of Eq. (3.7.3), which was just listed as a comment in passing, continues to be true for high tune lattices having $Q_x >> 1$. Sometimes one even contemplates lattices having $\alpha = 0$ in order to obtain very short bunch lengths.

3.8. Chromaticity

Since the focusing strength of typical magnetic elements varies inversely with momentum, the tunes tend to become smaller as the momentum increases. This is known as "chromaticity". One defines "chromaticities":

$$Q'_x = \frac{dQ_x}{d\delta}, \quad Q'_x = \frac{dQ_y}{d\delta} \ . \tag{3.8.1}$$

If nothing is done both chromaticities are negative, but usually sextupoles are included in the lattice to compensate the chromticities to zero or slightly positive values. In the formulas given so far for a weak focusing magnet, no dependence of n has been exhibited, but one sees from the inverse R dependence indicated by Eq. (3.4.1), that $n(\delta) \approx n(1-\delta)$. Hence we have

$$Q'_x = \frac{-dn/d\delta}{2\sqrt{1-n}} = \frac{n}{2\sqrt{1-n}}, \quad Q'_y = -\frac{\sqrt{n}}{2} \ . \tag{3.8.2}$$

Chapter 4.
The Influence Of Synchrotron Radiation On a Storage Ring

4.1. Introduction

It is synchrotron radiation that distinguishes electron storage rings from proton storage rings. The large difference in mass of these particles causes radiation to be dominant for electrons and negligible for protons. The main radiative result is the simple frittering away of energy, and the strong dependence on energy of this energy loss sets a practical limit to the achievable energy of an electron ring. The e^+e^- ring LEP at CERN in Geneva, Switzerland has achieved an energy of about 100 GeV, but nothing beyond that is currently judged to be economically feasible.

In the context of this course, to provide local flavor, formulas will be specialized to CESR, and analyzed for possible modifications that could lead to reduced emittance, shortened bunch length, etc.

For existing electron accelerators, once paying the power bill needed to replenish radiated energy has been accepted and discounted, the remaining effects are a mixed blessing. The steady loss of energy has the desirable effect of damping betatron oscillations, which leads to a beneficial emittance reduction. (At the highest energies presently contemplated for protons, this benefit will be just beginning to be useful.) Unfortunately this radiation is emitted in the form of photons, at a random rate, and with random energies. This randomness, also known as "quantum fluctuation", causes the stochastic excitation of both transverse and longitudinal oscillations. There is a competition between this excitation mechanism and the previously mentioned damping. This leads to an eventual equilibrium in which all beam dimensions are determined, independent of prehistory. Beams smaller (larger) than this damp (are excited) to these dimensions in a time which is typically of the order of milliseconds. Furthermore, except for nonlinear effects, all beam distributions become Gaussian. Analysis of these effects is an interesting application of stochastic physics, and makes up the main content of this section.

Many of the results to be discussed were first derived by Sands[1] and are described in his article referred to previously. In writing this chapter, for the sake of continuity, all results have been re-derived but, if questions arise, returning to Sands's article may be a useful step. As far as I know, there are no essential differences from Sands of formulas presented here.

4.2. Statistical Properties of Synchrotron Radiation

For the highly relativistic situation to be analysed in this section it will not be necessary to distinguish between energy E and momentum P (or Pc if you insist).

There is a central energy E_0 which, in a large machine like LEP, varies noticeably around the ring because of radiation. For typical synchrotron light sources this effect is negligible. Nevertheless, any particular electron energy $E = Pc$ will deviate from the nominal energy $E_0(s)$. The energy of a particular emitted photon will be denoted by u, the total energy radiated in one turn by U_0 and the revolution period by $T_0 = 1/f_0$. Stochasticity results because of the randomness of u, and U_0 is also, in principle, a random variable, but the number of photons emitted per revolution is so great that it will be legitimate to treat U_0 as non-probablistic.

The radius of curvature R and magnetic field B are related by

$$R = \frac{E/e}{cB} \tag{4.2.1}$$

In what follows, B and R will be treated as necessarily positive quantities. (Reverse bends are rare except in wigglers, where they are essential and where radiation effects are intentionally important. In those cases (straightforward) alterations must be applied to all formulas to follow.) In the same spirit it is common to make an "isomagnetic" assumption, probably introduced by Sands; the magnetic field, wherever it is non-vanishing, has the same value B. This causes the gross orbit to be a perfect circle except for interpolated straight line segments. For now we will be even more extreme, ignoring even the straight sections. Since these assumptions is never entirely valid, it will be necessary to generalize the formulas later to analyse practical accelerators. Remedying these oversimplifications will test our understanding of the statistics of the phenomenon under study.

The steps to be taken in analyzing emittance growth due to photon radiation in a certain sector of the accelerator are: first, calculate the total energy radiated in the sector; second, knowing the mean energy of radiated photons, calculate the number of photons radiated; third, calculate the effect of a single photon emission on the betatron amplitude of a particle in the accelerator; and fourth, using the fact that emission times are random, find the accumulated effect of these excitations. To find the final equilibrium it remains necessary to analyze the damping effect accompanying the replenishment of the radiated energy.

The total energy radiated for all photon energies from a single electron during time T is given [2-3] by

$$U_T = P_\gamma T = \frac{e^2 c^3}{2\pi} C_\gamma E^2 B^2 \, T = \sum_{j=1}^{\mathcal{N}_T} u_j, \tag{4.2.2}$$

where $C_\gamma = 0.885 \times 10^{-4}\,\mathrm{m/GeV^3}$, the total number of photons emitted is $\mathcal{N}_T$, and their energies are $u_j, j = 1, \ldots, \mathcal{N}_T$. The instantaneous radiated power P_γ is made up of forward-traveling photons; for all present purposes they will be taken as emitted exactly in the forward direction.

In a practical accelerator the field B varies with arc length coordinate s and the nominal energy radiated in one turn U_0^{rad} can be expressed as an integral along the central orbit

$$U_0^{\mathrm{rad}} = \frac{e^2 c^2}{2\pi} C_\gamma E^2 \oint B^2 \, (s) \, ds. \tag{4.2.3}$$

If B is constant, the total energy U_0 radiated in a complete revolution is

$$U_0^{\mathrm{rad}} = C_\gamma \, \frac{E^4}{R}, \tag{4.2.4}$$

which is as simple as it is because it is actually the defining equation for constant C_γ. It is necessary to be careful when using this equation since, conventionally, C_γ is given, as above, in other than M.K.S. units. A "typical" or "critical" photon energy parameter u_c is defined by†

$$u_c = \frac{3}{2} \hbar c \gamma^3 \frac{cB}{E/e}. \tag{4.2.5}$$

† Note that the "critical energy" u_c, defined in Eq. (4.2.5), corresponds to the "critical frequency" ω_c, defined in Eq. (2.3.2).

Beware that there is not necessarily universal agreement on the definition of u_c, which is neither the maximum nor the mean nor any other statistical parameter. (From Fig. 4.2.2, to be introduced shortly, it looks to be in a vicinity such that about half the energy is radiated below it and half above.) To serve as an example, a consistent set of numerical values for the physical parameters and for u_c and other derived quantities are listed in Table 4.2.1.

When the energy u is expressed in units of u_c by $\xi = u/u_c$ the probability distribution of photon energies $n_\xi(\xi)$ is a universal function. Because the number of photons in the energy range du at u is $n_u(u)du = n_\xi d\xi$, one has $n_u(u) = n_\xi d\xi/du = n_\xi(u/u_c)/u_c$. When $\mathcal{N}$ photons are emitted altogether, the number in the range du at u is $\mathcal{N}n_u(u)du$. These definitions imply that n_u and n_ξ are normalized† according to

$$\int_0^\infty n_u\left(u'\right)du' = \int_0^\infty n_\xi\left(\xi'\right)d\xi' = 1. \tag{4.2.6}$$

An analytic expression for $n_\xi(\xi)$ is given in Sands and in Landau and Lifshitz, in terms of a function called the "MacDonald" function which is itself expressed in terms of Bessel functions. Those formulas will not be replicated here, but an approximation will be given in the problems and in Fig. 4.2.1 and Fig. 4.2.2. All subsequent results will be expressed in term of moments of $n_\xi(\xi)$; Sands states these average energy and average energy-squared quantities to be

$$\begin{aligned} < u >_\gamma &= \int_0^\infty u' n_u\left(u'\right)du' = 0.3079u_c, \\ < u^2 >_\gamma &= \int_0^\infty u'^2 n_u\left(u'\right)du' = 0.4074u_c^2. \end{aligned} \tag{4.2.7}$$

Here the symbol $<>_\gamma$ implies averaging over photon energies, as contrasted, say, with averaging over lattice elements, or over particle turns, or betatron phase. (The γ subscript is unconnected with the relativistic factor γ.) This radiation is simpler to treat than, say, bremstrahlung, in one respect—there is no "infrared divergence" causing the number spectrum to be divergent near zero energy. This allows the number of photons emitted to remain finite, which allows $< u >_\gamma$ to be defined. Apart from normalization (4.2.7) lists the only important parameters (moments) of the emission distribution. For example,

† It is obviously unphysical for the upper limit to be set to ∞, but for present purposes all integrands will be sufficiently convergent that it does not matter.

Table 4.2.1: Parameters describing synchrotron radiation in a hypothetical pure FODO lattices resembling the CESR storage ring.

Quantity	Symbol	Formula or reference	Value	Units
rest energy	mc^2		0.511	MeV
relativistic factor	γ_0		10^4	
central energy	E_0	$\gamma_0 mc^2$	5.11	GeV
magnetic field	B		0.1939	T
radius of curvature	R	(4.2.1)	87.9	m
magnet length	l_B		6.57	m
one magnet bend angle	$\Delta\theta_1$	l_B/R	0.0747	
time in one magnet	T_1	l_B/c	21.9	ns
energy radiated per magnet	U_{T_1}	(4.2.2)	8.166	keV
critical energy	u_c	(4.2.5)	3.367	keV
mean energy	$< u >_\gamma$	(4.2.7)	1.037	keV
mean squared energy	$< u^2 >_\gamma$	(4.2.7)	4.619	keV^2
energy radiated per turn	$U_0^{\rm rad}$	$2\pi U_{T_1}/\Delta\theta_1$	0.687	MeV
photons per turn		$U_0^{\rm rad}/ < u >_\gamma$	662	
number of bend half-cells	n		84	
half-cell length	ℓ	(4.9.2)	8.44	m
phase advance per cell	2ϕ	(4.9.2)	70	deg.
"curly-H"	$\mathcal{H}$	(4.9.4)	0.250	m
emittance	ϵ_x	(4.9.5)	0.1089	μ m
maximum β_x in arcs	$\hat{\beta}_x$		45	m
σ_x at $\hat{\beta}_x$	$\hat{\sigma}_x$		2.22	mm
fractional energy spread	σ_δ	(4.7.11)	0.47	$\times 10^{-3}$

except for fluctuations, the total number of photons radiated, $\mathcal{N}_T$, is related to the total energy radiated U_T by

$$\mathcal{N}_T = \frac{U_T}{< u >_\gamma}. \tag{4.2.8}$$

If, $\mathcal{N}_0$, the total number of photons of photons radiated per revolution, is divided by γ_0, the result is independent of E and B, leading to the simple formula

$$\mathcal{N}_0 = \frac{U_0^{\rm rad}}{< u >_\gamma} = \frac{ecC_\gamma E^3 B}{0.3079\gamma u_c} = 0.0662\gamma. \tag{4.2.9}$$

This can be the basis for a curious bit of *trivia*: every electron in every present or planned symmetric B-factory—just a cute way of saying that $\gamma = 10^4$—radiates 662 photons, plus or minus, every turn. Half, 331, are radiated in each half, and the same would be true if the magnets in one half were replaced by bend fields one tenth as long, ten times as strong. Redistributing the bends to make the whole ring homogeneous would not make a difference. It seems then that for any guide field whatsoever (with no reverse bends, as for example in a wiggler) there will be 662 photons. Having "established" this remarkable result, one must acknowledge that it has little value—it is the energy radiated, not the number of photons, by which the radiation does its dirty work.

From the approximate distributions described in the problems it will be seen that the *number* of photons is dominated by low energies and their *r.m.s.* energy is dominated by high energies. This renders correct calculation somewhat tricky.

Problem 4..1. An approximation to n_ξ (developed for these notes, inaccurate at the $\pm 20\%$ level, and not intended for accurate calculation) is

$$n_\xi(\xi) = \begin{cases} 0.413\xi^{-2/3} & \text{if } 0 < \xi < 0.2; \\ (0.308/\xi)\left(1 - e^{-9\xi}\right)e^{-\xi} & \text{if } 0.2 < \xi < 1.7; \\ 0.239\xi^{-1/2}e^{-\xi} & \text{if } 1.7 < \xi < \infty. \end{cases} \tag{4.2.10}$$

Check the normalization condition Eq. (4.2.6) for this approximate form. These functions are plotted in Fig. 4.2.1. The distributions multiplied by $\xi/0.308$ to convert to a normalized distribution function for radiated energy are plotted in Fig. 4.2.2. (By Eq. (4.2.7) $\int_0^\infty \xi' \, n_\xi(\xi')d\xi' = 0.308$ is the normalizing factor). The low-ξ and high-ξ approximations given as the first and third forms in Eq. (4.2.10) are analytically correct limiting forms given by Sands for the energy spectrum in those limits. From the figure they can be seen

individually to be bad approximations everywhere except in their legitimate regions of validity. The level of inaccuracy of the approximation can be estimated by the failure of normalization of the patched together form in this figure. It can also be judged by comparing with the few points crudely read from the intentionally low resolution "exact" graph given by Sands and plotted in the same figure.

Problem 4..2. Extending the central approximation of the previous problem over the full range from zero to infinity (though not a very good approximation it is at least adequately convergent in both limits) adjust the normalization to satisfy Eq. (4.2.6) and calculate the moments $< u >_\gamma$ and $< u^2 >_\gamma$. With units as always a likely source of blunders, a useful numerical check is on the dimensionless ratio $< u^2 >_\gamma / < u >_\gamma^2$ for which the correct value is 4.297. Naturally your value will disagree because the distribution is somewhat wrong.

Problem 4..3. Find a superior interpolating function, and forward it to me, so I can improve the first problem. At some stage, to get accurate results, one is forced to use the correct analytic MacDonald function discussed by Jackson and by Landau and Lifshitz.

Because individual photons carry energy u, the radiated power $P_{\gamma u}$, expressed as a distribution of energy such that $P_{\gamma u}(u)du$ is the time rate of emission of energy in the range du at u is given by

$$P_{\gamma u}\left(u\right) = \frac{\mathcal{N}_T}{T} u\, n_u\left(u\right). \tag{4.2.11}$$

Except for a normalizing factor, this is the integrand of Eq. (4.2.7) for $< u >_\gamma$; it is plotted in Fig. 4.2.2.

Fluctuations in photon emission are central to electron dynamics in the accelerator. Since the main source of fluctuation is the random-in-time emission of photons, these quantum fluctuations would lead to emittance growth even if every photon carried the same energy. In that (unrealistic) case every photon emitted would have energy $< u >_\gamma$ and the r.m.s. energy would satisfy $< u^2 >_\gamma = < u >_\gamma^2$. In fact, the growth mechanism is stronger by a factor equal to the true ratio which is $\frac{<u^2>_\gamma}{<u>_\gamma^2} = 4.297$. This reflects the fact that the emittance growth due to a single emission is proportional to u^2 and that a relatively small number of higher energy photons can make a relatively large contribution to the average square.

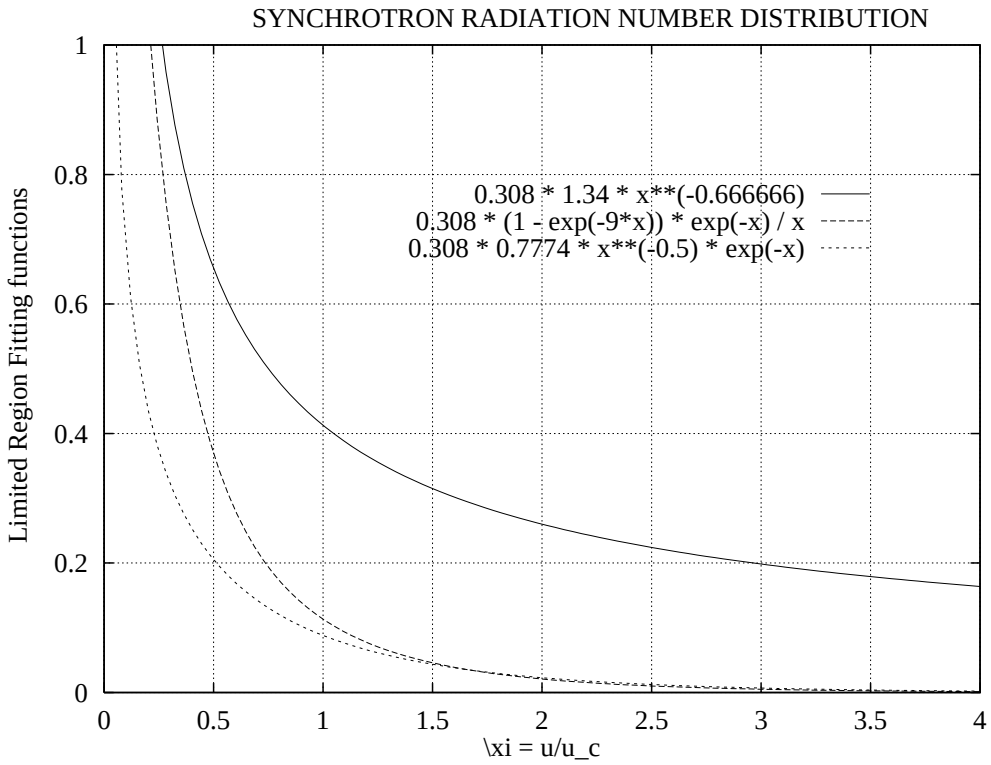

Figure 4.2.1: Approximating functions for the normalized probability distribution function $n_\xi(\xi)$ of number of photons as a function of normalized photon energy. The functions, indicated with the key, are those given in Problem 4..1. The upper curve valid only at low energy, is connected by the intermediate curve to the lower curve which is valid only at high energy.

The total energy U_T radiated during time T can be obtained from Eq. (4.2.2) and from that an almost unambiguous total number of photons $\mathcal{N}_T = U_T / < u >_\gamma$ can be obtained. To obtain this formula it has been assumed that individual emissions are uncorrelated and that $\mathcal{N}_T$ and U_T are not subject to fluctuations. For times too short (say comparible to the mean time between emissions) this would not be valid, but it will be applied only for times long enough to assure small fractional deviation of $\mathcal{N}_T$; *i.e.* $\sqrt{\mathcal{N}_T} >> 1$.

The emittance growth rate due to radiation will turn out to be proportional to the average sum of squares of photon energies for the $\mathcal{N}_T$ photons with total energy U_T radiated during time T. From formulas derived so far this is given by

$$< \sum_{j=1}^{\mathcal{N}_T} u_j^2 >_\gamma = \mathcal{N}_T < u^2 >_\gamma = \frac{U_T}{< u >_\gamma} < u^2 >_\gamma = 1.323 u_c U_T . \qquad (4.2.12)$$

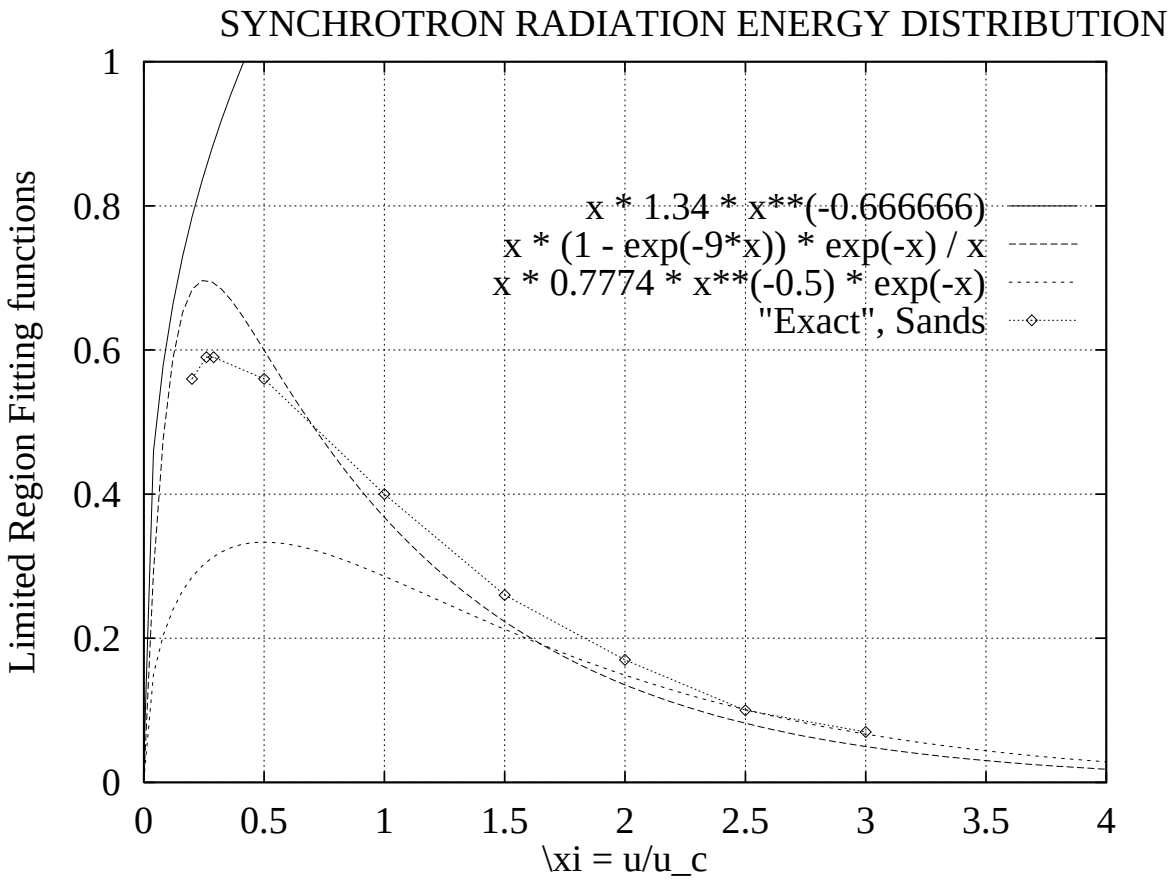

Figure 4.2.2: Approximating functions for energy radiated per unit energy $(\xi/0.308)n_\xi(\xi)$, where the numerical value 0.308 comes from Eq. (4.2.7), as a function of normalized energy ξ. The functions, indicated with the key, correspond to the functions given in Problem 4..1. Except for normalization this is also the radiated energy distribution, $P_{\gamma u}(u)$.

The important dependencies of this simple result are that (by Eq. (4.2.2)) U_T is proportional to E^2B^2 and that (by Eq. (4.2.5)) u_c is proportional to $\gamma^3 B/E$. The dependence on B forces regions of different magnetic fields to be treated independently, *i.e.* integrated over. The dependence on E is important in determining the final beam equilibrium.

4.3. The Damping Rate Sum Rule: Robinson's Theorem

The phase space coordinates of an individual particle at location s in the ring are given by (x, f, y, g, z, h), where $f \equiv x'$, $g \equiv y'$, and $h \equiv \delta$ have been introduced to suppress the primes. In passing through a differential longitudinal path element ds, evolution of these components is described by a transfer matrix:

$$\mathbf{X}\,(ds) = \mathbf{1} + \mathbf{dX} = \begin{pmatrix} 1+dX_1 & dX_2 & dX_3 & dX_4 & dX_5 & dX_6 \\ dF_1 & 1+dF_2 & dF_3 & dF_4 & dF_5 & dF_6 \\ dY_1 & dY_2 & 1+dY_3 & dY_4 & dY_5 & dY_6 \\ dG_1 & dG_2 & dG_3 & 1+dG_4 & dG_5 & dG_6 \\ dL_1 & dL_2 & dL_3 & dL_4 & 1+dL_5 & dL_6 \\ dH_1 & dH_2 & dH_3 & dH_4 & dH_5 & 1+dH_6 \end{pmatrix} . \tag{4.3.1}$$

The differential "d" symbols indicate elements that are first order in ds. For example, in pure drift regions $dX_2 = dY_4 = ds$. Eventually ds will be made arbitrarily small. The determinant $|\mathbf{X}(ds)|$ is not necessarily equal to 1 since the matrix represents radiation effects as well as the effects of bending and accelerating elements.

Transfer through a sequence of intervals ds_1, ds_2, ..., is represented by the matrix

$$\mathbf{X} = \prod_i \mathbf{X}\,(ds_i) \ . \tag{4.3.2}$$

The determinant of $\mathbf{X}$ can be organized into ascending powers of the ds_i:

$$|\mathbf{X}| = 1 + \sum_i \left(dX_{1,i} + dF_{2,i} + dY_{3,i} + dG_{4,i} + dL_{5,i} + dH_{6,i}\right) + \ldots . \tag{4.3.3}$$

Note that the off-diagonal matrix elements do not enter in lowest order.

For "Hamiltonian" elements such as magnets, drifts, and RF cavities, it is known that the determinant $\det|\mathbf{X}(ds)| = 1$ and that the six eigenvalues of $\mathbf{X}(ds)$ come in three complex conjugate pairs lying on the unit circle. Supposing these eigenvalues to have been determined for the interval ds, and generalizing slightly by permitting their magnitudes to deviate slightly from 1, we define them as[†]

$$\lambda_x\,(ds) = e^{-d\alpha_x \pm i d\beta_x}, \quad \lambda_y\,(ds) = e^{-d\alpha_y \pm i d\beta_y}, \quad \lambda_s\,(ds) = e^{-d\alpha_s \pm i d\beta_s} \ . \tag{4.3.4}$$

[†] Labeling the eigenvalues by x, y, and s suggests the theory is valid only if coupling is absent, but that is not the case. x, s, and s, are simply labels for the three local eigenvectors. Of course the α's and β's introduced here have nothing to do with the lattice Twiss functions.

Since the determinant of a matrix is equal to the product of the eigenvalues,

$$\begin{aligned} |\mathbf{X}\,(ds)\,| &= \left(e^{-d\alpha_x+id\beta_x}e^{-d\alpha_x-id\beta_x}\right)\left(e^{-d\alpha_y+id\beta_y}e^{-d\alpha_y-id\beta_y}\right)\left(e^{-d\alpha_s+id\beta_s}e^{-d\alpha_s-id\beta_s}\right) \\ &\approx 1-2\,(d\alpha_x+d\alpha_y+d\alpha_z)\ , \end{aligned} \tag{4.3.5}$$

where higher order terms have again been dropped. Letting $\alpha_x = \int d\alpha_x$, $\alpha_y = \int d\alpha_y$, and $\alpha_s = \int d\alpha_s$, all assumed to be small compared to 1, and proceeding to the $ds = 0$ limit, we obtain

$$|\mathbf{X}| = 1-2\,(\alpha_x+\alpha_y+\alpha_s)\ . \tag{4.3.6}$$

The determinant $\mathbf{X}$ given in Eq. (4.3.6) can also be approximated directly from Eq. (4.3.1), keeping only terms of first order in ds, and that is how we will evaluate the damping decrements α_i. If a term in the expansion of the determinant contains one off-diagonal element then it must contain at least one other, making it at least second order in ds. This limits our task to calculating the on-diagonal perturbations. In evaluating the on-diagonal elements $dX_1, dF_2, \ldots$, knowing that magnetic elements and drifts leave the value of the determinant unaltered (at 1), we need only include energy altering effects.

Within the path length ds under discussion there is the possibility of energy change due to radiation or due to RF cavities, or other longitudinal electric fields,

$$dE = -dU^{\rm rad} - dU^{\rm rf}.$$

The rationale for the sign of $dU^{\rm rad}$ (previously defined as a positive quantity) is that it should lead to a reduction in E. The rationale for the sign of $dU^{\rm rf}$ is that this quantity is to be regarded as a potential energy, such that moving to lower potential increases E.

We now evaluate the six on-diagonal perturbations one-by-one. It will turn out that dX_1, dY_3 and dL_5 vanish (to first order in ds) but to see this it is useful to first understand what contributes to dF_2, dG_4, and dH_6.

If the particle energy increases from E to $E - dU^{\rm rf}$ by virtue of potential energy change $dU^{\rm rf}$ (a negative quantity), its momentum will change as shown in Fig. 4.3.1. Though the particle's transverse momentum is unchanged, its longitudinal momentum increases by factor $1 - dU^{\rm rf}/E$, and this has the effect $x' \to x'(1 + dU^{\rm rf}/E)$; i.e. a slight reduction in slope. The vertical slope y' is reduced by the same factor. This means that

$$F_2 = G_4 = 1 + \frac{1}{E_0}\frac{\partial U^{\rm rf}}{\partial s}\,ds\ , \tag{4.3.7}$$

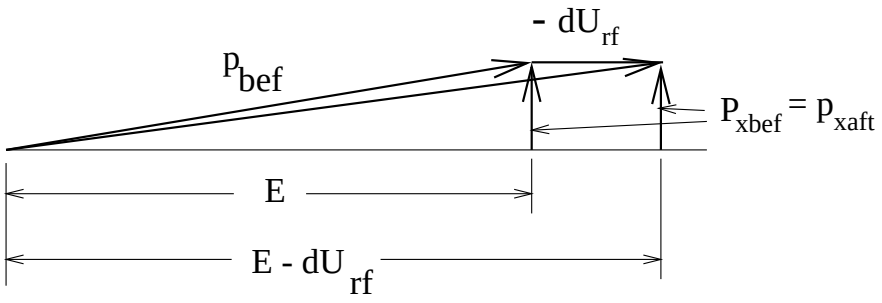

Figure 4.3.1: Damping of transverse amplitude accompanying longitudinal acceleration.

where it has been adequate to replace the slightly-variable quantity E by the central value E_0. These changes of slope also alter x and y, but the changes are proportional to ds^2. For this reason $dX_1 = dY_3 = 0$.

The remaining term to be evaluated, dH_6, accounts for change in longitudinal momentum (which at high energy is, except for units, equivalent to energy). Superficially it might seem one should account for the change $E \to E - dU^{\rm rf}$ due to RF acceleration, but since the reference orbit is also accelerated, $E_0 \to E_0 - dU^{\rm rf}$, the ratio $h \simeq E/E_0$ is unaffected by RF acceleration. This can be repeated for emphasis: though the energy of the reference orbit suffers a discontinuous change at an RF cavity, any particular particle suffers the same change, causing $E - E_0$ to be continuous. This may seem unnatural, and one could imagine a consistent perturbative formalism that would reckon the RF acceleration as perturbative, but that is not the case for our formalism. One also expects a term $(dU^{\rm rf}/dt)(l/c)$ to correspond to the intentional time varying RF field needed for stable synchrotron oscillations. But because this term contributes only to off-diagonal term dH_5, in the present context it can also be left out.

There is, however, a significant contribution to dH_6 from $dU^{\rm rad}$. Referring to Eq. (4.2.2) one notes that $dU^{\rm rad}$ is proportional to E^2, causing the mean energy radiated by a particular particle to deviate from the mean energy radiated by the reference particle. We have $dU^{\rm rad} = dU_0^{\rm rad}(E/E_0)^2 \simeq (1+2\delta)dU_0^{\rm rad}$ and

$$\delta = \frac{E - E_0}{E_0} \longrightarrow \frac{E_0\,(1+\delta) - dU_0^{\rm rad}\,(1+2\delta) - E_0 + dU_0^{\rm rad}}{E_0} = \left(1 - 2\frac{dU_0^{\rm rad}}{E_0}\right)\delta, \quad (4.3.8)$$

which implies

$$dH_6 = 1 - 2\frac{dU_0^{\rm rad}}{E_0} \quad (4.3.9)$$

(The earlier result that there is no contribution to dH_6 from accelerating fields could have been obtained in this same calculation using the fact that $dU^{\rm rf}$ is independent of δ.)

Combining results the transfer matrix is given to adequate accuracy by

$$\mathbf{X}\,(ds) \simeq \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+dU^{\rm rf}/E_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+dU^{\rm rf}/E_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2dU_0^{\rm rad}/E_0 \end{pmatrix}. \tag{4.3.10}$$

Evaluating the determinant yields

$$\det\left|\mathbf{X}\,(ds)\right| = 1 + 2dU_0^{\rm rf}/E_0 - 2dU_0^{\rm rad}/E_0. \tag{4.3.11}$$

(From here on it will be unnecessary to distinguish between dU_0 and dU since their difference is of higher order than ds.) Provided the terms on the right hand side remain small compared to 1, concatenating over successive intervals ds is equivalent to integrating over $dU^{\rm rf}$ and $dU_0^{\rm rad}$. Furthermore, when integrated over one complete revolution, assuming the RF is adjusted to just replace the radiated energy, one must obtain

$$\oint dU_0^{\rm rf} + \oint dU_0^{\rm rad} = 0\ . \tag{4.3.12}$$

In equilibrium this adjustment is automatic. Combining Eq. (4.3.12) with Eq. (4.3.5) yields

$$\alpha_1 + \alpha_2 + \alpha_3 = 2\frac{U_0^{\rm rad}}{E_0}. \tag{4.3.13}$$

This analyzis was first performed by Robinson, Phys. Rev. **111**, 373 (1958), and independently (and in less generality) by Orlov, JETP **35** 525 (1958). It is known as "Robinson's theorem"—in words: for any lattice the sum of the damping decrements is twice the fractional synchrotron radiation energy emitted per turn. It does not depend on the statistical properties of the radiated photons—only the fractional accumulated energy loss matters.

Historically Robinson's theorem played a very significant role in the early days of electron positron storage rings. The lattices of early electron rings were constructed from combined function magnets. As we shall see, such lattices exhibit extremely strong longitinal damping and hence, by Robinson's theorem, horizontal anti-damping. This problem was overcome by separated function lattices in which the bending and focusing is performed by separate elements.

4.3.1. Vertical Damping

By Robinson's theorem the sum of damping rates can be calculated easily, even for a complicated lattice, but the individual damping rates depend sensitively on lattice details. One simple, and usually realizeable, case has no "cross-plane coupling" between x and y or "synchrobetatron coupling" between longitudinal and transverse. In such an ideally uncoupled accelerator the once-around transfer matrix takes the form

$$\mathbf{X} = \begin{pmatrix} X_1 & X_2 & 0 & 0 & X_5 & X_6 \\ F_1 & F_2 & 0 & 0 & F_5 & F_6 \\ 0 & 0 & Y_3 & Y_4 & 0 & 0 \\ 0 & 0 & G_3 & G_4 & 0 & 0 \\ L_1 & L_2 & 0 & 0 & L_5 & L_6 \\ H_1 & H_2 & 0 & 0 & H_5 & H_6 \end{pmatrix}, \tag{4.3.14}$$

and one of the three eigenvalues is unambiguously identifiable with vertical coordinate y. In this case the preceding analysis can be applied to the 2×2 vertical transfer matrix to determine α_y, independent of horizontal and longitudinal motion. (See problem.)

Almost invariably, in practice, the numerical values of the matrix elements are such that the smallest eigenvalue corresponds to longitudinal oscillation and the other two eigenvalues to transverse oscillation, but this feature cannot be inferred from the algebraic structure of Eq. (4.3.14).

Problem 4..4. For the case of an uncoupled lattice, just discussed, complete the calculation and show that the vertical damping rate is given by

$$\alpha_y = \frac{1}{2}\frac{U_0^{\rm rad}}{E_0}. \tag{4.3.15}$$

In this case vertical damping accounts for 1/4 of the damping assured by the Robinson sum rule.

Problem 4..5. It can be argued that the discussion of damping of pure vertical oscillations in the text and in the previous problem is internally inconsistent because the magnetic field has been assumed to be independent of y and yet stable motion with non-zero y implies non-zero focusing fields that depend on y. For the case that all focusing is performed by independent quadrupoles (for which the on-axis field vanishes), show that the damping due to radiation in the quadrupoles is negligible for sufficiently small y-amplitudes.

4.3.2. Longitudinal Damping

Oscillations of fractional momentum variable δ, along with corresponding variation of longitudinal variable ℓ, are known as "synchrotron oscillations". It turns out that the frequency of longitudinal oscillation is typically much less than the frequency of betatron oscillation, so the phase advance per revolution is much less. It is perhaps for this reason that it is more customary to analyse longitudinal motion by differential equations than by transfer matrices, and this is the way we will begin. Later an almost equivalent transfer matrix analysis will be given.

As regards damping, horizontal and longitudinal motion are more intertwined with each other than either is with vertical motion. Specific lattice details complicate this coupling. There is an especially important distinction between combined function lattices having on-axis fields where focusing occurs, and separated function lattices for which all focusing occurs in quadrupoles having zero on-axis magnetic fields. Problem 4..5 was intended to provide and advanced hint of the importance of this distinction. Here we concentrate on isolating the longitudinal damping, beginning with a digression into simple harmonic motion.

One knows that δ does not retain its same non-zero value, but rather executes synchrotron (longitudinal/momentum) oscillation, which satisfies †

$$\ddot{\delta} + \mu_s^2 \delta = \; F_d = -2\alpha_s \dot{\delta}. \tag{4.3.16}$$

A particular approximate solution, having maximum offset δ_0 initially, is

$$\begin{aligned} \delta &\simeq \; \left(\delta_0 e^{-\alpha_s t}\right) \cos \mu_s t, \\ \dot{\delta} &\simeq \; -\mu_s \left(\delta_0 e^{-\alpha_s t}\right) \sin \mu_s t - \alpha_s \delta, \\ \ddot{\delta} &\simeq \; -\mu_s^2 \delta - 2\alpha_s \dot{\delta}, \end{aligned} \tag{4.3.17}$$

where $\alpha_s << 1$, which has allowed α_s^2 terms to be dropped. (You should check this.) For many purposes, even the terms proportional to α_s can also be neglected and the damping

† Since the quantity oscillating is δ it would be consistent with our other notation to use δ as parameter subscript. But since "s" (for synchrotron) is a standard notation, and the variable conjugate to δ is s which shares the same parameters, we use "s". Also, as will be explained later, α_s may vary as a function of phase in the longitudinal oscillation cycle. This will make it necessary to replace α_s by an appropriate average value later on. We will continue to use the same symbol however, and the significance of the quantity α_s will correspond to the damping rates α_x, α_y, etc. introduced previously.

factors $e^{-\alpha_s t}$ set to 1. This then supports a "phase space" graph with δ as abscissa and $\dot{\delta}/\mu_s$ as ordinate, in which the phase space point stays on a circle of radius δ_0.

We have anticipated the answer by using the symbol α_s here—it can be to be equivalent, though for the whole ring, to the quantity $d\alpha_s$ appearing in Eq. (4.3.4), provided the unit of t is taken to be the revolution period. For this choice of time, for a slowly varying quantity such as δ, the change in one revolution $\Delta\delta$ and the time derivative $\dot{\delta}$ are equal (to the intended accuracy.)

$$\Delta\delta \equiv \dot{\delta}_0. \tag{4.3.18}$$

(Where the symbol Δ appears in the following discussion it will always have this same significance.) With this time unit, μ_s is related to the quantity Ω_s, conventionally called the "synchrotron frequency" by $\Omega_s = \mu_s f_0$ where $\omega_0 = 2\pi f_0$ is the angular revolution frequency, and the "synchrotron tune" is given by $Q_s = \mu_s/(2\pi)$. With non-zero α_s the phase space "circle" slowly spirals in, its radius shrinking by factor $e^{-\alpha_s t}$ each turn.

For the calculations to be performed now, δ is to be held constant during one complete turn—this is a sensible approximation only if $\mu_s << 2\pi$, as we assume. It is only the approximate validity of this assumption and "multiple time scale approximation", $Q_x >> Q_s$, that makes it meaningful to separate out an explicit longitudinal damping rate. While its momentum is essentially constant, the motion of an off-momentum particle can be plotted as in Fig. 4.3.2 and regarded as the superposition of a betatron amplitude x_β and a synchrotron amplitude $\eta\delta$.†

In Eq. (4.3.16) the "dissipative force" $F_d = -2\alpha_s\dot{\delta}$ has been written on the right hand side of the equation in order to encourage treating it perturbatively. To calculate the damping caused by this perturbation, one can calculate the "work" it does during a complete cycle of lossless motion; such motion is described by Eq. (4.3.16) with $\alpha_s = 0$, and the work is given by

$$\oint F_d\dot{\delta}dt = -2\alpha_s \oint \dot{\delta}^2 dt = -\alpha_s\delta_0^2\mu_s^2. \tag{4.3.19}$$

† For our purposes the separation of longitudinal and horizontal oscillation based on their different time scales is a satisfactory approximation. This formalism is so firmly fixed in the minds of most accelerator physicists as to make it difficult to suppress it for phenomena such as synchrobetatron oscillations, where the approximation is not really appropriate.

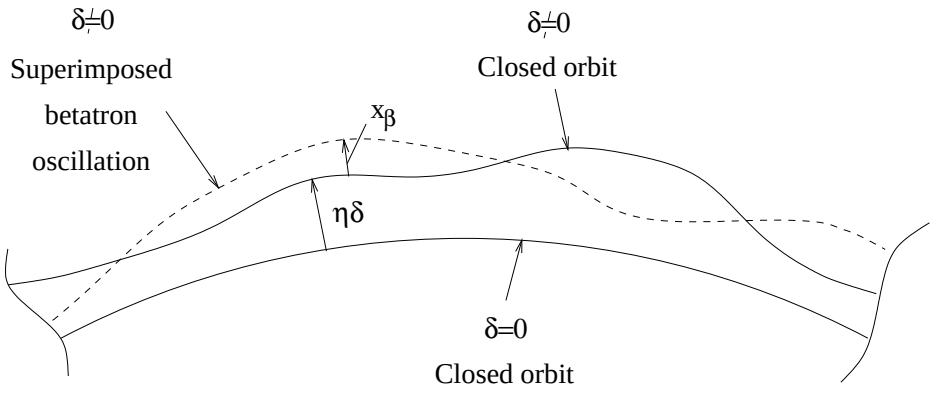

Figure 4.3.2: The betatron amplitude x_β of an off-momentum particle oscillates relative to the off-momentum closed orbit.

Putting completely out of mind the fact that δ is itself derived from the particle energy, we define an entirely independent "oscillator energy".

$$E_{\rm osc} = \frac{1}{2}\mu_s^2\delta_0^2 = \frac{1}{2}\dot{\delta}^2 + \frac{1}{2}\mu_s^2\delta^2. \tag{4.3.20}$$

(It would not be wrong to work instead with to the "action", or "longitudinal Courant-Snyder invariant" but the elementary treatment of simple harmonic motion of mass $m = 1$ and spring constant $k = \mu_s^2$, first understood in one's youth, may make it more familiar to relate work and energy.) In the same spirit in which F_d has been called "force" the terms in Eq. (4.3.20) can be called "total", "kinetic" and "potential" energy respectively. The force F_d causes a systematic reduction of $E_{\rm osc}$ according to work done

$$\overline{\Delta E_{\rm osc}} = \oint F_d\dot{\delta}dt = -\alpha_s\delta_0^2\mu_s^2. \tag{4.3.21}$$

The overline indicates averaging over a complete synchrotron oscillation period, and $\dot{E}_{\rm osc}$ has been replaced by change per turn $\Delta E_{\rm osc}$ which, as explainded previously, means the same thing. The integration in (4.3.21) has been performed using Eq. (4.3.17). Combining these formulas and using $E_{\rm osc} \sim \delta_0^2$,

$$\alpha_s = -\frac{1}{2}\frac{\overline{\Delta E_{\rm osc}}}{E_{\rm osc}} = -\frac{\overline{\Delta\delta}}{\delta_0}. \tag{4.3.22}$$

This may seem to have been an unnecessarily roundabout procedure until one appreciates that the dissipative force F_d could have had any dependence whatsoever on δ and $\dot{\delta}$ (provided it and its accumulated effects are small) and the result Eq. (4.3.22) would still be valid. We will use Eq. (4.3.22) below to obtain α_s from a calculation of $\Delta\delta$.

Problem 4..6. Look up in an introductory physics text, or rederive, the "work-energy theorem" of mechanics and confirm that its use here has been justified.

For a one dimensional eigenmotion having $\lambda_s = e^{-\alpha_s \pm i\mu_s}$ as eigenvalue, the phase space radius acquires a factor $e^{-\alpha_s}$ each time t increases by 1, which agrees with Eq. (4.3.22), and is consistent with our interpretation of α_s. Note though that the attenuation in Eq. (4.3.17) is uniform—the fractional shrinking per turn of the phase space radius is independent of oscillation phase. For synchrotron radiation damping this will not be true, but that will be overcome by averaging—the rate α_s will be strictly applicable only when interpreted as attenuation averaged over a complete period of synchrotron oscillation. This completes the digression concerning simple harmonic motion.

We have seen in the previous section that emittance changes can result when the radiated energy depends on the particle energy and path. Starting again from Eq. (4.2.2), the possible dependencies are through E^2, B^2 and C_B, where C_B is the path length in regions of nonzero magnetic field B. Also it will not be possible to avoid the effect of transverse spatial variation of B needed for transverse focusing since, especially in quadrupoles, the off-momentum closed orbit passes through regions of nonvanishing field even if the central orbit does not.

We wish to calculate the average damping of pure longitudinal motion. In principle this could be done by concatenating the differential transfer matrices of Eq. (4.3.10), but dispersion mixes x and δ motions, which makes that difficult. Instead the loss of "oscillator energy" of an off-momentum particle will be calculated using the formula (4.3.22) just derived.

Consider the motion of an off-momentum particle having no betatron component and traveling along the solid off-momentum closed orbit shown in Fig. 4.3.2. In one turn its energy will be sapped by the excess energy radiated (over and above that radiated on the central orbit),

$$\Delta U^{\rm rad} \simeq K \int_0^{C_B+\Delta C_B} E_0^2 \left(1+2\delta\right) B^2 \left(1+2\frac{\Delta B\left(s\right)}{B}\right) ds + K \int_0^{C_Q} E_0^2 \left(\Delta B\right)^2 ds - U_0^{\rm rad}, \tag{4.3.23}$$

where $K = \frac{e^2c^2}{2\pi}C_\gamma$ is an abbreviation for a constant defined in Eq. (4.2.2), all quantities with subscript 0 are evaluated on the central orbit, and $\Delta B(s)$ is the inescapable, off-axis, extra magnetic field needed for focusing—it is first order in δ and can be written

$$\Delta B \simeq \frac{\partial B_y}{\partial x}\Big|_0 \frac{dx}{d\delta}\,\delta = K_1(s)\,\eta_x(s)\;\delta, \tag{4.3.24}$$

with η_x being the usual horizontal closed-orbit dispersion function, and the field gradient can be expressed in terms of the quadrupole gradient coefficient $K_1 = \frac{1}{pc/e}\frac{\partial B_y}{\partial x}|_0$. Understanding the long term stability of electron storage rings comes down to understanding all contributions to Eq. (4.3.23).

An allowance has been made for radiation in quadrupoles by including an integral over the partial circumference C_Q that contains quadrupoles. (Actually, for consistency, terms for higher multipole elements should be included as well, but they would also be negligible.) Since only terms of first order in δ have been retained and the integrand of the quadrupole term is proportional to δ^2, this term will not, in fact, contribute importantly to damping. Radiation in quadrupoles has been included here to permit making this point plus three others. (i) If, because of errors, the closed orbit does not pass through the center of a quadrupole there will be corresponding radiation and some resultant damping—in a large ring this can require serious attention to preserve the intended damping. (ii) The denominator factor B in the $(B + \Delta B)^2$ Taylor expansion cannot cause trouble by vanishing—by definition of C_B. (iii) Finally, and most important, in a strong focusing lattice, the excess magnetic field needed to bend an excess momentum particle through 2π can be, and is, found partially in quadrupoles. Still they do not contribute importantly to damping. This simple fact has a profound effect on electron storage rings. It might even be said to be what makes them possible, as it overcomes the anti-damping of horizontal oscillations.

Having dispensed with quadrupoles, according to Eq. (4.3.23), there are three contributors to extra radiation: the particle's energy deviates from nominal, it finds itself in a higher than nominal magnetic field, or it has excess path length in the bend field. The first of these is dealt with first, since it comes from the E^2 factor in the master radiation formula (4.2.2), which has expanded to $E_0^2(1 + 2\delta)$ in the integrand of Eq. (4.3.23). If this were the only source of excess radiation accompanying momentum offset δ, the fractional

excess energy loss per unit momentum offset would be given by $(1/U_0^{\rm rad})(dU^{\rm rad}/d\delta) = 2$. Following Sands, this part is subtracted off, and the fractional excess energy is represented by a lattice dependent quantity $\mathcal{D}$, known as "curly-D", that is defined by

$$\mathcal{D} = \frac{1}{U_0^{\rm rad}} \frac{dU^{\rm rad}}{d\delta} - 2 \ . \tag{4.3.25}$$

Recombining all contributions, the damping decrement of δ can then be expressed in terms of $\mathcal{D}$;

$$\Delta\delta \simeq -\frac{\Delta U^{\rm rad}}{E_0} = -\frac{dU^{\rm rad}}{d\delta}\frac{1}{E_0}\delta = -\left(2+\mathcal{D}\right)\frac{U_0^{\rm rad}}{E_0}\,\delta. \tag{4.3.26}$$

There are important cases, where contributions from the other terms in Eq. (4.3.23) are unimportant, in which one has $|\mathcal{D}| << 1$. In this case, the value of $\Delta\delta$ is immediately known. In general, for any practical lattice, separated or combined-function, isomagnetic or not, $\mathcal{D}$ can be calculated by numerical integration, and that what is mainly done in practice. Here we will make some simplifications and approximations, mainly rather good ones, to get simple analytic approximations and develop intuition.

To simplify the calculation somewhat, without essential error in most cases, and again following Sands, we make the isomagnetic approximation according to which the magnetic field on the design orbit is either zero or has the same value B_0. Manipulation of Eq. (4.3.23) yields

$$\frac{\Delta U^{\rm rad}}{U_0^{\rm rad}} = 2\delta + \frac{2}{B_0 C_B}\int_0^{C_B} \Delta B\, ds + \frac{\Delta C_B}{C_B} \ . \tag{4.3.27}$$

Segments of the perturbed orbit that newly pass through the nominal field B_0 because $\delta \neq 0$ are included in the $\Delta C_B/C_B$ term.† There is one global requirement that $\Delta B(s)$ must meet—the total bend after a full turn must remain equal to 2π. Because the rate of bend is inversely proportional to $1+\delta$, the existence of $\delta > 0$ must be accompanied by contributions from $\Delta C_B > 0$ and/or $\Delta B(s) > 0$. Quantitatively

$$\delta = \frac{\Delta C_B}{C_B} + \frac{1}{B_0 C_B}\int_0^{C_B} \Delta B\, ds + \frac{1}{B_0 C_B}\int_0^{C_Q} \Delta B\, ds \ ; \tag{4.3.28}$$

the terms being as simple as they are because of the isomagnetic assumption. This formula shows how quadrupoles can "help" in the bending even though they don't "hurt" by causing

† In strong focusing lattices the term $\Delta C_B/C_B$ is typically very small, because of large "momentum compaction".

radiation. This relation can be used to re-express Eq. (4.3.27) by eliminating the first integral;

$$\frac{\Delta U^{\rm rad}}{U_0^{\rm rad}} = 4\delta - \frac{\Delta C_B}{C_B} - \frac{2}{B_0 C_B}\int_0^{C_Q} \Delta B ds. \tag{4.3.29}$$

In the isomagnetic approximation then, $\mathcal{D}$ is given by two equivalent forms

$$\begin{aligned} \mathcal{D} &= \frac{1}{C_B}\frac{dC_B}{d\delta} + \frac{2}{B_0 C_B}\int_0^{C_B} \frac{dB}{d\delta} ds, \\ &\stackrel{\rm or}{=} 2 - \frac{1}{C_B}\frac{dC_B}{d\delta} - \frac{2}{B_0 C_B}\int_0^{C_Q} \frac{dB}{d\delta} ds. \end{aligned} \tag{4.3.30}$$

The ranges of integration are along the on-momentum, central orbit, and the integrands are evaluated on that orbit. The second expression is handy (because the integral drops out) if there are no quadrupoles, as in a combined function lattice.

Useful further reduction of $\mathcal{D}$ is different for different styles of accelerator. For example, a common, not unnatural, but certainly not universal design, uses only uniform field "sector magnets". In these magnets the closed orbit enters and leaves all dipoles normal to the poles, $dB/d\delta = 0$ (because the field is uniform), and the path length in drift regions is independent of δ. This permits elimination of $C_B(\delta)$ in favor of the (more easily known) total circumference $C(\delta)$, which is greater by the accumulated lengths of all drift sections. One then defines "transition gamma", γ_t, (its inverse square γ_t^{-2} is called the "momentum compaction factor") by

$$\frac{1}{\gamma_t^2} = \frac{1}{C_0}\frac{dC\,(\delta)}{d\delta}\Big|_{\delta=0}\ , \tag{4.3.31}$$

where $dC/d\delta = dC_B/d\delta$ because there is no linear dependence of path length on δ in drift regions. Then we have

$$\frac{1}{C_B}\frac{dC_B}{d\delta} = \frac{C_0}{C_B}\frac{1}{C_0}\frac{\Delta C}{\delta} = \frac{C_0}{C_B}\frac{1}{\gamma_t^2}, \tag{4.3.32}$$

and

$$\mathcal{D} = \frac{C_0}{C_B}\frac{1}{\gamma_t^2}. \tag{4.3.33}$$

With quadrupole elements present, the circumference increases far less than proportional to $1+\delta$, —which is the source of the name "momentum compaction". In high tune lattices one commonly has $\gamma_t \simeq Q_x$, which leads to $\gamma_t^{-2} << 1$. This is why longitudinal damping in separated function accelerator is much less than in a combined function accelerator. Superficially undesirable, this is in fact good since it saves some of the Robinson ration to be available for horizontal damping.

We have seen the sort of calculation needed to calculate $\mathcal{D}$ and seen that it depends on lattice details such as whether the lattice is combined function (focusing built into bending magnets) or separated function (separate quadrupole magnets). All this detail is distilled into the value of $\mathcal{D}$ and we can turn to studying the effect $\Delta\delta$, as given by Eq. (4.3.26), has on synchrotron oscillation. Fig. 4.3.3 is supposed to be helpful. The radius of the phase space circle is equal to δ_0, the maximum value that δ achieves, and the instantaneous synchrotron oscillation phase angle is given by $\mu_s t$, where $Q_s = \mu_s/(2\pi)$ is the "synchrotron tune". Typically $Q_s << 1$ so the synchrotron phase advance, as the electron completes a whole turn of the storage ring, is much less than 2π. The deviation $\Delta\delta$ reduces the radius δ_0, but the effect is proportional to $\cos^2 \mu_s t$, with one factor $\cos \mu_s t$ being due to proportionality to δ and the other due to the shift being parallel to the δ-axis in phase space rather than radial. Averaging over one synchrotron period the mean fractional-decrement of δ_0 per turn is

$$\alpha_s = -\frac{\overline{\Delta\delta_0}}{\delta_0} = (2 + \mathcal{D})\,\frac{U_0^{\rm rad}}{2E_0}. \tag{4.3.34}$$

Here we have used Eq. (4.3.22) to complete our task of evaluating α_s. As the signs have been introduced α_s is a "decay rate" rather than a "growth rate."

Problem 4..7. Calculate α_s for an (impractical) lattice for which the entire central orbit is in a constant uniform magnetic field. Also calculate α_s for the (weak focusing) lattice, for which (according to Eq. (3.4.1)) the magnetic field is longitudinally constant on the entire central orbit, but has radial dependence $B_y(r) = B_{y0}(1 - \frac{n}{R}\,r)$.

Problem 4..8. In the text an analogy has been established between synchrotron oscillations and simple harmonic motion of a viscous-damped, point-mass and spring combination. In this analogy δ corresponds to point coordinate x and $\dot{\delta}$ to $\dot{x}$. What is there about the damping that violates this analogy, why does that not invalidate the result Eq. (4.3.34), and how could the analogy be improved?

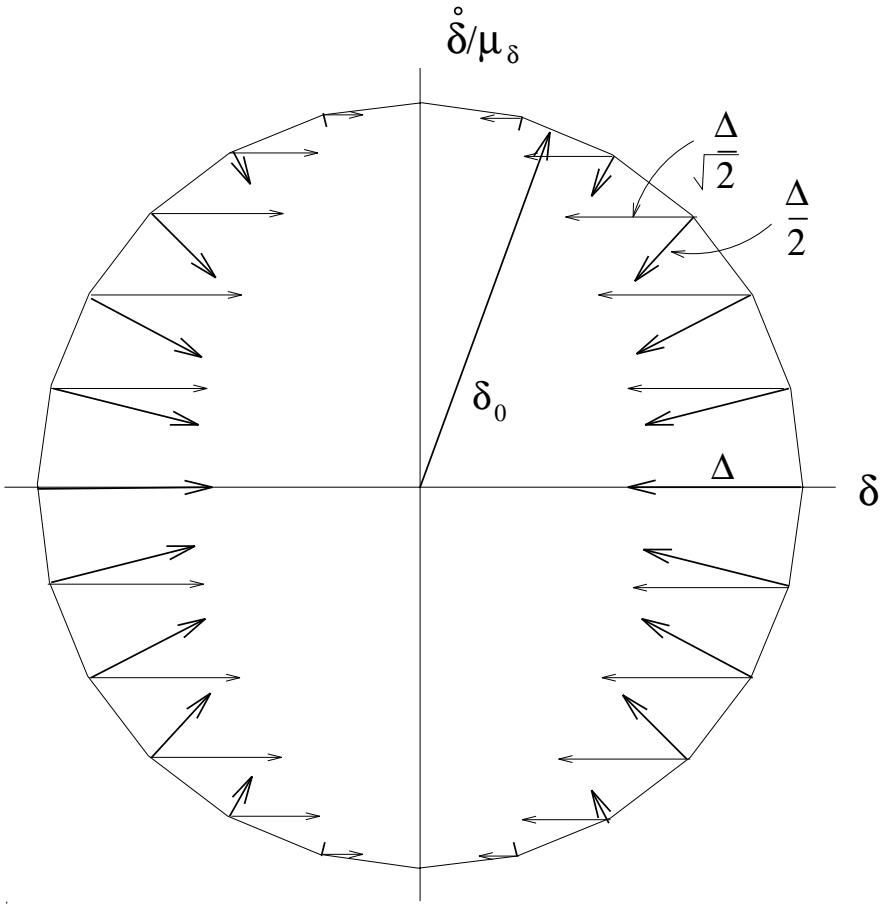

Figure 4.3.3: Longitudinal phase space plot with δ as horizontal axis and $\dot{\delta}/\mu_s$ as vertical axis, for $Q_s = 1/24$, small compared to 1 as assumed in the text. Light horizontal arrows (exaggerated in length) are the decrement of δ during that particle revolution—the decrement is proportional to δ. Heavy radial arrows represent the corresponding decrement of the radius of the phase space circle.

4.3.3. Horizontal Damping and Partition Numbers

We have obtained relations Eq. (4.3.13) for the sum of damping rates $\alpha_x + \alpha_y + \alpha_s$ Eq. (4.3.15) for vertical damping rate α_y (in the uncoupled case we now assume) and Eq. (4.3.34) for the longitudinal damping rate α_s. Evidently these can be combined to obtain α_x.

$$\alpha_x = (1 - \mathcal{D})\,\frac{U_0^{\rm rad}}{2E_0}, \quad \alpha_y = (1)\,\frac{U_0^{\rm rad}}{2E_0}, \quad \alpha_s = (2 + \mathcal{D})\,\frac{U_0^{\rm rad}}{2E_0}. \tag{4.3.35}$$

Sands felt (correctly as it turned out) that it would be easier to remember these rates if they were measured in units of $U_0^{\rm rad}/(2E_0)$,† and quoted as three "partition numbers" J_x, J_y, forced by Robinson's theorem to add up to 4. Then Eq. (4.3.35) becomes

$$\begin{aligned} \alpha_x =& J_x \frac{U_0^{\rm rad}}{2E_0}, \quad \alpha_y = J_y \frac{U_0^{\rm rad}}{2E_0}, \quad \alpha_s = J_s \frac{U_0^{\rm rad}}{2E_0}, \\ J_x =& 1-\mathcal{D}, \quad J_y = 1, \quad J_s = 2+\mathcal{D}. \end{aligned} \tag{4.3.36}$$

For stable operation, all partition numbers must be positive. We can now understand a comment made earlier, that too much energy damping is bad—it can cause J_x to be negative.

If there were no "heating effect" and all the J's positive, all beam emittances would decay to zero. But there is heating, which we investigate next.

4.4. Equilibrium Between Damping and Fluctuation.

We return to the subject of radiation fluctuations, discussing vertical, horizontal, and longitudinal motion separately. Actually vertical motion can be dispensed with in one sentence because, with photons radiated in the exact forward direction (an excellent approximation in this context) there is no mechanism for exciting vertical oscillations. Since there is however damping, the bunch height damps down to a negligible value. In the immediate section the competition between damping and growth caused by quantum fluctuations for horizontal betatron motion is analyzed, and after that longitudinal motion is analyzed. Though the treatments are entirely parallel, both are given in some detail, and largely independent of each other. Since the longitudinal case is somewhat simpler, if some obscurity is encountered in the transverse case, the reader might consider skipping to the longitudinal case in hopes of finding an argument less cluttered with "helpful" verbiage.

There is a well-established stochastic formalism using the Fokker-Planck equation for analyzing problems like this. I prefer not to use that formalism at this time, instead using "physical arguments". It would be more accurate to say "arguments presented in terms likely to be persuasive to physicists" as, other than in degree of rigor, the arguments are much the same as could be used to establish the Fokker-Planck equation. (That will in

† Actually, since the quantities α_i as Sands defines them are rates per unit time rather than rates per revolution, his rates are measured in units of $U_0^{\rm rad}/(2T_0E_0)$ where T_0 is the revolution period.

fact be done in a later section in order to describe the approach to equilibrium and particle loss mechanisms.) Only one result will be assumed from probability theory and that is the Central Limit Theorem. In oversimplified physicist's terms, this theorem states that repeatedly convoluted distributions become Gaussian and errors combine "quadratically".

4.5. Horizontal Equilibrium and Beam Width

This discussion of beam equilibrium commencing now is complicated by the fact that three separate averagings are to be performed: over betatron phase, over photon energies, and over lattice elements. (Over lattice elements it is actually a sum rather than an average, and over photons it is both an average and a sum, but that is getting ahead of the story.) Consider the on-momentum particle illustrated in Fig. 4.5.1. It is free of betatron motion and traveling on the central orbit at the instant a photon of energy u is emitted exactly in the forward direction. The emission changes the particle energy but not its direction;

$$\delta \to \delta - \frac{u}{E_0} \ . \tag{4.5.1}$$

The coordinates of the local closed orbit corresponding to particle with fractional energy offset $\delta = -u/E_0$ are $(-\eta_x u/E_0, -\eta'_x u/E_0)$. But the physical location of the particle is exactly the same as before the emission, which is to say on the closed orbit appropriate to the previous momentum. This means that horizontal betatron phase space components $(\eta_x u/E_0, \eta'_x u/E_0)$ have been impulsively imparted to the particle.

In general, a beam particle already has non-zero betatron components corresponding to some Courant-Snyder invariant

$$\epsilon_{x0} = \frac{x^2 + (\beta_x x' + \alpha_x x)^2}{\beta_x} \quad \overset{\text{also}}{=} \quad \gamma_x x^2 + 2\alpha_x x\, x' + \beta_x x'^2 \ , \tag{4.5.2}$$

before the emission takes place. Its coordinates at location s are

$$x_0 = \sqrt{\beta_x(s)\,\epsilon_{x0}}\cos\psi_x, \quad x'_0 = -\sqrt{\frac{\epsilon_{x0}}{\beta_x(s)}}\left(\sin\psi_x + \frac{\alpha_x}{\beta_x}\cos\psi_x\right), \tag{4.5.3}$$

where $\alpha_x = -\beta'_x/2$ and $\psi_x(s)$ is its horizontal betatron phase. After the photon emission the phase space coordinates have become

$$x_1 = \sqrt{\beta_x \epsilon_{x0}}\cos\psi_x + \eta_x u/E_0, \quad x'_1 = -\sqrt{\frac{\epsilon_{x0}}{\beta_x}}\left(\sin\psi_x + \frac{\alpha_x}{\beta_x}\cos\psi_x\right) + \eta'_x u/E_0. \tag{4.5.4}$$

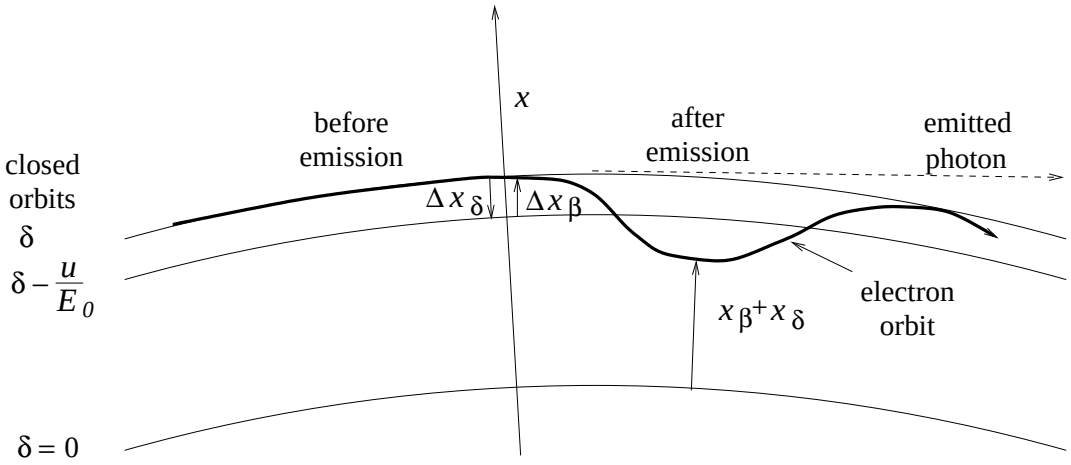

Figure 4.5.1: Betatron oscillation induced by forward emission of photon of energy u for an initially off-momentum particle which has negligible betatron amplitude.

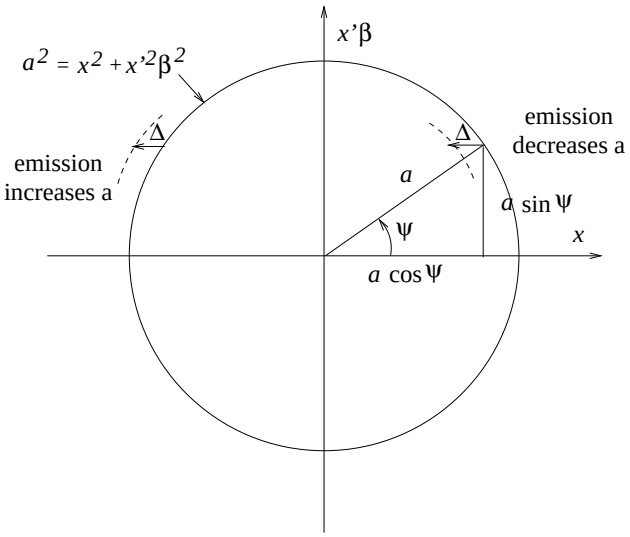

Figure 4.5.2: An impulsive increment Δ to the betatron displacement can increase or decrease the Courant-Snyder invariant. But, averaged over phase, there is a net increase that is proportional to Δ^2.

It can be seen from Fig. 4.5.2 that a photon emission can increase or decrease the Courant-Snyder invariant, depending upon the betatron phase. Averaging over phase angle ψ proceeds as

$$\overline{(a\cos\psi + \Delta)^2 + a^2\sin^2\psi - a^2} = \overline{2a\Delta\cos\psi + \Delta^2} = \Delta^2 \ . \qquad (4.5.5)$$

Because the value of ψ_x where radiation occurs is random, terms proportional to $\sin\psi_x$ or $\cos\psi_x$ average to zero when averaging over ψ_x. (Averaging over betatron phase is to be indicated by an overhead line.) The phase-averaged change in the Courant-Snyder invariant, $\overline{\Delta\epsilon_x^u} = \overline{\epsilon_{x,1} - \epsilon_{x,0}}$, due to the emission of a photon of energy u at location s is defined to be $\mathcal{H}(s)(u/E_0)^2$, where $\mathcal{H}(s)$ goes by the name "curly H" and is given by

$$\begin{aligned}\mathcal{H}(s) = \frac{\overline{\Delta\epsilon_x^u}}{(u/E_0)^2} =& \frac{\gamma_x\overline{x_1^2 - x_0^2} + 2\alpha_x\overline{x_1x_1' - x_0x_0'} + \beta_x\overline{x_1'^2 - x_0'^2}}{(u/E_0)^2} \\ =& \gamma_x\eta_x^2 + 2\alpha_x\eta_x\eta_x' + \beta_x\eta'^2 \equiv \frac{\eta_x^2 + (\beta_x\eta_x' + \alpha_x\eta_x)^2}{\beta_x}.\end{aligned} \tag{4.5.6}$$

It is important to appreciate that $\mathcal{H}(s)$ depends only on lattice functions.

The averaging so far has been over betatron phase, permitting the expectation value of growth due to the emission of a single photon of energy u to be calculated. Since the growth due to uncorrelated emission of many photons is proportional to the number of photons, the total number of photons with energy u must be calculated next and the result then summed over photons of all energies. It is also necessary to fold in the damping effect calculated in an earlier section. In equilibrium the growth and damping cancel. It turns out that in practical situations the time constant governing approach to that equilibrium is long compared to the revolution period $T_0 = 1/f_0$ of the accelerator. This *a priori* unguaranteed feature will be used to simplify the calculation at this point, and justified *a posteriori*.

Because the lattice functions and the spectral functions depend on s the instantaneous rates should be calculated first and then integrated over lattice elements. Unfortunately our earlier discussion of fluctuations made assumptions that break down for too-short path intervals. But the long time constant mentioned in the previous paragraph makes it legitimate to treat ϵ_x as essentially constant for the duration T_0 of a single turn, say the i-th turn, and to accumulate changes over a full revolution. Permission is now requested to permit the clumsy, but temporary, notation, $\overline{\Delta\epsilon_x(u)}$, to stand for the betatron-phase-averaged, cumulative for energies less than u, increment of the Courant-Snyder invariant during one full turn. (The symbol u has been moved from being a superscript to being an argument as a lame indication that u is now the upper limit of a range rather than a running variable.) From calculations performed so far, the increment $d\overline{\Delta\epsilon_x(u)}$ coming

from photon emissions in the range u to $u + du$ can be given in terms of $\mathcal{H}$ and radiation formulas;

$$d\overline{\Delta\epsilon_x(u)} = du\,\frac{u^2}{E_0^2}\oint \mathcal{H}(s)\,\frac{P_\gamma(s)\,ds/c}{<u(s)>_\gamma} n_u(u)\,. \tag{4.5.7}$$

Here $P_\gamma ds/c$ has given the energy emitted in path ds as in Eq. (4.2.2); this energy has then been converted to total number of photons by Eq. (4.2.8), and their distribution as an energy spectrum expressed as $n_u(u)$. The energy accumulated, Courant-Snyder one turn increment will be denoted

$$\Delta\epsilon_x^P = \int_0^\infty \frac{d}{du'}\overline{\Delta\epsilon_x(u')}du' = \overline{\Delta\epsilon_x(\infty)}. \tag{4.5.8}$$

On the symbol $\Delta\epsilon_x^P$ the superscript P for "particle" has been introduced to make the point that this is an individual particle parameter, and to distinguish it from a parameter ϵ_x^B, with B for "beam", to be introduced shortly. I have also taken the opportunity to restrain the proliferation of averaging and overline symbols. The averaging over betatron phase and integration over photon energies just described will be implicitly assumed to have occurred in what follows.

A few sentences of justification may be in order. In a varying magnetic field the length over which B can be treated as constant is so short as to make the fluctuation in number of photons emitted fractionally significant, contrary to assumption. But if one visualizes the integal of Eq. (4.5.7) as broken into a "Riemann" sum of terms that are re-arranged and gathered together into groups of terms for intervals on which the field has (essentially) the same value, then the fractional fluctuation in number of photons emitted within each such group is much suppressed—let's say made negligible. Since rearranging the terms in the Riemann sum does not change the integral, and our model treats all emissions as uncorrelated, Eq. (4.5.7) has been legitimized. A further pedantic point, made only to encourage contemplation of this formula, is that the factor $n(u)$ cannot be taken outside the integral, even though u is being held fixed, because the distribution function $n(u)$ depends on the magnetic field, which depends on s. In an isomagnetic lattice $n(u)$ could be taken outside, making moot all the concerns of this paragraph.

It remains to integrate over all values of u, which yields

$$\begin{aligned}\Delta\epsilon_x^P =& \frac{1}{E_0^2}\oint \mathcal{H}(s)\,\frac{P_\gamma(s)\,ds/c}{<u(s)>_\gamma}\int_0^\infty u'^2 n_u\left(u'\right)du' \\ =& \frac{1}{E_0^2}\oint \mathcal{H}(s)\,\frac{P_\gamma(s)\,ds/c}{<u(s)>_\gamma} <u^2(s)>_\gamma\ .\end{aligned} \tag{4.5.9}$$

Note that this is independent of the pre-existing betatron amplitude.

This result was anticipated in Eq. (4.2.12). In particular the integrand is proportional to $u_c\ dU_T/ds$. However the present result is more general in that the lattice dependent factor $\mathcal{H}(s)$ has been correctly folded in under the integral. Nevertheless an approximate growth rate can be based on the estimate of the sum of squares of radiated energies by $1.323\,u_c\,U_0^{\rm rad}$—roughly speaking, the root mean square energy radiated is the geometric mean of the critical energy and the energy loss per turn. If all radiation in one turn occurred at a single location where $\eta_x' = \alpha_x = 0$ (so that $\mathcal{H} \approx \eta_x^2/\beta_x$) the expected horizontal emittance growth would be $\Delta\epsilon_x^P \approx 1.3\eta_x^2 u_c U_0^{\rm rad}/(E_0^2\beta_x)$.

At this point it is important to distinguish between the Courant-Snyder invariant ϵ_{sc}^P of an individual particle, defined in Eq. (4.5.2), with superscript P having been added to emphasize the point, and the "beam emittance" ϵ_x^B, encountered now for the first time (in this context);

$$\epsilon_x^B = \frac{\sigma_x^2(s)}{\beta_x(s)}\ , \tag{4.5.10}$$

where $\sigma_x(s)$ is the r.m.s. beam width at location s. Equivalently $\sigma_x^2(s)$ is the "variance" of x. Recall that ϵ_x^B is independent of s. (Choosing the same symbol ϵ for these two different quantities, as is universally done, is certainly confusing, and at this point it is misleading in a way that makes a factor-of-two error likely. The symbol σ_x is equally deserving of being decorated with the medal "B", but this is unnecessary since there is no conceivable misinterpretation of σ_x as an individual particle parameter.)

Because the transverse distribution is the result of a random walk, by the central limit theorem, the distribution is Gaussian, at least in the linear regime. There is an arbitrary, conventional, numerical factor in the definition of ϵ_x^B and the choice here of this number as 1 is a special "electron world" choice. (Since proton beams are not necessarily Gaussian, the conventional "proton world" definitions of ϵ_x^B define phase space areas containing "most" of the beam particles.)

The probability distributions of x and the quantity $\upsilon = \beta_x x' + \alpha_x x$ are assumed to be identical, normalized, independent, and Gaussian;

$$P_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}}, \quad P_\upsilon(\upsilon) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{\upsilon^2}{2\sigma_x^2}}. \tag{4.5.11}$$

Here, and in the rest of the section, we are observing the beam at a fixed reference location in the ring, where the lattice functions α_x and β_x are fixed and known. The two-dimensional joint probabilty distribution of x and υ is $P_{x,\upsilon}(x,\upsilon) = P_x(x)P_\upsilon(\upsilon)$. From this one can calculate the expectation value (*i.e.* average over all particles in the beam) on the i'th turn

$$< x_i^2 + \upsilon_i^2 >_{\text{beam}} = 2\sigma_x^2, \tag{4.5.12}$$

where σ_x^2 would depend on i only in dynamic situations. Calling the effect of radiation fluctuations "excitations", the trick now is to treat the excitations $(\Delta x, \Delta\upsilon)$ occurring during one turn as Gaussian distributed with shortly-to-be-determined variance κ^2,

$$P_{\Delta x}(\Delta x) = \frac{1}{\sqrt{2\pi}\kappa} e^{-\frac{\Delta x^2}{2\kappa^2}}, \quad P_{\Delta\upsilon}(\Delta\upsilon) = \frac{1}{\sqrt{2\pi}\kappa} e^{-\frac{\Delta\upsilon^2}{2\kappa^2}}. \tag{4.5.13}$$

These are the effective Gaussian-distributions of the one-turn accumulations of radiation fluctuations. Superposition of these deviations on a pre-existing (Gaussian) distribution will be handled by convolution.

(Even though individual emissions are far from Gaussian, by the Central Limit Theorem, for not-too-bizarre distributions, the sum of many emissions is Gaussian distributed with variance equal to the sum over individual variances. Like the distributions of x and υ, that are identical because of isotropy in phase space, the distributions of Δx and $\Delta\upsilon$ must be equal in order to preserve isotropy. It would scarcely be quibbling at this point to ask "Who says isotropy must be preserved?" A pleasing answer would be based on solving the stochastic equations and proving it to be true. A less satisfying answer is that decoherence causes tangential "shear" motion in phase space and that shear causes "mixing" or "filamentation" which maintains isotropy in phase space for practical tune spreads. Though this argument is somewhat of a swindle, coming as it does "from left field", it nevertheless is descriptive of actual beam behavior. Without pursuing the issue of which argument is correct, and conscious of the great simplification that will result, we shamelessly claim that our approach correctly describes the "physics", and proceed.)

The value κ^2 needed to match the expectation value $\Delta\epsilon_x^P$ (a quantity previously calculated in Eq. (4.5.9)) satisfies

$$\Delta\epsilon_x^P = \frac{< \Delta x^2 + \Delta\upsilon^2 >_{\text{excitation}}}{\beta_x} = \frac{2\kappa^2}{\beta_x} \quad \text{or} \quad \kappa^2 = \frac{\beta_x \Delta\epsilon_x^P}{2}. \tag{4.5.14}$$

There is no point in proceeding without having assimilated the purpose and meaning of this equation as it is the linchpin of the entire development. It links the beam-related quantity κ^2 to the particle-related quantity $\Delta\epsilon_x^P$. It also contains the answer to a question that is sometimes of practical importance, "What are the beam dimensions of an initially-zero emittance beam after one turn?"

The probability distributions of x_{i+1}, υ_{i+1} after one turn are convolutions of (4.5.11) and (4.5.13). Gaussian variances being additive, the new variances are both equal to $\sigma_x^2 + \kappa^2 = \sigma_x^2 + \beta_x \Delta\epsilon_x^P/2$. This is a timely point to include also the effect of damping, which has the effect $\sigma_x^2 \to \sigma_x^2(1 - 2\alpha_x)$, with α_x obtained from Eq. (4.3.36). Combining both effects

$$< x_{i+1}^2 + \upsilon_{i+1}^2 >_{\text{beam}} = 2\sigma_x^2 \left(1 - 2\alpha_x\right) + \beta_x \Delta\epsilon_x^P. \tag{4.5.15}$$

In equilibrium $\sigma_{x,i+1} = \sigma_{x,i}$, which implies

$$\epsilon_x^B = \frac{\sigma_x^2}{\beta_x} = \frac{\Delta\epsilon_x^P}{4\alpha_x}. \tag{4.5.16}$$

(If the final emittance ϵ_x^B were made up in equal steps of size $\Delta\epsilon_x^P$, the total number of steps would be $(4\alpha_x)^{-1}$, a quantity that can be referred to as the "relaxation time measured in turns".) Incorporating Eq. (4.5.9), the equilibrium beam emittance is given by

$$\begin{aligned} \epsilon_x^B = \frac{\sigma_x^2}{\beta_x} = \frac{\Delta\epsilon_x^P}{4\alpha_x} &= \frac{\oint \mathcal{H}\left(s\right) \frac{P_\gamma(s) ds/c}{<u(s)>_\gamma} < u^2\left(s\right) >_\gamma}{2E_0 J_x \oint P_\gamma\left(s\right) ds/c} \\ &= \frac{u_c}{B} \frac{1.323 \oint \mathcal{H}\left(s\right) B^3\left(s\right) ds}{2E_0 J_x \oint B^2\left(s\right) ds} = \frac{1.323 u_c}{2E_0 J_x} < \mathcal{H} >_{\text{isomag-only}}, \end{aligned} \tag{4.5.17}$$

where α_x has been obtained from Eq. (4.3.36), U_0^{rad} from Eq. (4.2.3), and the ratio of radiation moments from Eq. (4.2.7). Knowing from Eq. (4.2.5) that the factor $\frac{u_c(s)}{B(s)} = \frac{3}{2}\frac{e\hbar}{m}\gamma^2$ is independent of B, that factor has been moved outside the integral, and this leaves the emittance given by an integral weighted by B^3 divided by an integral weighted by B^2. This implies that increased damping due to excess radiation in local regions of specially high

magnetic field (increasing the denominator) cannot make up for the increased quantum fluctuations in the radiation that increase the numerator. This shows that the emittance is minimum in the isomagnetic case. The final form, valid only in the isomagnetic case, uses $\mathcal{H}$ averaged over dipoles.

4.6. Minimum Vertical Beam Size

Because of the near forward direction of radiated photons, the beam height tends to be tiny compared to its width. This section discusses this issue more quantitatively. Most readers are advised to skip the section.

When a photon of energy u is emitted with vertical angle ψ the electron recoils with vertical angle $\psi u/E_0$ and, as in Eq. (4.5.4), the vertical coordinates are altered to

$$y_1 = \sqrt{\beta_y \epsilon_{y0}}\cos\psi_y, \quad y_1' = -\sqrt{\beta_y \epsilon_{y0}}\left(\sin\psi_y + \frac{\alpha_y}{\beta_y}\cos\psi_y\right) + \frac{\psi u}{E_0}\ , \tag{4.6.1}$$

and (as in Eq. (4.5.6)) the change in vertical C-S invariant, averaged over betatron phase, is

$$\overline{\Delta\epsilon_y^{u\psi}} = \gamma_y\overline{y_1^2 - y_0^2} + 2\alpha_y\overline{y_1 y_1' - y_0 y_0'} + \beta_y\overline{y_1'^2 - y_0'^2} = \beta_y\frac{\psi^2 u^2}{E_0^2}\ . \tag{4.6.2}$$

As in Eq. (4.5.7), the change in C-S invariant over a full turn, from photons in the energy range u to $u+du$ and angular range ψ to $\psi + d\psi$ is given by

$$\frac{\partial^2}{\partial u \partial\psi}\overline{\Delta\epsilon_y(u,\psi)} = \frac{1}{E_0^2}\oint \beta_y(s)\, u^2\psi^2\,\frac{P_\gamma(s)\,ds/c}{<u(s)>_\gamma} n_{u,\psi}(u,\psi)\,. \tag{4.6.3}$$

As in Eq. (4.5.9), accumulating over all values of u and ψ, the increase in vertical C-S invariant after one turn is given by,

$$\begin{aligned}\Delta\epsilon_y^P =& \frac{1}{E_0^2}\oint \frac{P_\gamma(s)\,ds/c}{<u(s)>_\gamma}\int_0^\infty\int_0^\infty \beta_y(s)\; u'^2\psi'^2 n_u\left(u',\psi'\right)\, du'd\psi' \\ =& \frac{1}{E_0^2}\oint \beta_y(s)\;\frac{P_\gamma(s)\,ds/c}{<u(s)>_\gamma} < u^2\psi^2(s) >_\gamma\ .\end{aligned} \tag{4.6.4}$$

To this point I have tried to retain the dependencies on s, but for purposes of estimation let us assume all lattice functions to be independent of s. Also let us assume† $< u^2\psi^2(s) >_\gamma = < u^2 >_\gamma /\gamma^2$. Then the ratio of single-turn emittance growths is given by

$$\frac{\Delta\epsilon_y^P}{\Delta\epsilon_x^P} \approx \frac{\beta_y\beta_x}{\eta_x^2}\,\frac{1}{\gamma^2}\ . \tag{4.6.5}$$

† The value of $< u^2\psi^2(s) >_\gamma$ is presumably well known but, for the moment, not knowing what it is, I adopt the usual estimate of vertical angle, namely $1/\gamma$.

If the horizontal and vertical damping partition numbers are equal, the ratio of equilibrium beam emittances will retain the same value. For an ultra-low emittance lattice, values of the parameters might be $\beta_x = 1\,\mathrm{m}$, $\beta_y = 25\,\mathrm{m}$, $\eta_x = 5 \times 10^{-3}\,\mathrm{m}$, and the emittance ratio would be about one to one hundred; i.e. height/width=0.1.

4.7. Longitudinal Equilibrium and Energy Spread

Excitation of energy oscillations can be analyzed in much the same terms as horizontal betatron oscillations, but there are three simpler features in this case. One has to do with the longitudinal analog to $\beta_x(s)$. An analogous function, $\beta_\delta(s)$, can be defined to describe the modulation of energy oscillations as a function of circumferential coordinate s. But our model so far has legitimately assumed that longitudinal quantities vary so slowly as to be essentially independent of s; this is equivalent to taking $\beta_s =$ constant, and for present purposes we can take the constant to be 1. Hence we define an analog to ϵ_x^P by

$$\epsilon_\delta^P = \delta^2 + \varpi^2, \tag{4.7.1}$$

where ϖ is the phase-space coordinate complementary to δ, scaled to make the phase space trajectory circular. (In the linear regime ϖ is proportional to the longitudinal displacement from the reference particle, but that connection will be developed later.) The superscript P stands for "particle" as this is a parameter of an individual particle. As before the motivation behind analyzing ϵ_δ^P, rather than δ itself, is that its constancy in the unperturbed situation permits averaging over oscillation phases while calculating the effect of the emission of a photon. The second simpler feature is that a photon emission does not alter ϖ, and the third is that the radiation of the j-th photon, of energy u_j, is reflected directly by a change $\Delta\delta = -u_j/E_0$. As a result the synchrotron-phase averaged increase in single particle invariant is

$$\overline{\Delta\epsilon_\delta^u} = \frac{u^2}{E_0^2}, \tag{4.7.2}$$

without any need to introduce an s-dependent factor analogous to $\mathcal{H}(s)$. This formula, the analog of Eq. (4.5.6), can be described simply by saying that if an $\mathcal{H}$ had been introduced its value would be 1. Eq. (4.7.2) must be summed over photons emitted during one full turn to obtain $\Delta\epsilon_\delta^P$, the corresponding increase in single particle invariant. Analogous to

Eq. (4.5.9) the result is

$$\Delta\epsilon_\delta^P = \frac{1}{E_0^2}\oint \frac{P_\gamma(s)\,ds/c}{<u(s)>_\gamma}\int_0^\infty u'^2 n_u\left(u'\right)du' = \frac{1}{E_0^2}\oint \frac{<u^2(s)>_\gamma}{<u(s)>_\gamma}P_\gamma(s)\,ds/c. \tag{4.7.3}$$

In the isomagnetic case this becomes

$$\Delta\epsilon_\delta^P = \frac{U_0^{\rm rad}}{E_0^2}\frac{<u^2(s)>_\gamma}{<u(s)>_\gamma} = \frac{1.323u_cU_0^{\rm rad}}{E_0^2}. \tag{4.7.4}$$

This result is closely connected to Eq. (4.2.12) which can be expressed as

$$<\sum_{j=1}^{\mathcal{N}_0}\Delta\delta_j^2>_\gamma = \frac{1.323u_cU_0^{\rm rad}}{E_0^2}. \tag{4.7.5}$$

(At this point one might inquire why it is not $<u^2>_\gamma - <u>_\gamma^2$, the variance of the probability distribution of photon energies, rather than $<u^2>_\gamma$, that enters. This replacement might be appropriate if the electron were subject to no longitudinal restoring force and were simply dribbling away its energy in fluctuating lumps u_j. But, as was true for transverse excitations, it is only the term $<u^2>_\gamma$ that survives averaging over synchrotron oscillation phases.)

Corresponding to Eqs. (4.5.11), δ and ϖ distribution functions for the beam are

$$P_\delta(\delta) = \frac{1}{\sqrt{2\pi}\sigma_\delta^B}e^{-\frac{\delta^2}{2\sigma_\delta^{B2}}}, \quad P_\varpi(\varpi) = \frac{1}{\sqrt{2\pi}\sigma_\delta^B}e^{-\frac{\varpi^2}{2\sigma_\delta^{B2}}}, \tag{4.7.6}$$

where the superscript B stands for "beam". Fluctations in longitudinal amplitudes for one complete turn are distributed as in Eqs. (4.5.13)

$$P_{\Delta\delta}(\Delta\delta) = \frac{1}{\sqrt{2\pi}\kappa}e^{-\frac{\Delta\delta^2}{2\kappa^2}}, \quad P_{\Delta\varpi}(\Delta\varpi) = \frac{1}{\sqrt{2\pi}\kappa}e^{-\frac{\Delta\varpi^2}{2\kappa^2}}. \tag{4.7.7}$$

There is no harm in using the same symbol κ here as it will be eliminated immediately in this argument, as it was above. For that matter, the variable ϖ will also disappear shortly.

As in Eq. (4.5.14), the value of κ^2 needed to match $\Delta\epsilon_\delta^P$ is

$$\kappa^2 = \frac{\Delta\epsilon_\delta^P}{2}, \tag{4.7.8}$$

and, analogous to Eq. (4.5.15), the beam evolves during the i'th turn according to

$$<\delta_{i+1}^2 + \varpi_{i+1}^2>_{\rm beam} = 2\sigma_{\delta,i}^2(1-2a_s) + \Delta\epsilon_\delta^P. \tag{4.7.9}$$

Problem 4..9. Complete the preceeding argument to show that, in equilibrium, in the isomagnetic case, the longitudinal beam invariant is given by

$$\epsilon_\delta^B = \sigma_\delta^2 = \frac{\Delta\epsilon_\delta^P}{4\alpha_s}. \tag{4.7.10}$$

where $\Delta\epsilon_\delta^P$ comes from Eq. (4.7.3) and α_s comes from Eq. (4.3.36).

Problem 4..10. Continuing from the previous problem, derive the formula

$$\sigma_\delta = 0.813\sqrt{\frac{u_c}{(2+\mathcal{D})\,E_0}}, \tag{4.7.11}$$

for the r.m.s. beam energy spread in the isomagnetic case, where the partition number J_s, defined in Eq. (4.3.36), has been introduced, and replaced by $2+\mathcal{D}$. This formula is especially useful in the common case, *e.g.* separated function lattices, that $\mathcal{D} << 2$.

In general, calculating the energy spread requires integrating over the lattice as in Eq. (4.5.17),

$$\sigma_\delta^2 = \frac{\Delta\epsilon_\delta^P}{4\alpha_s} = \frac{\oint \frac{P_\gamma(s)ds/c}{<u(s)>_\gamma} < u^2\,(s) >_\gamma}{2E_0 J_s \oint P_\gamma\,(s)\,ds/c} = \frac{1.323}{2E_0 J_s}\,\frac{u_c}{B}\,\frac{< B^3 >_{\text{orbit}}}{< B^2 >_{\text{orbit}}}. \tag{4.7.12}$$

Here, as was true when calculating the horizontal emittance, the factor $\frac{u_c(s)}{B(s)}$, being independent of B, can be taken outside the integral, leaving σ_δ^2 proportional to $< B^3 > / < B^2 >$. Comparing with Eq. (4.5.17), this means there is a strong tendency of energy spread to be proportional to horizontal beam width. This tendency can however be overcome, for example to obtain small horizontal emittance, by designing the lattice in such a way that $\mathcal{H}$ is small at locations where B is large.

At this point it may be appropriate to look at the statistical discussion in Appendix A.

4.8. Bunch Length Determination

[Since this section has been grafted in from a different source without much care, it may not be very coherent.]

Some experiments for which synchrotron light beams are used study ultrafast time dependencies, and for these the duration of individual photon bursts is important. This duration is given by the photon bunch length (divided by c.) Though the burst observed at detection screen P from any one electron is much shorter than the electron bunch, because the electrons are traveling at the speed of light, and emissions from individual electrons are incoherent, the photon and electron bunch lengths are the same. Having calculated the electron energy spread, we are now able to calculate the electron bunch length.

The very understanding of longitudinal stability was judged sufficiently important to earn McMillen and Veksler the Nobel Prize. It is the basis of all high energy accelerators. The essential element needed for this stability (and to make up for the energy taken off in synchrotron radiation) is the RF cavity. When passing through an RF cavity a particle acquires extra energy ΔE that can be expressed in terms of ct which, (scaled by a factor c), is the arrival time t at the cavity relative to that of the reference particle,

$$\Delta E = e\hat{V}\left(t_a\right)\sin\left[\frac{\omega_{rf}\left(t_a\right)}{c}ct + \phi_0\left(t_a\right)\right] - U_0^{\mathrm{rad}}\left(t_a\right). \tag{4.8.1}$$

(In this formula, the energy loss per turn has been temporarily allowed to depend on an "absolute" time variable t_a to allow for the possiblity that the accelerator energy is changing, or "ramping" as the process is known.) The RF frequency $\omega_{rf}(t_a)$, amplitude $\hat{V}(t_a)$, and phase $\phi_0(t_a)$ are supposed to be adjusted so that the reference particle acquires precisely the correct acceleration to stay on the reference orbit. In Eq. (4.8.1) allowance has also been made for a possible lumped energy loss each turn given by $U_0^{\mathrm{rad}}(t_a)$. In practice this would be due to synchrotron radiation or wall-impedance loss, distributed continuously around the ring, but we assume that it can be adequately represented by a single loss occurring at the RF cavity. This same formula is applicable to the "ramping up" of energy during the acceleration process, but we will assume that a steady beam is being described. In this section we will set up analogies between longitudinal parameters and the more familiar Twiss parameters and emittances of transverse oscillation. The assumption

that lattice parameters are being held constant requires that the nominal $ct = 0$ particle receives no net energy so $e\hat{V}\sin\phi_0 = u$ in Eq. (4.8.1).

Reviewing earlier material, transverse (betatron) motion of a particle through one revolution of an accelerator lattice is conventionally represented using transfer matrices:

$$\begin{pmatrix} x \\ dx/ds_a \end{pmatrix}_{t+1} = \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ -\gamma_x \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x \end{pmatrix} \begin{pmatrix} x \\ dx/ds_a \end{pmatrix}_t \tag{4.8.2}$$

where $(x, dx/ds_a)$ are transverse phase space coordinates, with s_a being arc length along the reference orbit. (The subscript a is to distinguish from the symbol $s = -v(ct)/c$ which now represents displacement relative to the reference particle.) The Twiss parameters α_x, β_x, and γ_x have their customary meanings.

We wish to employ a similar notation for longitudinal motion but to use (ct, δ) instead of $(x, dx/ds_a)$ as phase space variables. In analysing motion in longitudinal phase space the following points should be noted

- Unlike dx/ds_a above, δ is not $d(ct)/ds_a$, though in a linearized treatment it is proportional to that quantity. (Derived below: $\frac{d(ct)}{ds_a} = \frac{\eta_{syn}}{v_0/c}\delta$.)
- Because of the externally imposed time dependent ramping and RF, it is natural to use time t_a rather than s_a as independent variable.
- If the longitudinal focusing were uniform around the ring it would lead to pure simple harmonic oscillation but that is not the case. The RF cavity acts like a "thin lens" for longitudinal motion, slowing down front runners and speeding up tardy particles. If we need to describe longitudinal evolution at intermediate times, while the particle is not located right at a solitary cavity, (as would be necessary with more than one cavity) "β-functions" that depend on the independent variable, are, in principle, required. But we assume that the focusing is weak (i.e. the effective "focal length" of the RF cavity is long compared to the ring radius, or equivalently $\mu_s << 1$) so that β_s can be assumed to be constant. This will be justified below.

Longitudinal evolution is then described in Twiss form by

$$\begin{pmatrix} ct \\ \delta \end{pmatrix}_{t+1} = a_s \begin{pmatrix} \cos\mu_s & \tilde{\beta}_s \sin\mu_s \\ -\sin\mu_s/\tilde{\beta}_s & \cos\mu_s \end{pmatrix} \begin{pmatrix} ct \\ \delta \end{pmatrix}_t \tag{4.8.3}$$

with an *ad hoc* damping factor $a_s = e^{-\delta_s}$ included. The analogy between $\tilde{\beta}_s$ and an ordinary beta-function will be established below. Consistent with $\tilde{\beta}_s$ being treated as constant, $\tilde{\alpha}_s$ has been taken to be zero and $\tilde{\gamma}_s = 1/\tilde{\beta}_s$.

To make this into a rotation the scale along the energy axis can be changed;

$$\begin{pmatrix} ct \\ \tilde{\beta}_s \delta \end{pmatrix}_{t+1} = a_s \begin{pmatrix} \cos\mu_s & \sin\mu_s \\ -\sin\mu_s & \cos\mu_s \end{pmatrix} \begin{pmatrix} ct \\ \tilde{\beta}_s \delta \end{pmatrix}_{t} \tag{4.8.4}$$

The numerical values of the parameters μ_s and $\tilde{\beta}_s$, parameterize the single turn analytic longitudinal oscillator description. In particular, σ_{ct}, the r.m.s. spread of ct, is related to σ_δ, the r.m.s. fractional momentum deviation, by

$$\sigma_{ct} = \tilde{\beta}_s \, \sigma_\delta . \tag{4.8.5}$$

These parameters are now to be related to the other important quantities:

- ω_{rev}: the revolution frequency; T_{rev}: the revolution period $= 2\pi/\omega_{rev}$,
- ω_{rf}: the RF frequency,
- $\mathcal{C}_0$: the circumference,
- η_{disp}: the dispersion function which is a function which, when multiplied by δ, gives the off-momentum closed orbit,
- R : the local radius of curvature,
- γ_t^{-2}: the momentum compaction factor, $= \mathcal{C}_0^{-1} \int_0^{\mathcal{C}_0} ds \, \eta_{disp}/R$; this quantity is $(d\mathcal{C}/\mathcal{C}_0)/d\delta$, the ratio of the fractional change in circumference, divided by the fractional change in momentum. For simple FODO lattices, a rough approximation is $<\eta_{disp}> \simeq \frac{\mathcal{C}_0}{2\pi\nu_x^2}$, which leads to $\gamma_t^{-2} \simeq \nu_x^{-2}$, where ν_x is the horizontal tune.
- γ^{-2}: $= (1/v_0)(dv/d\delta)$, the usual relativistic factor,
- η_{syn}: $= (d\mathcal{C}/\mathcal{C}_0 - dv/v_0)/d\delta = \gamma_t^{-2} - \gamma^{-2}$; "syn" stands for "synchronous" and this factor is frequently called the "slip factor". It is especially important in proton accelerators, since it can change sign "at transition" when $\gamma = \gamma_t$. For electron accelerators the term γ^{-2} can typically be dropped from γ_t^{-2} because $\gamma >> \gamma_t$. The main use for η_{syn} is in obtaining
- $\Delta(ct)$: (linearized) change in ct after a full turn (due to δ) $= cT_{rev}(\eta_{syn}\delta)$.

- v_0: velocity of reference particle, $= dE/dp$. (This formula, and others to appear imediately do not actually assume the particles are highly relativistic, and neither does Eq. (4.8.1). But, in present context, $E = pc$.)
- $e\hat{V}$: maximum possible energy gain in the RF cavity,
- Δp: (linearized) momentum change at RF, $= \frac{e\hat{V}\omega_{rf}\cos\phi_0}{cv_0}ct$. There are two possible choices for the angle ϕ_0, both of which lead to the same acceleration per turn of the reference particle. The ambiguity will be resolved below by the requirement that the motion be stable; the sign identification switches in passing through transition (where η_{syn} changes sign).

Assuming small amplitudes, the (linearized) change in δ at the RF can be expressed by a difference equation, using i as turn index

$$\delta_{i+\frac{1}{2}} - \delta_{i-\frac{1}{2}} = \frac{e\hat{V}\omega_{rf}\cos\phi_0}{p_0 c v_0} ct_i \tag{4.8.6}$$

(There is nothing special about $+\frac{1}{2}$ and $-\frac{1}{2}$. Any other points before and after the RF would be as good unless for some reason it was considered useful to treat δ as a smoothly variable function, in which case the present choice is a good one.) The longitudinal evolution for two consecutive turns is given by

$$\begin{aligned} ct_{i+1} - ct_i &= cT_{rev}\eta_{syn}\delta_{i+\frac{1}{2}}, \\ ct_i - ct_{i-1} &= cT_{rev}\eta_{syn}\delta_{i-\frac{1}{2}}, \end{aligned} \tag{4.8.7}$$

Subtracting these two equations, and applying Eq. (4.8.6) yields

$$ct_{i+1} - 2ct_i + ct_{i-1} = \frac{T_{rev}\eta_{syn}e\hat{V}\omega_{rf}\cos\phi_0}{p_0 v_0} ct_i \tag{4.8.8}$$

The synchrotron tune μ_s can be obtained directly from this equation because the coefficient of ct_i is $2\cos\mu_s$;

$$\cos\mu_s = 1 + \frac{T_{rev}\eta_{syn}e\hat{V}\omega_{rf}\cos\phi_0}{2p_0 v_0} \tag{4.8.9}$$

In all practical cases μ_s is sufficiently small to allow the small angle approximation, so that

$$\mu_s^2 = -\frac{T_{rev}\eta_{syn}e\hat{V}\omega_{rf}\cos\phi_0}{v_0 p_0} \tag{4.8.10}$$

For stable motion it is required that μ_s be real, which implies that the choice of angle ϕ_0 must be such that $\cos\phi_0$ and η_{syn} have opposite signs.

Another important parameter operationally is the "overvoltage factor" $e\hat{V}/U_0^{\rm rad}$ which may be required to be as large as, say, 5, to insure adequate particle lifetime. It is important because the RF cost depends primarily on $\hat{V}$.

The description has been in terms of difference equations, rather than differential equations; with the RF concentrated at one point this constitutes a correct description. But, because the synchrotron tune is usually small, the angular steps in phase space are small, and it is customary to introduce a "smoothed" description in which the longitudinal variables execute simple harmonic motion. In that spirit, the "synchrotron frequency" ω_s can be obtained from the synchrotron tune;

$$\omega_s = \frac{\mu_s}{T_{rev}}. \tag{4.8.11}$$

In the same spirit, the smoothed rate of change of the longitudinal coordinate ct with respect to t_a can be approximated by

$$\frac{d\,(ct)}{dt_a} \simeq \frac{\Delta\,(ct)}{T_{rev}} = c\eta_{syn}\delta. \tag{4.8.12}$$

It is convenient to change the scale of the "y-axis", i.e. of the second component, so that the phase space motion is circular. For this purpose, as in simple harmonic motion, the phase space coordinates need to be $(ct, \frac{1}{\omega_s}\frac{d(ct)}{dt_a})$. On the other hand, for using the beta-function terminology, they need to be $(ct, \beta_s\frac{d(ct)}{ds_a})$. Utilizing the relation $\mathcal{C}_0 = v_0 T_{rev}$, and substituting from Eq. (4.8.11), we obtain the relation

$$\beta_s = \frac{\mathcal{C}_0}{\mu_s}. \tag{4.8.13}$$

This should be regarded as the "longitudinal beta-function". But for analysing longitudinal motion, the phase space coordinate pair introduced in Eq. (4.8.4) is the convenient phase space coordinate pairs, because δ is explicit in it. For this the vertical axis in phase space, going with the horizontal axis ct, is

$$\beta_s\frac{d\,(ct)}{ds_a} = \frac{\mathcal{C}_0\eta_{syn}}{\mu_s v_o/c}\delta \simeq \frac{\mathcal{C}_0\eta_{syn}}{\mu_s}\delta \tag{4.8.14}$$

or in the notation of Eq. (4.8.4) ,

$$\tilde{\beta}_s = \frac{\mathcal{C}_0\eta_{syn}}{\mu_s}. \tag{4.8.15}$$

Knowing $\tilde{\beta}_s$, are now able to calculate the "bunch length" σ_{ct} given by Eq. (4.8.5), for example using numbers typical for CESR:

$$\sigma_{ct} = \frac{\mathcal{C}_0}{\mu_s \gamma_t^2} \sigma_\delta = \frac{768}{2\pi \times 0.04 \times 10^2} 0.6 \times 10^{-3} = 1.8\,\text{cm}. \tag{4.8.16}$$

This is a number that has been used previously in discussing undulator radiation.

The tune μ_s is rarely greater than 0.1 and is normally far less than that. As a result the "longitudinal beta function" β_s normally exceeds the ring radius R. This justifies some of the assumptions that have been made, such as neglecting α_s and treating β_s as constant. It can also be noted, since the transverse beta function satisfies $<\beta_x> \simeq \mathcal{C}_0/\mu_x$, that

$$\frac{\beta_s}{<\beta_x>} \simeq \frac{\mu_x}{\mu_s}, \tag{4.8.17}$$

the ratio typically being in the range of hundreds or thousands.

The entire analysis to this point has been linearized, but description of large amplitude motion is also important. In general this requires numerical treatment, but analytic formulas for features of the separatrix, such as the maximum excursions of the variables and the "bucket area", can be obtained. Since this discussion relies on the smoothed description, it is only approximate. Eq. (4.8.8) can be repeated, but without being simplified by linearization, by substituting directly from Eq. (4.8.1);

$$\frac{ct_{i+1} - 2ct_i + ct_{i-1}}{T_{rev}^2} = \frac{\eta_{syn}}{p_0 v_0 T_{rev}} \left[e\hat{V} \sin\left(\frac{\omega_{rf}}{c} ct + \phi_0\right) - u \right] \tag{4.8.18}$$

Interpreting the left hand side as a difference approximation to the second derivative $d^2(ct)/dt_a^2$ one obtains the "Newton's law" equation satisfied by ct

$$\frac{d^2\,(ct)}{dt_a^2} = \frac{\eta_{syn}}{p_0 v_0 T_{rev}} \left[e\hat{V} \sin\left(\frac{\omega_{rf}}{c} ct + \phi_0\right) - u \right] \tag{4.8.19}$$

Points where the "force" on the right hand side of this equation vanishes,

$$e\hat{V} \sin\left(\frac{\omega_{rf}}{c} ct + \phi_0\right) = u, \tag{4.8.20}$$

are the fixed points of the motion. The angle ϕ_0 has previously been chosen so that $ct = 0$ is the stable fixed point. As a result the unstable fixed point, which is also the maximum excursion of ct, is given by

$$(ct)_{max} = \frac{c}{\omega_{rf}} \left(\sin^{-1} \frac{u}{e\hat{V}} - \phi_0 \right) \tag{4.8.21}$$

The separatrix separating stable and unstable motion passes through the ct axis at this point. The rest of the separatrix is determined by "conservation of energy". To exploit the fact that the right hand side of Eq. (4.8.18) depends only on the dependent variable ct, it can be written $d^2(ct)/dt_a^2 = -\partial\mathcal{V}/\partial(ct)$, where $\mathcal{V}$ is a "potential energy" function given by

$$\mathcal{V} = \frac{\eta_{syn}}{p_0 v_0 T_{rev}} \left[\frac{ec\hat{V}}{\omega_{rf}} \cos\left(\frac{\omega_{rf}}{c} ct + \phi_0\right) + u\ ct \right]. \tag{4.8.22}$$

Then "energy" $\mathcal{E}$ defined by

$$\begin{aligned} \mathcal{E} &= \left(\frac{d\,(ct)}{dt_a}\right)^2 + \mathcal{V} \\ &= (c\eta_{syn})^2\,\delta^2 + \frac{\eta_{syn}}{p_0 v_0 T_{rev}} \left[\frac{ec\hat{V}}{\omega_{rf}} \cos\left(\frac{\omega_{rf}}{c} ct + \phi_0\right) + u\ ct \right] \end{aligned} \tag{4.8.23}$$

is a constant of motion. In this language, the term $(\frac{d(ct)}{dt_a})^2$ can be thought of as "kinetic energy". Since $\delta = 0$ at the previously determined maximum excursion point $(ct)_{max}$, the maximum value of $\mathcal{E}$ is given by

$$\mathcal{E}_{max} = \frac{\eta_{syn}}{p_0 v_0 T_{rev}} \left[\frac{ec\hat{V}}{\omega_{rf}} \cos\left[\frac{\omega_{rf}}{c}\,(ct)_{max} + \phi_0\right] + u\ \ (ct)_{max} \right]. \tag{4.8.24}$$

The maximum value of δ is then obtained from the stuation when the "energy is all kinetic" by

$$\delta_{max} = \frac{\sqrt{\mathcal{E}_{max}}}{c\eta_{syn}}. \tag{4.8.25}$$

The approximate bucket area is $\pi\delta_{max}(ct)_{max}$ if the phase space axes are δ and ct. Traditionally the bucket area $\mathcal{A}$ is the area of a graph of energy versus time and is given in electron-volt-seconds. As a result

$$\mathcal{A} \simeq \pi\delta_{max}\,(ct)_{max}\,p_0 v_0/c. \tag{4.8.26}$$

4.9. Tune Dependence of Emittance

Some of the important lattice parameters for a pure FODO arc are listed in Table 4.9.1 and plotted in Fig. 4.9.1. The quantity $\sin^{-3}\phi$, proportional to natural emittance, is also shown, as is the quantity $\sqrt{\sin^{-3}\phi\beta^{\max}}$, proportional to the the maximum beam width.

Table 4.9.1: Dependence of some lattice properties on phase advance per cell 2ϕ. FODO lattice with 50 cells, $l = 7.68\,\mathrm{m}$, $l_B = 6.528\,\mathrm{m}$, $\Delta\theta = 2\pi/100$, $R = 103.9\,\mathrm{m}$, $B = 0.1641\,\mathrm{T}$, $\gamma = 10^4$.

2ϕ	$\check{\beta}/\hat{\beta}$	$\check{\eta}/\hat{\eta}$	$1/\sin^3\phi$	$\mathcal{H}_{av}$	η^2/β	$\mathcal{H}_{av}$	q
cell phase	beta min/max	eta min/max		numer-ical		(4.9.4)	1/2 quad strength
degrees	m	m		m	m	m	1/m
63.4	8.25/25.9	1.300/2.20	7.010	0.1941	0.1923+/-0.0055	0.2090	0.0705
77.6	5.94/25.2	0.853/1.61	4.065	0.1112	0.1086+/-0.0060	0.1232	0.0842
90.0	4.57/25.8	0.632/1.30	2.828	0.0756	0.0722+/-0.0064	0.0858	0.0950
91.8	4.44/26.0	0.614/1.27	2.700	0.0729	0.0693+/-0.0065	0.0819	0.0961
106.0	3.25/28.3	0.460/1.05	1.963	0.0513	0.0461+/-0.0072	0.0595	0.1076
120.4	3.39/32.7	0.369/0.914	1.530	0.0406	0.0329+/-0.0081	0.0464	0.1168
135.0	1.68/41.1	0.310/0.822	1.268	0.0363	0.0240+/-0.0092	0.0385	0.1244

Calculation of the beam emittance begins with $\mathcal{H}(s)$ defined in Eq. (4.5.6), In a pure FODO lattice, with no "η-mismatch", The η-function and β_x function are related to good approximation by

$$\eta(s) = \mathrm{const.}\beta_x^{1/2}(s) = \frac{\eta^{\mathrm{typ}}}{\sqrt{\beta_x^{\mathrm{typ}}}}\beta_x^{1/2}(s) = \Delta\theta\,\ell^{1/2}(\sin\phi)^{-3/2}\,\beta_x^{1/2}(s), \tag{4.9.1}$$

where ϕ, the phase advance per half-cell. It satisfies

$$\sin\phi = q\ell, \tag{4.9.2}$$

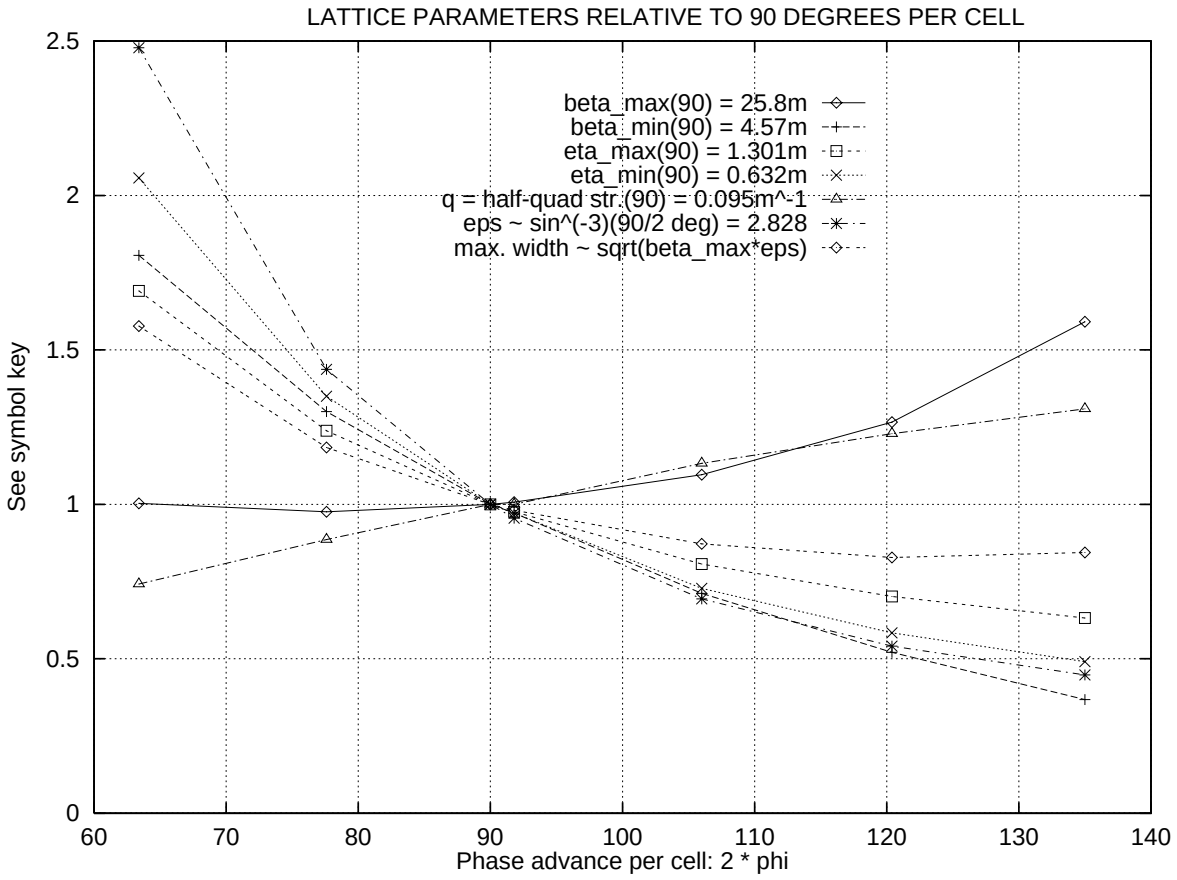

Figure 4.9.1: Lattice parameters of a pure FODO lattice, referred to the 90° degrees per cell ($\phi = 45°$) case. absolute values at 90° are given with the symbol key.

where q is the inverse focal length of a regular arc "half-quad", ℓ is the length of a half-cell, and $\Delta\theta$ is the bend angle per half-cell. Typical values have been obtained from

$$\beta_x^{\rm typ} = \sqrt{\beta_x^{\rm max}\beta_x^{\rm min}} = \frac{1}{q}, \quad \eta^{\rm typ} = \frac{1}{2}\left(\eta^{\rm max} + \eta^{\rm min}\right) = \frac{\Delta\theta}{q^2\ell}. \tag{4.9.3}$$

Substitution into Eq. (4.5.6) yields

$$\mathcal{H}\left(s\right) = \frac{\eta^2 + \left(\beta_x\eta' + \alpha_x\eta\right)^2}{\beta_x} \simeq \frac{\Delta\theta^2\ell}{\sin^3\phi}, \tag{4.9.4}$$

surprisingly independent of longitudinal coordinate s. Dependence of some lattice parameters as a function of phase advance per cell 2ϕ are listed in Table 4.9.1. Included is a numerical determination of $\mathcal{H}$. It can be seen that Eq. (4.9.4) gives a reasonably good estimate over the full range (considerably better than the approximation of Eq. (4.9.1) on which it is based.

With $\Delta\theta^2\ell$ being held fixed the emittance varies proportional to $\sin^{-3}\phi$. This dependence is shown in Table 4.9.1 and in Fig. 4.9.1. This shows, for example, that doubling the phase advance per cell from 67.5° to 135° reduces the (ideal FODO) emittance by a factor 4.6. The reduction factor for an actual accelerator will differ from this due to irregular sections such as the high bend regions of CESR. For a fixed total machine tune any deviation from regularity has a strong tendency to increase the emittance.

We obtain the natural emittance of a pure, isomagnetic FODO lattice by substitution from Eq. (4.9.4) into Eq. (4.5.17), and approximating $J_x = 1$,

$$\epsilon_x^{eq} = \frac{\sigma_x^2}{\beta_x} = \frac{1.323}{2J_x}\,\mathcal{H}\,\frac{u_c}{E_0} = \frac{1.323}{2}\,\frac{\Delta\theta^2\ell}{\sin^3\phi}\,\frac{u_c}{E_0}. \tag{4.9.5}$$

References

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2. J.D. Jackson, *Classical ELectrodynamics,* Third Edition, John Wiley, 1999.
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Chapter 5.
Statistical Methods

5.1. Brownian Motion, Stochastic and Fokker-Planck Equations, etc.

The processes that determine beam distributions in electron storage rings can be related to Brownian motion. This chapter contains a survey of this subject, following Chandrasekhar[1] or Stratonovich[2] or Rice[3] closely in places. The purpose is to develop tools for analysing statistical probems. Sophisticated statistical methods are needed to analyse many of the problems of importance for electron rings, such as beam lifetime, response to external stimulation, and intrabeam scattering.

Small, but visible, macroparticles suspended in colloids are observed to jiggle around randomly and this is now understood to be due to molecular collisions occuring at rapid rate $\mathcal{N}_1$ collisions per second. The molecular force $F(t)$ varies extremely rapidly, but one supposes that the impulse $I_j = \int F(t)\,dt$ delivered in a single collisions is random and occurs at random time t_j. The duration of each collision is short compared to $1/\mathcal{N}_1$, the typical interval between collisions. Statistically, these impulses resemble the photons emitted from a circulating electron. Through quantum mechanics, the radiated power $P(t)$ makes its presence felt through individual photons of energy $u_j = \int P(t)\,dt$ which are radiated at random times t_j. Their rate $\mathcal{N}_1$ is high, and their duration is short compared to $1/\mathcal{N}_1$. To enhance this analogy let us suppose the Brownian motion is one-dimensional. Then, the only essential difference is that the u_j are necessarily positive, while the I_j can have either sign. Because the radiated energy is made up "on the average" by accelerating structures in the ring, even this difference is largely superceded.

In statistical terminology the points t_j are said to form a "system of random points" and if, as we assume, the times t_j are uncorrelated, as a "Poisson system". Statistical properties of such systems are discussed in the next section. When such a process acts over a finite, but short, time interval Δt, the fluctuations it causes tend to be smoothed out to some extent due to averaging. Here we consider distributions on time scales long enough for such averaging to have occurred. In particular, it is assumed that enough